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## Wave Equation—Wave Mechanics of Particles in Curved Space-Time (Part I)

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### Abstract

In this paper, a universal wave equation of the motion of microscopic particles in curved space-time is established, which satisfies the conservation of probability. Under the classical limit conditions, the wave equation returns to the dynamic equation of special relativity. The application of the wave equation in the evolution of a flat expanding universe has preliminarily proved the correctness of the equation, and the new wave equation may open a window for us to understand physics within the Planck scale.

### Full Text

#### Preamble

#### Wave Equation—Wave Mechanics of Particles in Curved Spacetime (Part I)

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In this paper, a universal wave equation for the motion of microscopic particles in curved spacetime is established that satisfies probability conservation. Under classical limit conditions, the wave equation reduces to the dynamical equations of special relativity. The application of this wave equation to the evolution of a flat expanding universe provides preliminary verification of its correctness, and this new wave equation may open a window for understanding physics within the Planck scale.

### Introduction

General relativity and quantum mechanics represent two epoch-making contributions to modern physics, each achieving brilliant success in its respective

domain. General relativity governs the motion of matter in macroscopic, large-scale spacetime, where quantum effects can be neglected. Quantum mechanics, conversely, describes the microscopic world of atoms and subatomic particles, where gravitational effects are completely negligible. However, in scenarios involving both strong gravity and microscopic spacetime scales—such as black hole interiors or the early universe—both theories must be invoked simultaneously. Describing such phenomena requires a synthesis of general relativity and quantum mechanics [?].

How can these two theories be combined? One approach attempts to quantize the gravitational field itself, as in canonical quantum gravity [?]. However, the resulting quantum field theory cannot be renormalized, leading to divergences. Another approach seeks to quantize spacetime itself [?], offering a novel perspective but lacking experimental support.

In this work, my objective is neither to quantize the gravitational field nor spacetime directly, but rather to address a fundamental question: In general relativity or curved spacetime, what universal wave equation governs the matter waves of microscopic particles? I impose five essential conditions on this wave equation: (1) It must preserve the statistical interpretation of the wave function, where the squared amplitude is proportional to the probability of finding the particle—in other words, the wave must be a probability wave. (2) It must guarantee probability conservation in curved spacetime. (3) It must maintain the validity of the superposition principle. (4) It must explicitly contain the spacetime metric tensor  $g_{\mu\nu}$ . (5) It must simultaneously incorporate three fundamental physical constants: the gravitational constant  $G$ , Planck's constant  $\hbar$ , and the speed of light  $c$ , thereby representing a true synthesis of general relativity and quantum mechanics.

Clearly, neither the Schrödinger equation nor the Dirac equation can satisfy all these conditions simultaneously. In this paper, I establish the wave equation for microscopic particle motion in a gravitational field (curved spacetime), laying the foundation for further developments in quantum mechanics.

## II. Establishment of the Wave Equation

My approach extends Feynman's path integral theory [?, ?] to curved spacetime and then derives the differential wave equation. Let  $\psi(x, y, z, t)$  represent the probability amplitude for a particle to appear at point  $(x, y, z)$  at time  $t$ , and let  $\psi(x', y', z', t')$  represent the probability amplitude at point  $(x', y', z', t')$ . These amplitudes are related through the propagator  $K(x, y, z, t; x', y', z', t')$ :

$$\psi(x, y, z, t) = \int K(x, y, z, t; x', y', z', t') \psi(x', y', z', t') d^4x'$$

where  $K(x, y, z, t; x', y', z', t')$  is called the propagator, describing the probability amplitude for a particle to travel from point  $A(x', y', z', t')$  to point

$B(x, y, z, t)$ . According to Feynman's path integral theory [?, ?], this propagator is given by:

$$K(B, A) = C \sum_{\text{all paths}} \exp \left\{ \frac{iS[r(t)]}{\hbar} \right\}$$

where  $S[r(t)] = \int L(r, \dot{r}, t) dt$  represents the action along path  $r(t)$ ,  $L$  is the particle's Lagrangian, and  $C$  is an appropriate normalization constant. The factor  $\hbar$  in the exponent renders the phase dimensionless, and the sum includes all possible paths from  $A$  to  $B$ .

In curved spacetime, I introduce the wave function  $\psi(x, y, z, t; \tau)$  to represent the probability amplitude for a particle to appear at spacetime point  $(x, y, z, t)$  at proper time  $\tau$ . The proper time  $\tau$  is related to spacetime coordinates through the metric:

$$d\tau^2 = -ds^2 = -g_{\mu\nu} dx^\mu dx^\nu$$

where  $g_{\mu\nu}$  is the spacetime metric tensor. Extending the path integral formalism to curved spacetime yields:

$$\psi(x, y, z, t; \tau) = \int K(x, y, z, t; \tau; x', y', z', t'; \tau') \psi(x', y', z', t'; \tau') \sqrt{-g} d^4x'$$

where  $g = |g_{\mu\nu}|$  is the determinant of the metric tensor and  $\sqrt{-g} d^4x$  is the invariant volume element.

To calculate the propagator in curved spacetime, I further extend Feynman's path integral formula. Let the propagator from point  $A(x', y', z', t')$  to point  $B(x, y, z, t)$  be:

$$K(B, A) = C \sum_{\text{all paths}} \exp \left\{ \frac{iS[r(\tau)]}{l_P^2} \right\}$$

where the action is:

$$S[r(\tau)] = \int_{\tau_A}^{\tau_B} L d\tau = \int_{\tau_A}^{\tau_B} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$$

Here  $L \equiv g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$  is the particle's Lagrangian,  $C$  is a normalization constant, and  $l_P = (G\hbar/c^3)^{1/2} = 1.61 \times 10^{-33}$  cm is the Planck length. Introducing the Planck length renders the phase factor dimensionless. As in the flat spacetime case, the sum includes all possible paths from  $A$  to  $B$ .

Equations (4) and (5) constitute the extension of Feynman's path integral theory to curved spacetime. From them, we can derive the differential wave equation for particle motion.

Consider the particle state  $\psi(x, y, z, t; \tau + \epsilon)$  at proper time  $\tau + \epsilon$ , which relates to the state  $\psi(x', y', z', t'; \tau)$  at proper time  $\tau$  through:

$$\psi(x, y, z, t; \tau + \epsilon) = \int K(x, y, z, t; \tau + \epsilon; x', y', z', t'; \tau) \psi(x', y', z', t'; \tau) \sqrt{-g} d^4 x'$$

In the limit  $\epsilon \rightarrow 0$ , for this infinitesimal proper time interval, the propagator can be expressed as:

$$K(x, y, z, t; \tau + \epsilon; x', y', z', t'; \tau) = C \exp \left\{ \frac{ig_{\mu\nu} \Delta x^\mu \Delta x^\nu}{l_P^2 \epsilon} \right\}$$

where  $C$  is a constant to be determined and  $\Delta x^\mu = x^\mu - x'^\mu$ .

Substituting this into the previous equation and expanding to first order in  $\epsilon$  while performing a Taylor expansion of the integrand in  $\Delta x^\mu$ , we obtain after careful calculation:

$$\psi(x, y, z, t; \tau) + \epsilon \frac{\partial \psi}{\partial \tau} = \psi(x, y, z, t; \tau) + \frac{il_P^2}{\sqrt{-g}} \left[ \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial \psi}{\partial x^\nu} \right) \right] \epsilon$$

This yields the differential wave equation for microscopic particle motion in curved spacetime (gravitational field):

$$il_P^2 \frac{\partial \psi}{\partial \tau} = - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial \psi}{\partial x^\nu} \right)$$

This complex equation preserves the statistical interpretation of the wave function in curved space. For a normalized wave function satisfying:

$$\int |\psi(x, y, z, t; \tau)|^2 \sqrt{-g} d^4 x = 1$$

the probability density  $\rho = |\psi|^2$  obeys a probability conservation equation (to be proven in the next section). The wave equation is linear, meaning that if  $\psi_1$  and  $\psi_2$  are solutions, then any linear combination  $a\psi_1 + b\psi_2$  is also a solution, consistent with quantum mechanics' superposition principle. In natural units where  $l_P = t_P c$ , with Planck time  $t_P = 5.38 \times 10^{-44}$  s, the wave equation can be expressed by replacing  $l_P$  with  $t_P$ .

Equation (20) contains three fundamental constants— $G$ ,  $\hbar$ , and  $c$  (since  $l_P = (G\hbar/c^3)^{1/2}$ )—making it a wave equation that truly combines general relativity with quantum mechanics. If the Dirac equation unified special relativity with quantum mechanics, then wave equation (20) represents the unification of general relativity with quantum mechanics.

### III. Conservation of Probability and Classical Limit

I define the four-dimensional operator:

$$\nabla \equiv \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} + \hat{e}_t \frac{\partial}{\partial t}$$

where  $\hat{e}_x, \hat{e}_y, \hat{e}_z, \hat{e}_t$  are unit vectors in four-dimensional rectangular coordinates. The wave equation (20) can then be rewritten as:

$$il_P^2 \frac{\partial \psi}{\partial \tau} = -\frac{1}{\sqrt{-g}} \nabla \cdot (\sqrt{-g} \nabla \psi)$$

Separating the wave function into amplitude and phase, we let:

$$\psi = Re^{iS}$$

where  $R$  and  $S$  are real functions. Substituting this into the wave equation yields two coupled equations:

$$\begin{aligned} \frac{\partial R}{\partial \tau} &= -\frac{1}{2l_P^2} \left[ R \nabla^2 S + 2 \nabla R \cdot \nabla S + \frac{1}{\sqrt{-g}} (\nabla \sqrt{-g}) \cdot (\nabla S) R \right] \\ \frac{\partial S}{\partial \tau} &= -\frac{1}{2l_P^2} \left[ (\nabla S)^2 - \frac{\nabla^2 R}{R} + \frac{1}{\sqrt{-g}} \nabla \sqrt{-g} \cdot \nabla S \right] \end{aligned}$$

These equations are completely equivalent to the original wave equation (24). I now discuss their physical significance.

First, I prove that equation (26) expresses probability conservation in curved spacetime. The probability density  $\rho$  and probability current density  $J^\mu$  are defined as:

$$\rho = |\psi|^2 = R^2$$

$$J^\mu = \frac{i}{2l_P^2} [\psi^* \nabla^\mu \psi - \psi \nabla^\mu \psi^*]$$

Substituting  $\psi = Re^{iS}$  into the definition of  $J^\mu$  gives:

$$J^\mu = R^2 \nabla^\mu S$$

Multiplying equation (26) by  $2R$  and using these definitions, we obtain the continuity equation:

$$\frac{\partial \rho}{\partial \tau} + \nabla_\mu J^\mu = 0$$

where  $\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} J^\mu)$  is the covariant divergence of the four-dimensional probability current density.

Integrating this continuity equation over any volume element in curved spacetime and applying Gauss's theorem yields:

$$\frac{d}{d\tau} \int \rho \sqrt{-g} d^4x + \oint J^\mu d^3S_\mu = 0$$

which is precisely the conservation equation for probability in curved spacetime.

Now consider the classical limit of equation (27). Under classical conditions, the Planck length  $l_P \rightarrow 0$ , so the  $l_P^2$  terms become negligible, simplifying equation (27) to:

$$(\nabla S)^2 + \frac{1}{\sqrt{-g}} \nabla \sqrt{-g} \cdot \nabla S = 0$$

Since the four-velocity is  $U^\mu = \nabla^\mu S$ , this can be rewritten as:

$$U^2 + \frac{1}{\sqrt{-g}} U \cdot \nabla \sqrt{-g} = 0$$

Taking the four-dimensional gradient of both sides and using  $U^\mu = \nabla^\mu S$ , we obtain:

$$\frac{dU^\mu}{d\tau} + (U \cdot \nabla) U^\mu = 0$$

This is exactly the equation of motion for free particles in special relativity. Thus, when the Planck length approaches zero, both gravitational and quantum effects vanish simultaneously, and the wave equation reduces to the classical dynamical equations of special relativity.

## IV. The Evolution of a Flat Expanding Universe—A Simple Application

Wave equation (20) is a complex partial differential equation that is generally difficult to solve, though special cases admit solutions. In this section, I apply the wave equation to discuss the dynamical evolution of a flat expanding universe.

In a flat, expanding universe, the metric tensor is given by the Robertson-Walker (RW) metric [?]:

$$g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t))$$

where  $a(t)$  is the scale factor of the expanding universe.

Substituting the RW metric into Einstein's field equations yields the Friedmann equations:

$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p)$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2$$

where  $\rho$  and  $p$  are the energy density and pressure of the cosmic medium, uniform in space and varying only with time:  $\rho = \rho(t)$ ,  $p = p(t)$ . These two independent equations involve three unknown functions— $\rho(t)$ ,  $p(t)$ , and  $a(t)$ —making the system incomplete. Traditional cosmological models must artificially add an equation of state  $p = p(\rho)$  to close the system. In a complete theory, the equation of state should emerge naturally from the fundamental equations.

Since wave equation (20) applies only to massive matter, during epochs of the universe dominated by non-zero mass matter, the Friedmann equations combined with the new equation of motion (20) form a complete set of dynamical equations.

In comoving coordinates,  $d\tau = dt$ . Separating the wave function into amplitude and phase as  $\psi(t) = R(t)e^{iS(t)}$  and substituting the RW metric into equations (26) and (27) yields:

$$\dot{R} = -\frac{1}{2l_p^2} \left[ R\ddot{S} + 2\dot{R}\dot{S} + 3\frac{\dot{a}}{a}R\dot{S} \right]$$

$$\dot{S} = -\frac{1}{2l_p^2} \left[ \dot{S}^2 - \frac{\ddot{R}}{R} + 3\frac{\dot{a}}{a}\dot{S} \right]$$

Together with the Friedmann equations (50) and (51), we have a complete system of four independent equations for four unknown functions:  $\rho(t)$ ,  $p(t)$ ,  $a(t)$ , and  $R(t)$  (with  $S(t)$  determined up to an additive constant). The statistical interpretation  $\rho(t) = |\psi|^2 = R^2(t)$  ensures the system is closed and solvable.

The exact solution to this system is:

$$\rho(t) = \frac{1}{6\pi Gt + C_1}$$

$$p(t) = 0$$

$$a(t) = C_2(6\pi Gt + C_1)^{1/3}$$

$$R(t) = \frac{1}{\sqrt{6\pi Gt + C_1}}$$

$$S(t) = t + C_3$$

where  $C_1$ ,  $C_2$ , and  $C_3$  are integration constants determined by initial conditions. This solution shows that both the equation of state  $p = 0$  and the relation  $\rho a^3 = \text{const.}$  emerge naturally as consequences of the complete dynamical system. Furthermore, the four-velocity  $U^0 = \dot{S} = 1$  and the normalization  $\dot{S}^2 = U^0 U_0 = -1$  follow directly from the equations.

## V. Equations of Spacetime Structure and Particle Motion

In general relativity, particles move along geodesics in curved spacetime, with the metric tensor  $g_{\mu\nu}$  obtained by solving Einstein's field equations. The standard procedure assumes a known matter distribution (i.e., a specified energy-momentum tensor  $T_{\mu\nu}$ ), computes  $g_{\mu\nu}$  from the field equations, and then determines particle motion from the geodesic equation. However, particle motion itself contributes to the energy-momentum tensor, which in turn affects the spacetime geometry. In the most general case, both  $T_{\mu\nu}$  and  $g_{\mu\nu}$  should be treated as unknown. General relativity cannot solve for both simultaneously because the equations of motion are not independent—they are corollaries of the field equations. Thus, we have only one independent field equation, insufficient to determine both  $g_{\mu\nu}$  and  $T_{\mu\nu}$  simultaneously.

Microscopic particle motion, however, fundamentally obeys quantum mechanical laws. With the new wave equation (20) for particle motion, which is independent of Einstein's field equations, we can now formulate a complete system to solve for both  $g_{\mu\nu}$  and  $T_{\mu\nu}$  simultaneously.

Let the particle wave function be  $\psi = Ae^{iS}$ , where  $A$  and  $S$  satisfy:

$$i\ell_P^2 \frac{\partial A}{\partial \tau} = -\frac{1}{2}A\nabla^2 S - \nabla A \cdot \nabla S - \frac{1}{2\sqrt{-g}}(\nabla\sqrt{-g}) \cdot (\nabla S)A$$

$$\frac{\partial S}{\partial \tau} = -\frac{1}{2\ell_P^2}(\nabla S)^2 + \frac{1}{2\ell_P^2} \frac{\nabla^2 A}{A} - \frac{1}{2\sqrt{-g}}\nabla\sqrt{-g} \cdot \nabla S$$

The energy-momentum tensor of the particle can then be expressed as:

$$T_{\mu\nu} = \rho U_\mu U_\nu = A^2(\nabla_\mu S)(\nabla_\nu S)$$

When the wave function  $\psi$  evolves, the energy-momentum tensor  $T_{\mu\nu}$  changes accordingly through equation (60). Assuming Einstein's field equations remain valid during this dynamical evolution, any local change in  $T_{\mu\nu}$  immediately induces a corresponding change in the metric tensor to preserve the validity of the field equations.

Substituting equation (60) into Einstein's field equations and combining them with the motion equations (58) and (59) yields the coupled system:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GA^2(\nabla_\mu S)(\nabla_\nu S)$$

$$i\ell_P^2 \frac{\partial A}{\partial \tau} = -\frac{1}{2}A\nabla^2 S - \nabla A \cdot \nabla S - \frac{1}{2\sqrt{-g}}(\nabla\sqrt{-g}) \cdot (\nabla S)A$$

$$\frac{\partial S}{\partial \tau} = -\frac{1}{2\ell_P^2}(\nabla S)^2 + \frac{1}{2\ell_P^2} \frac{\nabla^2 A}{A} - \frac{1}{2\sqrt{-g}}\nabla\sqrt{-g} \cdot \nabla S$$

where  $R_{\mu\nu}$  is the Ricci tensor and  $R \equiv g^{\mu\nu}R_{\mu\nu}$  is the curvature scalar. Solving this system simultaneously yields both the particle wave function (through  $A$  and  $S$ ) and the spacetime metric tensor  $g_{\mu\nu}$ . The solution represents the true physical state—a state of equilibrium between particle motion and spacetime geometry.

Several important considerations emerge: (1) The system contains 12 unknown functions: 10 metric components  $g_{\mu\nu}$ , plus the wave function amplitude  $A$  and phase  $S$ . When the Bianchi identities  $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$  hold, only 8 independent equations remain, leaving the system underdetermined. Following standard methods in general relativity, we can impose 4 gauge or coordinate conditions to complete the system. A particularly convenient choice is the harmonic coordinate condition:

$$\Gamma^\lambda \equiv g^{\mu\nu}\Gamma_{\mu\nu}^\lambda = 0$$

- (2) The new wave equation (20) contains the Planck length  $l_P$ . When  $l_P \rightarrow 0$ , it reduces to classical special relativistic dynamics, but within the Planck scale,  $l_P$  cannot be ignored. It is therefore reasonable to propose that equation (20) describes matter motion at all spacetime scales, including the Planck scale. If matter motion always affects the spacetime metric through Einstein's field equations, then the coupled system (61) remains valid within the Planck scale. At such scales, quantum fluctuations of spacetime may violate the Bianchi identities (62), potentially yielding 12 independent equations for 12 unknown functions—a complete, self-contained system. Thus, solving the coupled equations (61) may provide profound insights into Planck-scale physics, warranting further investigation.

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