Revealing the effects of curvature on the cosmological models (Postprint)

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Abstract

In this paper we consider the effects of adding curvature in extended cosmologies involving a free-to-vary neutrino sector and different parametrizations of Dark Energy (DE). We make use of the Planck 2018 cosmic microwave background temperature and polarization data, Baryon Acoustic Oscillations and Pantheon type Ia Supernovae data. Our main result is that a non-flat Universe cannot be discarded in light of the current astronomical data, because we find an indication for a closed Universe in most of the DE cosmologies explored in this work. On the other hand, forcing the Universe to be flat can significantly bias the constraints on the equation of state of the DE component and its dynamical nature.

Full Text

Preamble

Revealing the Effects of Curvature on Cosmological Models

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In this paper we consider the effects of adding curvature in extended cosmologies involving a free-to-vary neutrino sector and different parametrizations of Dark Energy (DE). We make use of the Planck 2018 cosmic microwave background temperature and polarization data, Baryon Acoustic Oscillations and Pantheon type Ia Supernovae data. Our main result is that a non-flat Universe cannot be discarded in light of the current astronomical data, because we find an indication for a closed Universe in most of the DE cosmologies explored in this work. On the other hand, forcing the Universe to be flat can significantly bias the constraints on the equation of state of the DE component and its dynamical nature.

Introduction

On large scales, our Universe is almost homogeneous and isotropic, but as far as its curvature is concerned we cannot firmly conclude that it is spatially flat [?]. Even though observational probes in past years were in agreement with spatial flatness \cite{2–7}, this result has recently been questioned by some experiments. Investigations with Cosmic Microwave Background (CMB) temperature and polarization spectra from the Planck 2018 team using the baseline Plik likelihood suggest that our Universe could have a closed geometry at more than three standard deviations \cite{8–11} (4σ when the physical curvature density is considered [?]), with indications coming mainly from temperature data affected by an otherwise inexplicable excess of lensing (the A_{lens} problem [?, ?, ?]). Furthermore, complementary examinations using the alternative CamSpec likelihood [?, ?] also support a closed geometry of the Universe at more than 99% CL, and this indication persists even when considering CMB temperature-only data from the new Planck PR4 analysis [?]. Additionally, an indication for a closed universe is present in BAO data using Effective Field Theories of Large Scale Structure, once the (hidden) assumptions of flatness (in the fiducial cosmology, reconstruction process, and covariance matrix) are removed from the beginning [?]. These outcomes challenge the assumptions of flatness present in the standard ΛCDM model and spark debate about the flatness of our Universe. In fact, if there is no solid theoretical argument, then observations in agreement with a small curvature cannot be used as proof for a spatially flat universe [?], and the assumption of a spatially flat Universe may significantly affect cosmological parameters when the underlying geometry is curved [?]. Therefore, in this article we systematically investigate several well-known cosmological models, leaving the curvature of the Universe as a free parameter and allowing observational data to select the best possibilities, removing possible biases due to the flatness assumption. Moreover, we are interested in investigating what happens to current cosmological tensions [?] when curvature is allowed (also see \cite{22–30}). In particular, we examine the value of the Hubble constant and the $>5\sigma$ tension between the ΛCDM-based Planck 2018 estimate [?] and the SH0ES (Supernovae and H0 for the Equation of State of dark energy) measurement [?, ?], known as the Hubble tension [?, ?] (see also Refs. \cite{35–69} aiming to alleviate the H0 tension in various ways), together with the $>3\sigma$ tension in the S8 parameter [?] defined as a combination of the amplitude of the matter power spectrum $ $8 with the matter density at present Ωm0 (S8 = $ $8√($\Omega$m0/0.3)) between CMB data (within the ΛCDM assumption) and weak lensing experiments \cite{71–74} (see Refs. \cite{40,54,63,75–83} offering possible routes to alleviate the S8 tension in various alternatives to ΛCDM).

Following this approach, we consider a variety of cosmological models in a curved background (i.e., allowing the curvature of the Universe as a free parameter), namely: the standard ΛCDM, wCDM, the Chevallier-Polarski-Linder parametrization [?, ?] where dark energy has a dynamical equation of state parameter, and a recently introduced emergent dark energy model known as the phenomenologically emergent dark energy model [?, ?][1], along with various extensions allowing a free-to-vary neutrino sector characterized by the total neutrino mass and the effective number of relativistic degrees of freedom. These models, together with their neutrino sector extensions, have been constrained using CMB data from Planck 2018, Baryon Acoustic Oscillation distance measurements, and the Pantheon sample of Type Ia Supernovae. This work complements similar approaches used in Refs. [?, ?, ?, ?], but differs in the combination of parameters explored, observational datasets, and extended analysis in the dark energy sector. In particular, for the first time we include in the analysis a dynamical dark energy equation of state together with a non-zero curvature parameter that is free to vary.

The paper is structured as follows. In Section II we introduce the framework; in Section III we describe the observational data and statistical analysis method; in Section IV we present our main results. Finally, in Section V we summarize our findings and conclusions.

II. Dark Energy in a Curved Universe

As argued in the Introduction, we start with a homogeneous and isotropic spacetime characterized by the Friedmann-Lemaître-Robertson-Walker (FLRW) line element:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

where $a(t)$ is the expansion scale factor of the universe and K denotes its curvature scalar. The values $K = 0, +1, -1$ correspond to spatially flat, closed, and open geometries respectively. In the context of General Relativity, we assume that the matter distribution of the universe is minimally coupled to gravity and that none of the fluids interact with any component. The dynamics of the universe are then described by Einstein's gravitational equations:

$$\dot{H} + 3H^2 = -\frac{K}{a(t)^2} + \frac{8\pi G}{3}\sum_i \rho_i,$$

$$2\dot{H} + 3H^2 = -\frac{K}{a(t)^2} - 8\pi G\sum_i p_i,$$

where an overhead dot represents time derivative, $H \equiv \dot{a}(t)/a(t)$ is the Hubble function of the FLRW universe, ρ_i and p_i are respectively the energy density and pressure of the i-th component, and G is Newton's gravitational constant. Introducing the critical density $\rho_c = 3H^2/8\pi G$, equation (2) can be expressed as:

$$1 = \Omega_K + \sum_i \Omega_i,$$

where $\Omega_i = \rho_i/\rho_c$ is the density parameter for the i-th fluid and $\Omega_K = -K/(a^2H^2)$ is the curvature density parameter.[2] The evolution of the energy density of each component can be found by solving the balance equation:

$$\sum_i \dot{\rho}_i + 3H\sum_i (p_i + \rho_i) = 0,$$

which follows from the field equations (2) and (3). Since the fluids do not interact, we have $\dot{\rho}_i + 3H(p_i + \rho_i) = 0$ for each i.

Here we consider that the total energy density of the Universe is distributed among radiation (r), baryons (b), neutrinos (ν), cold dark matter (CDM), and a dark energy (DE) component. For each component $i = r, b, \nu, \mathrm{CDM}, \mathrm{DE}$, one can find the evolution law of the energy density by solving the conservation equation for the i-th fluid. Concerning the dark energy sector, using the conservation equation (4) for its equation-of-state $w_{\mathrm{DE}} = p_{\mathrm{DE}}/\rho_{\mathrm{DE}}$, the evolution of the energy density is:

$$\rho_{\rm DE} = \rho_{\rm DE,0} \exp\left[\int_0^z \frac{3[1+w_{\rm DE}(z')]}{1+z'} dz'\right],$$

where $\rho_{\rm DE,0}$ is the present value of the dark energy density. For $w_{\rm DE} = -1$, the energy density becomes constant, representing a cosmological constant. As the nature of dark energy is not yet clearly understood, one might investigate how different variants of dark energy models affect the expansion history along with the curvature of the Universe. We select several well-known choices:

- The cosmological constant as dark energy candidate characterized by $w_{\rm DE} = -1$.
- Dark energy with a constant equation of state other than the cosmological constant (i.e., $w_{\rm DE} \neq -1$).[3]
- Dynamical dark energy equation of state assuming the Chevallier-Polarski-Linder parametrization [**?**, **?**]:

  $$w_{\rm DE} = w_0 + w_a(1-a),$$

  where w_0 is the present value of $w_{\rm DE}$ and $w_a = dw_{\rm DE}/da$ at $a = 1$ is another free parameter.[4]
- The phenomenologically emergent dark energy (PEDE) model, where DE has no effective presence in the past but emerges at late times. In this scenario, the dark energy density is directly parametrized as [**?**, **?**]:

  $$\rho_{\rm DE} = \rho_{c,0}\, \Omega_{\rm DE,0}\left[1 - \tanh(\log_{10}(1+z))\right],$$

  where $\rho_{c,0}$ is the critical energy density at present and $\Omega_{\rm DE,0}$ is the present value of the DE density parameter $\Omega_{\rm DE} \equiv \rho_{\rm DE}/\rho_{\rm crit}$. Note that even though this parametrization has a dynamical nature, it offers no additional degree of freedom or free parameter.

With these models established, we can now constrain them in the presence of curvature.

III. Observational Data and Statistical Process

We describe the main observational datasets used to constrain the cosmological models and our statistical methodology.

Cosmic Microwave Background (CMB) Observations: We use measurements from the final release of the Planck 2018 team, specifically the CMB temperature and polarization power spectra (plikTTTEEE+lowl+lowE) [**?**, **?**].

Baryon Acoustic Oscillations (BAO): We employ various BAO measurements from 6dFGS [**?**], SDSS-MGS [**?**], and BOSS DR12 [**?**] as considered by the Planck 2018 team [**?**].

Pantheon Sample from Supernovae Type Ia (SNIa): Along with CMB and BAO measurements, we use the Pantheon sample [?]—a recent compilation of SNIa distributed in the redshift interval $z \in [0.01, 2.3]$.

To constrain the parameter spaces of these cosmological scenarios, we use the publicly available Markov Chain Monte Carlo code CosmoMC [?, ?] (see http://cosmologist.info/cosmomc/), which supports the Planck 2018 likelihood [?] and includes convergence diagnostics following Gelman-Rubin statistics [?]. We adopt flat priors on the parameters as listed in Table I.

Parameter Spaces:

The scenarios $\Lambda\mathrm{CDM} + \Omega_K$, $\Lambda\mathrm{CDM} + \Omega_K + M_\nu$, $\Lambda\mathrm{CDM} + \Omega_K + N_{\mathrm{eff}}$, and $\Lambda\mathrm{CDM} + \Omega_K + M_\nu + N_{\mathrm{eff}}$ contain respectively 7, 8, 8, and 9 free parameters:

$$\begin{gathered}
\mathcal{L}_1 \equiv \{\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, n_s, \log[10^{10} A_s], \Omega_K\}, \\
\mathcal{L}_2 \equiv \mathcal{L}_1 \cup \{M_\nu\}, \\
\mathcal{L}_3 \equiv \mathcal{L}_1 \cup \{N_{\mathrm{eff}}\}, \\
\mathcal{L}_4 \equiv \mathcal{L}_1 \cup \{M_\nu\} \cup \{N_{\mathrm{eff}}\}.
\end{gathered}$$

The scenarios $w\mathrm{CDM} + \Omega_K$, $w\mathrm{CDM} + \Omega_K + M_\nu$, $w\mathrm{CDM} + \Omega_K + N_{\mathrm{eff}}$, and $w\mathrm{CDM} + \Omega_K + M_\nu + N_{\mathrm{eff}}$ contain respectively 8, 9, 9, and 10 free parameters:

$$\begin{gathered}
\mathcal{W}_1 \equiv \{\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, n_s, \log[10^{10} A_s], \Omega_K, w\}, \\
\mathcal{W}_2 \equiv \mathcal{W}_1 \cup \{M_\nu\}, \\
\mathcal{W}_3 \equiv \mathcal{W}_1 \cup \{N_{\mathrm{eff}}\}, \\
\mathcal{W}_4 \equiv \mathcal{W}_1 \cup \{M_\nu\} \cup \{N_{\mathrm{eff}}\}.
\end{gathered}$$

The scenarios $w_0 w_a\mathrm{CDM} + \Omega_K$, $w_0 w_a\mathrm{CDM} + \Omega_K + M_\nu$, $w_0 w_a\mathrm{CDM} + \Omega_K + N_{\mathrm{eff}}$, and $w_0 w_a\mathrm{CDM} + \Omega_K + M_\nu + N_{\mathrm{eff}}$ contain respectively 9, 10, 10, and 11 free parameters:

$$\begin{gathered}
\mathcal{C}_1 \equiv \{\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, n_s, \log[10^{10} A_s], \Omega_K, w_0, w_a\}, \\
\mathcal{C}_2 \equiv \mathcal{C}_1 \cup \{M_\nu\}, \\
\mathcal{C}_3 \equiv \mathcal{C}_1 \cup \{N_{\mathrm{eff}}\}, \\
\mathcal{C}_4 \equiv \mathcal{C}_1 \cup \{M_\nu\} \cup \{N_{\mathrm{eff}}\}.
\end{gathered}$$

Finally, the scenarios $\mathrm{PEDE} + \Omega_K$, $\mathrm{PEDE} + \Omega_K + M_\nu$, $\mathrm{PEDE} + \Omega_K + N_{\mathrm{eff}}$, and $\mathrm{PEDE} + \Omega_K + M_\nu + N_{\mathrm{eff}}$ contain respectively 7, 8, 8, and 9 free parameters:

$$\mathcal{E}_1 \equiv \{\Omega_b h^2, \Omega_c h^2, 100\theta_{MC}, \tau, n_s, \log[10^{10} A_s], \Omega_K\},$$

$$\mathcal{E}_2 \equiv \mathcal{E}_1 \cup \{M_\nu\},$$

$$\mathcal{E}_3 \equiv \mathcal{E}_1 \cup \{N_{\rm eff}\},$$

$$\mathcal{E}_4 \equiv \mathcal{E}_1 \cup \{M_\nu\} \cup \{N_{\rm eff}\}.$$

TABLE I. Flat priors assumed on independent parameters.

Parameter	Prior
$\Omega_b h^2$	[0.005, 0.1]
$\Omega_c h^2$	[0.001, 0.99]
$100\theta_{MC}$	[0.5, 10]
τ	[0.01, 0.8]
$\log(10^{10}A_s)$	[1.61, 3.91]
n_s	[0.8, 1.2]
Ω_K	[-0.3, 0.3]
w	[-3, 0]
w_0	[-3, 3]
w_a	[0, 1]
M_ν	[0, 2.2]
$N_{\rm eff}$	[2.2, 4]

IV. Observational Constraints

This section describes observational constraints on various dark energy scenarios in a curved Universe. We report constraints on each model and its extensions.

A. Non-flat ΛCDM and its extensions

1. ΛCDM + Ω_K Table II shows constraints on the ΛCDM + Ω_K scenario for various datasets, with 1D and 2D contour plots for key parameters in Figure 1 [Figure 1: see original paper].

TABLE II. 68% and 95% CL constraints on free and derived parameters of ΛCDM + Ω_K.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02261^{+0.00017+0.00033}_{-0.00015-0.00031}$	$0.02239^{+0.00017+0.00030}_{-0.00033-0.00067}$	$0.02245^{+0.00016+0.00033}_{-0.00031-0.00063}$	$0.02240^{+0.00015+0.00030}_{-0.00031-0.00062}$
$\Omega_c h^2$	$0.1181^{+0.0015+0.0029}_{-0.0015-0.0029}$	$0.1197^{+0.0014+0.0029}_{-0.0014-0.0028}$	$0.1191^{+0.0015+0.0030}_{-0.0015-0.0029}$	$0.1196^{+0.0014+0.0028}_{-0.0014-0.0028}$
$100\theta_{MC}$	$1.04117^{+0.00034+0.00066}_{-0.00034-0.00066}$	$1.04095^{+0.00032+0.00062}_{-0.00015-0.00030}$	$1.04103^{+0.00032+0.00061}_{-0.00016-0.00031}$	$1.04098^{+0.00031+0.00062}_{-0.00031-0.00061}$
τ	$0.0487^{+0.0079+0.016}_{-0.0076-0.016}$	$0.0551^{+0.0076+0.016}_{-0.0076-0.015}$	$0.054^{+0.0076+0.016}_{-0.0076-0.015}$	$0.055^{+0.0074+0.017}_{-0.0082-0.015}$
n_s	$0.9706^{+0.0047+0.0091}_{-0.0046-0.0096}$	$0.9660^{+0.0045+0.0089}_{-0.0045-0.0088}$	$0.9676^{+0.0045+0.0093}_{-0.0044-0.0090}$	$0.9663^{+0.0044+0.0088}_{-0.0045-0.0087}$
$\ln(10^{10}A_s)$	$3.028^{+0.016+0.032}_{-0.016-0.035}$	$3.045^{+0.016+0.032}_{-0.016-0.031}$	$3.042^{+0.016+0.033}_{-0.016-0.032}$	$3.045^{+0.016+0.034}_{-0.016-0.031}$
Ω_{K0}	$-0.043^{+0.018+0.033}_{-0.015-0.034}$	$0.0008^{+0.0019+0.0040}_{-0.0019-0.0039}$	$-0.0060^{+0.0063+0.011}_{-0.0054-0.012}$	$0.0008^{+0.0019+0.0039}_{-0.0019-0.0038}$

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
Ω_{m0}	$0.481^{+0.057+0.13}_{-0.067-0.12}$	$0.3096^{+0.0066+0.013}_{-0.0066-0.012}$	$0.336^{+0.020+0.044}_{-0.024-0.041}$	$0.309^{+0.0062+0.013}_{-0.0068-0.012}$
σ_8	$0.775^{+0.016+0.028}_{-0.014-0.030}$	$0.8109^{+0.0081+0.016}_{-0.0085-0.017}$	$0.805^{+0.0097+0.019}_{-0.0097-0.019}$	$0.811^{+0.0081+0.016}_{-0.0083-0.016}$
H_0 [km/s/Mpc]	$54.5^{+3.3+7.5}_{-3.9-7.2}$	$67.90^{+0.67+1.3}_{-0.67-1.3}$	$65.2^{+2.1+4.4}_{-2.2-4.1}$	$67.97^{+0.65+1.3}_{-0.65-1.3}$
$r_{\rm drag}$ [Mpc]	$147.34^{+0.31+0.60}_{-0.31-0.61}$	$147.16^{+0.33+0.58}_{-0.30-0.62}$	$147.24^{+0.31+0.61}_{-0.31-0.61}$	$147.177^{+0.30+0.60}_{-0.30-0.59}$

For CMB alone, we find strong indication of a closed Universe at more than 95% CL ($\Omega_{K0} = -0.043^{+0.033}_{-0.034}$). The Hubble constant takes a very low value ($H_0 = 54.5^{+3.3}_{-3.9}$ km/s/Mpc at 68% CL), increasing the Hubble tension, and due to the strong correlation between H_0, Ω_K, and Ω_{m0}, the matter density takes a higher value as expected. We also notice that the tension in S_8 increases. Thus, CMB alone gives strong indication for a closed Universe at the expense of increased tensions in both H_0 and S_8.

When BAO data are added to CMB (CMB+BAO), Ω_K becomes consistent with spatial flatness and the Hubble constant rises toward the Planck ΛCDM value. However, this result is obtained with datasets that disagree at more than 3σ [**?**, **?**, **?**], making the CMB+BAO combination unsafe and the results not completely reliable. The results for CMB+Pantheon and CMB+BAO+Pantheon remain similar, though with a mildly lower Hubble constant in the CMB+Pantheon case ($H_0 = 65.2^{+2.1}_{-4.1}$ km/s/Mpc at 68% CL). In both cases, spatial flatness is consistent with the data.

2. ΛCDM + Ω_K + M_ The first non-flat ΛCDM extended scenario incorporates massive neutrinos (ΛCDM + Ω_K + M_). Table III provides observational constraints, with posterior distributions shown in Figure 2 [Figure 2: see original paper].

TABLE III. 68% and 95% CL constraints on ΛCDM + Ω_K + M_ .

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02254^{+0.00018+0.00036}_{-0.00018-0.00036}$	$0.02238^{+0.00015+0.00030}_{-0.00033-0.00069}$	$0.02246^{+0.00016+0.00032}_{-0.00031-0.00063}$	$0.02241^{+0.00015+0.00031}_{-0.00032-0.00065}$
$\Omega_c h^2$	$0.1183^{+0.0015+0.0030}_{-0.0015-0.0030}$	$0.1197^{+0.0014+0.0027}_{-0.0014-0.0028}$	$0.1190^{+0.0014+0.0028}_{-0.0014-0.0028}$	$0.1196^{+0.0014+0.0029}_{-0.0013-0.0028}$
M_ν [eV]	< 0.46 (95% CL)	< 0.072 (95% CL)	< 0.066 (95% CL)	< 0.071 (95% CL)
Ω_{K0}	$-0.073^{+0.043+0.060}_{-0.022-0.071}$	$0.0007^{+0.0020+0.0044}_{-0.0022-0.0041}$	$-0.0061^{+0.0064+0.011}_{-0.0056-0.012}$	$0.0007^{+0.0020+0.0048}_{-0.0024-0.0045}$
H_0 [km/s/Mpc]	$48.3^{+5.7+11}_{-5.9-11}$	$67.84^{+0.67+1.4}_{-0.67-1.3}$	$65.3^{+2.2+4.4}_{-2.3-4.4}$	$67.97^{+0.66+1.4}_{-0.66-1.3}$

Even in this extended model, Planck CMB data alone show preference for a curved cosmological spacetime at more than 95% CL ($\Omega_{K0} = -0.073^{+0.060}_{-0.071}$ at

95% CL). This preference is strongly reduced when Planck data are combined with BAO datasets that are in tension with it, and their combination prefers spatial flatness within one standard deviation. Concerning the expansion rate today, due to its degeneracy with neutrino mass, CMB measurements prefer even lower H_0 values, giving $H_0 = 48.3^{+5.7}_{-5.9}$ km/s/Mpc at 68% CL—strongly tensioned with the SH0ES independent measurement [?] at the level of 4.3σ. Combining CMB with BAO and Pantheon data, the tension between datasets remains statistically significant. In particular, CMB+BAO gives $H_0 = 67.84 \pm 0.67$ km/s/Mpc at 68% CL, similar to the flat ΛCDM result, while CMB+Pantheon yields $H_0 = 65.3^{+2.2}_{-2.3}$ km/s/Mpc at 68% CL. Both values remain in tension with SH0ES at 4.3σ and 3.2σ respectively. Due to the strong anti-correlation between H_0 and Ω_{m0}, lower H_0 values prefer higher Ω_{m0}, resulting in severe tension also for the S_8 parameter. Comparing S_8 from CMB alone versus CMB+BAO+Pantheon reveals tension at the level of 3σ.

Concerning neutrino masses, CMB data alone constrain the total mass to $M_\nu <$ 0.79 eV at 95% CL—a factor of three more relaxed than the flat ΛCDM+M_ scenario. However, neutrinos suppress structure formation at small scales, making astrophysical galaxy clustering measurements crucial for improving constraints. Including BAO and Pantheon significantly improves the upper bound to $M_\nu < 0.17$ eV at 95% CL. This should be compared with the flat ΛCDM+M_ scenario bound of $M_\nu < 0.13$ eV at 95% CL for CMB+BAO. Thus, the flatness assumption produces a much stronger upper limit on total neutrino mass than when curvature can vary, biasing conclusions important for laboratory experiments.

3. ΛCDM + Ω_K + N_{eff} This extended model accounts for a larger effective number of relativistic degrees of freedom $N_{\rm eff}$ at recombination. $N_{\rm eff}$ is defined by $\rho_{\rm rad} = N_{\rm eff}(7/8)(4/11)^{4/3}\rho_\gamma$, with ρ_γ the present CMB energy density. Within standard ΛCDM, $N_{\rm eff} = 3.044$ \cite{113–118}, consisting of three massless neutrino species plus corrections from non-instantaneous neutrino decoupling. Here we treat $N_{\rm eff}$ as an additional free parameter.

TABLE IV. 68% and 95% CL constraints on ΛCDM + Ω_K + N_{eff}.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02260^{+0.00025+0.00051}_{-0.00024-0.00049}$	$0.02232^{+0.00024+0.00045}_{-0.00044-0.00086}$	$0.02245^{+0.00024+0.00047}_{-0.00044-0.00086}$	$0.02233^{+0.00022+0.00045}_{-0.00044-0.00085}$
$\Omega_c h^2$	$0.1181^{+0.0031+0.0060}_{-0.0030-0.0060}$	$0.1185^{+0.0029+0.0062}_{-0.0032-0.0058}$	$0.1187^{+0.0030+0.0061}_{-0.0030-0.0058}$	$0.1185^{+0.0030+0.0062}_{-0.0032-0.0058}$
$N_{\rm eff}$	$3.04^{+0.20+0.40}_{-0.19-0.39}$	$2.96^{+0.19+0.39}_{-0.19-0.38}$	$3.03^{+0.19+0.40}_{-0.20-0.38}$	$2.97^{+0.20+0.38}_{-0.19-0.38}$
Ω_{K0}	$-0.044^{+0.020+0.035}_{-0.016-0.038}$	$0.0012^{+0.0021+0.0044}_{-0.0021-0.0041}$	$-0.0060^{+0.0066+0.011}_{-0.0057-0.012}$	$0.0012^{+0.0020+0.0043}_{-0.0021-0.0041}$
H_0 [km/s/Mpc]	$54.4^{+3.6+7.6}_{-4.0-7.6}$	$67.5^{+1.2+2.4}_{-1.2-2.3}$	$65.1^{+2.2+4.6}_{-2.4-4.2}$	$67.6^{+1.1+2.2}_{-1.1-2.2}$

As in previous extended models, CMB data alone suggest a non-flat background geometry with $\Omega_{K0} = 0$ excluded at more than 95% CL. Conversely, including

CMB-independent datasets eliminates this preference, giving indication for flatness within 1σ—again as a result of tension with Planck.

Concerning N_{eff}, the reference value $N_{\text{eff}} = 3.044$ is always consistent within one standard deviation for all data combinations. We find no significant evidence of deviations from the standard model of elementary particles, and current observations constrain additional contributions to $\Delta N_{\text{eff}} \equiv N_{\text{eff}} - 3.044 \lesssim 0.4$ at 95% CL, analogous to the standard spatially flat case.[5] As shown in Figure 3 [Figure 3: see original paper], there is no correlation between N_{eff} and Ω_K. Modifying the relativistic energy density changes the sound horizon at recombination, which is partly degenerate with late-time geometry. Higher N_{eff} values lead to smaller sound horizons, affecting σ_8 amplitude and potentially alleviating the Hubble tension through preference for higher expansion rates. This does not occur in this extended model, where the same tensions between early and late-time H_0 measurements persist at 5σ (CMB only vs. SH0ES) and 3.3σ (CMB+BAO/Pantheon). Similarly, different data combinations provide S_8 values in tension at more than 95% CL.

4. ΛCDM + Ω_K + M_ + N_{eff} This final extended non-flat model simultaneously varies both total neutrino mass and effective relativistic degrees of freedom. Results are given in Table V and posterior distributions in Figure 4 [Figure 4: see original paper].

TABLE V. 68% and 95% CL constraints on ΛCDM + Ω_K + M_ + N_{eff}.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02252^{+0.00025+0.00050}_{-0.00025-0.00049}$	$0.02232^{+0.00024+0.00046}_{-0.00045-0.00089}$	$0.02244^{+0.00024+0.00048}_{-0.00044-0.00084}$	$0.02233^{+0.00023+0.00046}_{-0.00047-0.00087}$
$\Omega_c h^2$	$0.1182^{+0.0029+0.0060}_{-0.0032-0.0056}$	$0.1184^{+0.0030+0.0061}_{-0.0031-0.0058}$	$0.1187^{+0.0031+0.0061}_{-0.0031-0.0059}$	$0.1185^{+0.0030+0.0059}_{-0.0030-0.0059}$
M_ν [eV]	$<$ 0.81 (95% CL)	< 0.18 (95% CL)	< 0.16 (95% CL)	< 0.18 (95% CL)
N_{eff}	$3.04^{+0.19+0.39}_{-0.21-0.38}$	$2.96^{+0.20+0.39}_{-0.12-0.38}$	$3.02^{+0.20+0.40}_{-0.20-0.38}$	$2.97^{+0.20+0.38}_{-0.20-0.39}$
Ω_{K0}	$-0.074^{+0.041+0.059}_{-0.023-0.070}$	$0.0011^{+0.0023+0.0053}_{-0.0025-0.0046}$	$-0.0061^{+0.0065+0.011}_{-0.0057-0.012}$	$0.00113^{+0.0022+0.0051}_{-0.0026-0.0048}$
H_0 [km/s/Mpc]	$48.1^{+5.2+11}_{-6.0-10}$	$67.4^{+1.2+2.3}_{-1.2-2.3}$	$65.1^{+2.1+4.6}_{-2.4-4.2}$	$67.6^{+1.1+2.2}_{-1.1-2.4}$

Simultaneously varying total neutrino mass and effective relativistic degrees of freedom produces no significant change in Planck's preference for a closed Universe or the H_0 tension. From CMB alone we obtain $\Omega_{K0} = -0.074^{+0.059}_{-0.070}$ at 95% CL and $H_0 = 48.1^{+5.2}_{-6.0}$ km/s/Mpc at 68% CL, tensioned with local measurements at 4.7σ. Including BAO (or Pantheon) likelihood—datasets that disagree with Planck in this extended model—eliminates the preference for a closed Universe, with constraints on the expansion rate tensioned at 3.3σ with SH0ES. The H_0 tension drives S_8 values to disagree between 3 and 4 standard deviations depending on data combination. Bounds on N_{eff} show no deviation from the Standard Model expectation and remain unchanged from the previous case without neutrinos. Upper bounds on total neutrino mass are similarly unaffected: CMB alone gives $M_\nu < 0.81$ eV at 95% CL (with a 1σ indication for non-zero mass), while including BAO and Pantheon improves this to $M_\nu < 0.18$ eV (CMB+BAO+Pantheon) and $M_\nu < 0.16$ eV (CMB+Pantheon), all at 95% CL.

B. Non-flat wCDM and its extensions

1. wCDM + Ω_K Table VI summarizes constraints on wCDM + Ω_K, with posterior distributions in Figure 5 [Figure 5: see original paper].

TABLE VI. 68% and 95% CL constraints on wCDM + Ω_K.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02261^{+0.00017+0.00034}_{-0.00017-0.00032}$	$0.02240^{+0.00015+0.00030}_{-0.00032-0.00060}$	$0.02239^{+0.00015+0.00031}_{-0.00033-0.00062}$	$0.02257^{+0.00016+0.00033}_{-0.00032-0.00063}$
$\Omega_c h^2$	$0.1180^{+0.0014+0.0029}_{-0.0014-0.0028}$	$0.1197^{+0.0014+0.0028}_{-0.0014-0.0028}$	$0.1183^{+0.0014+0.0028}_{-0.0014-0.0028}$	$0.1197^{+0.0014+0.0029}_{-0.0015-0.0027}$
w	$-1.31^{+0.98+1.1}_{-0.49-1.4}$	$-1.22^{+0.10+0.17}_{-0.08-0.18}$	$-1.046^{+0.099+0.19}_{-0.089-0.19}$	$-1.026^{+0.039+0.074}_{-0.038-0.078}$
Ω_{K0}	$-0.046^{+0.041+0.050}_{-0.015-0.073}$	$-0.028^{+0.012+0.019}_{-0.009-0.020}$	$-0.0001^{+0.0027+0.0067}_{-0.0035-0.0061}$	$0.0001^{+0.0022+0.0042}_{-0.0022-0.0045}$
H_0 [km/s/Mpc]	61^{+10+33}_{-22-26}	$61.2^{+2.4+4.8}_{-2.4-4.6}$	$68.7^{+1.6+3.5}_{-1.9-3.4}$	$68.30^{+0.84+1.8}_{-0.84-1.6}$

For CMB alone, we find evidence of a closed Universe at more than 68% CL, though spatial flatness is recovered within 95% CL. The dark energy equation of state is perfectly consistent with a cosmological constant ($w = -1.31^{+0.98}_{-0.49}$ at 68% CL). However, even when varying this parameter, unlike in phantom dark energy models where $w < -1$ increases the expansion rate, here H_0 remains almost unconstrained with a low mean value ($H_0 = 61^{+10}_{-22}$ km/s/Mpc at 68% CL), reducing tension with SH0ES to 1.2σ due to large error bars. This lowering is driven by curvature effects.

Adding BAO to CMB changes constraints significantly due to their mutual tension in a curved universe, making the scenario very close to spatially flat ΛCDM. More interesting are the CMB+Pantheon results: we find a closed Universe at more than 95% CL ($\Omega_K = -0.028^{+0.019}_{-0.020}$ at 95% CL) together with a phantom Universe at more than 95% CL ($w = -1.22^{+0.17}_{-0.18}$ at 95% CL)—an indication for a phantom closed universe, as noted in [?]. Similar to CMB alone, we obtain a smaller H_0 value ($H_0 = 61.2^{+2.4}_{-2.4}$ km/s/Mpc at 68% CL) and consequently a larger Ω_{m0} due to positive correlation.

For CMB+BAO+Pantheon, the presence of BAO data (tensioned with Planck) again strongly suggests spatial flatness, while w agrees with a cosmological constant within 68% CL. Notably, H_0 is mildly increased ($H_0 = 68.30^{+0.84}_{-0.84}$ km/s/Mpc at 68% CL) compared to the Planck 2018 ΛCDM prediction [?], but tension with SH0ES remains at 3.6σ.

2. wCDM + Ω_K + M_ This extension includes total neutrino mass as an additional free parameter (wCDM + Ω_K + M_). Constraints are summarized in Table VII and correlations shown in Figure 6 [Figure 6: see original paper].

TABLE VII. 68% and 95% CL constraints on wCDM + Ω_K + M_ .

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02253^{+0.00018+0.00036}_{-0.00018-0.00036}$	$0.02240^{+0.00016+0.00031}_{-0.00034-0.00068}$	$0.02255^{+0.00017+0.00035}_{-0.00034-0.00065}$	$0.02240^{+0.00016+0.00031}_{-0.00034-0.00068}$
$\Omega_c h^2$	$0.1183^{+0.0015+0.0030}_{-0.0015-0.0030}$	$0.1197^{+0.0015+0.0027}_{-0.0015-0.0029}$	$0.1184^{+0.0015+0.0028}_{-0.0015-0.0029}$	$0.1197^{+0.0014+0.0029}_{-0.0015-0.0027}$
w	$-1.5^{+1.1+1.2}_{-0.6-1.5}$	$-1.046^{+0.099+0.17}_{-0.085-0.19}$	$-1.28^{+0.16+0.25}_{-0.09-0.28}$	$-1.025^{+0.040+0.077}_{-0.040-0.080}$
M_ν [eV]	$<$ 0.83 (95% CL)	< 0.078 (95% CL)	< 0.212 (95% CL)	< 0.078 (95% CL)
Ω_{K0}	$-0.073^{+0.055+0.070}_{-0.027-0.086}$	$-0.0001^{+0.0028+0.0067}_{-0.0035-0.0062}$	$-0.030^{+0.012+0.020}_{-0.010-0.022}$	$0.0001^{+0.0023+0.0051}_{-0.0027-0.0049}$
H_0 [km/s/Mpc]	52^{+6+27}_{-16-20}	$68.7^{+1.5+3.5}_{-1.9-3.3}$	$60.2^{+2.6+5.2}_{-2.6-4.9}$	$68.27^{+0.81+1.7}_{-0.88-1.6}$

With CMB data alone, a spatially flat background is disfavored at slightly more than 95% CL ($\Omega_{K0} = -0.073^{+0.070}_{-0.086}$ at 95% CL). Interestingly, strong preference for curved spacetime appears when combining CMB+Pantheon ($\Omega_{K0} = -0.030^{+0.020}_{-0.022}$ at 95% CL). Including BAO measurements makes flatness consistent within one standard deviation for all other combinations, though these results should be treated cautiously due to data inconsistency in a curved universe.

Concerning the dark energy equation of state, CMB data alone allow both phantom ($w < -1$) and quintessential ($w > -1$) behaviors within 68% CL. Combining CMB with BAO (+Pantheon) shows no significant differences. However, CMB+Pantheon results ($w = -1.28^{+0.25}_{-0.28}$ at 95% CL) suggest preference for phantom dark energy at more than 95% CL. As in the baseline wCDM + Ω_K model, allowing phantom dark energy does not alleviate tensions: CMB+Pantheon indicates a phantom closed universe [**?**]. The H_0 value from CMB data ($H_0 = 52^{+6}_{-16}$ km/s/Mpc at 68% CL) tensions with SH0ES at 3.4σ. This preference for lower H_0 is driven by degeneracy with spatial curvature. When CMB+BAO are combined, no curvature evidence is found and H_0 constraints become similar to standard flat ΛCDM results. Conversely, for CMB+Pantheon, the preference for curved spacetime yields $H_0 = 60.2 \pm 2.6$ km/s/Mpc at 68% CL, in strong tension with local measurements at 4.6σ.

Bounds on total neutrino mass remain similar to previous scenarios: CMB measurements give $M_\nu < 0.83$ eV at 95% CL, while CMB+Pantheon relaxes this to $M_\nu < 0.46$ eV at 95% CL. The most significant impact comes from CMB+BAO, yielding $M_\nu < 0.19$ eV at 95% CL. Combining all three datasets leaves this bound robust and almost unchanged.

3. wCDM + Ω_K + N_{eff} This model varies N_{eff} instead of total neutrino mass. Results are in Table VIII and triangular plots in Figure 7 [Figure 7: see original paper].

TABLE VIII. 68% and 95% CL constraints on wCDM + Ω_K + N_{eff}.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02258^{+0.00024+0.00048}_{-0.00024-0.00047}$	$0.02230^{+0.00024+0.00046}_{-0.00045-0.00088}$	$0.02232^{+0.00025+0.00050}_{-0.00046-0.00085}$	$0.02231^{+0.00023+0.00045}_{-0.00046-0.00089}$
$\Omega_c h^2$	$0.1179^{+0.0030+0.0060}_{-0.0030-0.0057}$	$0.1182^{+0.0031+0.0060}_{-0.0030-0.0061}$	$0.1180^{+0.0030+0.0062}_{-0.0033-0.0058}$	$0.1182^{+0.0031+0.0062}_{-0.0031-0.0059}$
w	$-1.22^{+0.95+1.1}_{-0.42-1.4}$	$-1.05^{+0.11+0.19}_{-0.09-0.20}$	$-1.22^{+0.10+0.18}_{-0.08-0.19}$	$-1.028^{+0.040+0.078}_{-0.039-0.080}$
N_{eff}	$3.03^{+0.19+0.39}_{-0.19-0.36}$	$2.94^{+0.19+0.39}_{-0.20-0.37}$	$3.03^{+0.19+0.39}_{-0.21-0.38}$	$2.94^{+0.20+0.39}_{-0.20-0.37}$
Ω_{K0}	$-0.049^{+0.044+0.053}_{-0.016-0.076}$	$0.0003^{+0.0029+0.0071}_{-0.0037-0.0066}$	$-0.028^{+0.011+0.020}_{-0.010-0.021}$	$0.0005^{+0.0024+0.0049}_{-0.0027-0.0052}$
H_0 [km/s/Mpc]	59^{+9+33}_{-21-25}	$68.2^{+1.8+3.9}_{-2.2-3.8}$	$61.1^{+2.3+5.0}_{-2.6-4.8}$	$67.9^{+1.2+2.4}_{-1.2-2.4}$

Replacing total neutrino mass with N_{eff} changes curvature results, reducing Planck's preference for a closed Universe. The Planck limit $\Omega_{K0} = -0.049^{+0.053}_{-0.076}$ at 95% CL is consistent with flatness at 1.1σ. Interestingly, CMB+Pantheon still prefers a closed Universe ($\Omega_{K0} = -0.028^{+0.020}_{-0.021}$ at 95% CL), a preference that disappears with BAO inclusion.

Regarding the H_0 tension, due to correlation between expansion rate and N_{eff}, CMB bounds are less tight but still show preference for smaller values ($H_0 = 59^{+9}_{-21}$ km/s/Mpc at 68% CL). With large uncertainties, tension with SH0ES falls below 2σ. Combining CMB+BAO gives $H_0 = 68.2^{+1.9}_{-2.2}$ km/s/Mpc at 68% CL—close to the flat ΛCDM value but reducing tension to 2.2σ due to larger error bars. Conversely, CMB+Pantheon's preference for a closed Universe lowers H_0 to $61.1^{+2.3}_{-2.6}$ km/s/Mpc at 68% CL while uncertainty remains almost unchanged, resulting in 4.8σ tension.

These tensions strongly affect bounds on Ω_{m0} and consequently S_8, as both are pushed toward higher (lower) values when lower (higher) H_0 values are preferred. Concerning the dark energy equation of state, CMB measurements give a relaxed bound on w completely consistent with a cosmological constant, a result that remains true when combining CMB+BAO. As in previous models, CMB+Pantheon strongly suggests phantom dark energy, excluding a cosmological constant at more than 95% CL.

Bounds on N_{eff} remain basically unchanged, with additional contributions constrained to $\Delta N_{\text{eff}} \lesssim 0.4$ for all datasets, independent of curvature.

4. wCDM + Ω_K + M_ + N_{eff} This final wCDM extension simultaneously varies both total neutrino mass and effective relativistic degrees of freedom. Results are in Table IX and posterior distributions in Figure 8 [Figure 8: see original paper].

TABLE IX. 68% and 95% CL constraints on wCDM + Ω_K + M_ + N_{eff}.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02253^{+0.00025+0.00049}_{-0.00025-0.00047}$	$0.02230^{+0.00024+0.00046}_{-0.00045-0.00088}$	$0.02252^{+0.00025+0.00050}_{-0.00046-0.00085}$	$0.02231^{+0.00023+0.00045}_{-0.00046-0.00089}$
$\Omega_c h^2$	$0.1183^{+0.0030+0.0060}_{-0.0033-0.0059}$	$0.1182^{+0.0031+0.0060}_{-0.0030-0.0061}$	$0.1180^{+0.0030+0.0062}_{-0.0033-0.0058}$	$0.1182^{+0.0031+0.0062}_{-0.0031-0.0059}$
w	$-1.5^{+1.0+1.1}_{-0.6-1.5}$	$-1.05^{+0.11+0.19}_{-0.09-0.20}$	$-1.22^{+0.10+0.18}_{-0.08-0.19}$	$-1.028^{+0.040+0.078}_{-0.039-0.080}$
M_ν [eV]	< 0.81 (95% CL)	< 0.084 (95% CL)	< 0.203 (95% CL)	< 0.080 (95% CL)
$N_{\rm eff}$	$3.04^{+0.20+0.40}_{-0.20-0.39}$	$2.94^{+0.19+0.39}_{-0.20-0.37}$	$3.03^{+0.19+0.39}_{-0.21-0.38}$	$2.94^{+0.20+0.39}_{-0.20-0.37}$
Ω_{K0}	$-0.067^{+0.055+0.068}_{-0.024-0.086}$	$0.0003^{+0.0030+0.0071}_{-0.0037-0.0066}$	$-0.029^{+0.012+0.020}_{-0.010-0.021}$	$0.0005^{+0.0024+0.0052}_{-0.0027-0.0052}$
H_0 [km/s/Mpc]	54^{+7+29}_{-17-22}	$68.2^{+1.8+3.9}_{-2.2-3.8}$	$60.3^{+2.5+5.4}_{-2.9-5.0}$	$67.8^{+1.2+2.4}_{-1.2-2.4}$

In this extended scenario, Planck's preference for a closed Universe is reduced below 2σ, with CMB data constraining $\Omega_{K0} = -0.067^{+0.068}_{-0.086}$ at 95% CL. Conversely, CMB+Pantheon shows preference for curved geometry ($\Omega_{K0} = -0.029^{+0.020}_{-0.022}$ at 95% CL), as observed in previous extensions. H_0 results remain stable: CMB measurements give $H_0 = 54^{+29}_{-22}$ km/s/Mpc at 68% CL, reducing tension with SH0ES to <2σ due to large errors rather than better agreement. Combining CMB+BAO or CMB+Pantheon yields $H_0 = 68.2^{+1.8}_{-2.2}$ km/s/Mpc and $H_0 = 60.3^{+2.5}_{-2.9}$ km/s/Mpc respectively, producing the same tensions discussed previously.

For CMB+Pantheon, we still observe >95% CL preference for phantom dark energy, while other datasets agree with a cosmological constant within one standard deviation. Upper bounds on total neutrino mass are not significantly changed, with the most constraining bound from CMB+BAO ($M_\nu < 0.20$ eV at 95% CL) only slightly relaxed when also varying $N_{\rm eff}$. No significant change is observed in $N_{\rm eff}$ constraints.

C. Non-flat w_0w CDM and its extensions

1. w_0w CDM + Ω_K Table X summarizes constraints on this model, with posterior distributions in Figure 9 [Figure 9: see original paper].

TABLE X. 68% and 95% CL constraints on w_0w CDM + Ω_K.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02261^{+0.00017+0.00033}_{-0.00017-0.00034}$	$0.02244^{+0.00015+0.00031}_{-0.00015-0.00030}$	$0.02260^{+0.00017+0.00033}_{-0.00017-0.00034}$	$0.02243^{+0.00016+0.00032}_{-0.00016-0.00031}$
$\Omega_c h^2$	$0.1181^{+0.0015+0.0029}_{-0.0015-0.0028}$	$0.1194^{+0.0014+0.0028}_{-0.0014-0.0028}$	$0.1181^{+0.0014+0.0029}_{-0.0014-0.0029}$	$0.1195^{+0.0014+0.0028}_{-0.0014-0.0028}$
w_0	$-1.3^{+1.2+1.3}_{-0.4-1.7}$	$-0.51^{+0.29+0.44}_{-0.19-0.48}$	$-1.04^{+0.20+0.34}_{-0.16-0.36}$	$-0.89^{+0.09+0.21}_{-0.11-0.20}$

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
w_a	$-0.6^{+1.2+2.3}_{-1.8-2.4}$	$-1.8^{+0.3+1.4}_{-1.2-1.2}$	$-1.2^{+0.9}_{-1.1}$	$-0.73^{+0.61+1.0}_{-0.46-1.1}$
Ω_{K0}	$-0.038^{+0.031+0.039}_{-0.012-0.052}$	$-0.0042^{+0.0030+0.0067}_{-0.0036-0.0066}$	$-0.031^{+0.011+0.020}_{-0.011-0.021}$	$-0.0032^{+0.0030+0.0062}_{-0.0030-0.0065}$
H_0 [km/s/Mpc]	63^{+10+30}_{-19-24}	$64.6^{+1.9+4.6}_{-2.6-4.4}$	$61.0^{+2.5+5.1}_{-2.8-5.1}$	$68.01^{+0.83+1.7}_{-0.85-1.7}$

CMB alone indicates preference for a closed Universe at more than 68% CL, though Ω_K agrees with zero within 2σ ($\Omega_K = -0.038^{+0.031}_{-0.012}$ at 68% CL). As usual, we find a lower mean H_0 with large error bars ($H_0 = 63^{+10}_{-19}$ km/s/Mpc at 68% CL), reducing tension with SH0ES to 1σ. The current dark energy equation of state has a mean in the phantom region ($w_0 = -1.3^{+1.2}_{-0.4}$ at 68% CL) but agrees with a cosmological constant within 68% CL. The dynamical character of dark energy, quantified by $w_a = -0.6^{+1.2}_{-1.8}$ at 68% CL, is consistent with zero, i.e., dynamical DE is not favored.

Adding BAO to CMB allows non-zero curvature only at 68% CL ($\Omega_K = -0.0042^{+0.0030}_{-0.0036}$ at 68% CL) and shifts H_0 toward lower mean values with reduced errors ($H_0 = 64.6^{+1.9}_{-2.6}$ km/s/Mpc at 68% CL), restoring tension with local measurements at 3.9σ. Concerning w_0 and w_a, w_0 is strictly in the quintessence regime at >95% CL while w_a is non-zero at >95% CL, indicating evidence for dynamical quintessence dark energy at more than 2σ.

CMB+Pantheon recovers evidence for non-zero curvature at >95% CL ($\Omega_K = -0.031^{+0.020}_{-0.021}$ at 95% CL) while exacerbating H_0 tension. The dark energy parameters show w_0 perfectly consistent with a cosmological constant ($w_0 = -1.04^{+0.20}_{-0.16}$ at 68% CL), but the dynamical nature of DE is signaled through w_a which is non-zero within 68% CL ($w_a = -1.2^{+0.9}_{-1.1}$ at 68% CL) though consistent with zero at 95% CL.

The combined analysis CMB+BAO+Pantheon breaks all degeneracies, yielding mild improvement in H_0 agreement ($H_0 = 68.01^{+0.83}_{-0.85}$ km/s/Mpc at 68% CL). The curvature parameter is consistent with zero at slightly more than 68% CL. We recover the quintessence nature of w_0 at slightly more than 68% CL, similar to CMB+BAO, and w_a is only non-zero at >68% CL, showing a hint for dynamical nature while $w_a = 0$ remains within 95% CL.

2. $\mathbf{w_0}$w CDM + Ω_K + M_ Extending the parameter space with massive neutrinos ($\mathrm{w_0}$w CDM + Ω_K + M_), constraints are in Table XI and Figure 10 [Figure 10: see original paper].

TABLE XI. 68% and 95% CL constraints on $\mathrm{w_0}$w CDM + Ω_K + M_ .

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02252^{+0.00018+0.00036}_{-0.00018-0.00036}$	$0.02244^{+0.00016+0.00032}_{-0.00034-0.00067}$	$0.02256^{+0.00017+0.00035}_{-0.00032-0.00063}$	$0.02242^{+0.00016+0.00032}_{-0.00034-0.00067}$
$\Omega_c h^2$	$0.1183^{+0.0015+0.0029}_{-0.0015-0.0029}$	$0.1193^{+0.0015+0.0029}_{-0.0014-0.0029}$	$0.1183^{+0.0015+0.0030}_{-0.0015-0.0030}$	$0.1195^{+0.0014+0.0028}_{-0.0014-0.0029}$

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
w_0	unconstrained	$-0.51^{+0.28+0.43}_{-0.19-0.47}$	$-1.09^{+0.21+0.36}_{-0.16-0.38}$	$-0.90^{+0.10+0.22}_{-0.11-0.20}$
w_a	< 1.9 (95% CL)	< -0.4 (95% CL)	< 0.24 (95% CL)	$-0.75^{+0.65+1.1}_{-0.47-1.1}$
M_ν [eV]	$<$ 0.84 (95% CL)	< 0.092 (95% CL)	< 0.272 (95% CL)	< 0.090 (95% CL)
Ω_{K0}	$-0.065^{+0.051+0.066}_{-0.023-0.081}$	$-0.0041^{+0.0034+0.0071}_{-0.0034-0.0067}$	$-0.034^{+0.011+0.020}_{-0.011-0.021}$	$-0.0030^{+0.0032+0.0065}_{-0.0032-0.0061}$
H_0 [km/s/Mpc]	55^{+7+29}_{-16-21}	$64.6^{+1.9+4.6}_{-2.6-4.3}$	$59.7^{+2.4+5.5}_{-3.0-5.2}$	$67.98^{+0.86+1.7}_{-0.85-1.7}$

With CMB data alone, $\Omega_{K0} = -0.065^{+0.066}_{-0.081}$ at 95% CL shows that while flatness is disfavored at 68% CL, it remains consistent within 95% CL bounds, similar to the baseline case. For H_0, we obtain a lower mean value with respect to the baseline without neutrinos: $H_0 = 55^{+29}_{-21}$ km/s/Mpc at 68% CL. Due to large errors, this agrees with SH0ES within 2σ. Combining CMB+BAO gives $H_0 = 64.6^{+1.9}_{-2.6}$ km/s/Mpc at 68% CL, increasing tension to 3.9σ due to smaller uncertainties. CMB+Pantheon, with its preference for curved spacetime, yields $H_0 = 59.7^{+2.4}_{-3.0}$ km/s/Mpc at 68% CL, in strong tension with local measurements at 5σ.

Different datasets provide discordant predictions for the dark energy sector. Using only Planck measurements, w_0 is unconstrained while only an uninformative upper bound $w_a < 1.9$ at 95% CL can be derived. Including BAO measurements finds strong preference for quintessential dark energy ($w_0 = -0.51^{+0.43}_{-0.47}$ at 95% CL) with dynamical behavior, as the upper bound $w_a < -0.4$ rules out $w_a = 0$ at 95% CL. Conversely, CMB+Pantheon agrees with a cosmological constant within 68% CL, with $w_a < 0.24$ at 95% CL consistent with non-dynamical evolution. Combining CMB+BAO+Pantheon finds quintessential dynamical models preferred at $>1\sigma$, though both phantom models and non-dynamical dark energy are allowed within 95% CL.

Neutrino mass limits are not drastically changed by modifications to the dark energy sector. The 95% CL bound from CMB+BAO ($M_\nu < 0.22$ eV) is only slightly larger than limits obtained within curved ΛCDM or wCDM cosmologies. The full dataset remains completely consistent with a flat universe.

3. $\mathbf{w_0}$w CDM + Ω_K + N_{eff} Allowing N_{eff} to vary in curved dynamical dark energy cosmologies (w_0w CDM + Ω_K + N_{eff}), constraints are in Table XII and Figure 11 [Figure 11: see original paper].

TABLE XII. 68% and 95% CL constraints on w_0w CDM + Ω_K + N_{eff}.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02261^{+0.00025+0.00049}_{-0.00025-0.00047}$	$0.02237^{+0.00024+0.00047}_{-0.00043-0.00082}$	$0.02257^{+0.00026+0.00051}_{-0.00044-0.00086}$	$0.02235^{+0.00023+0.00046}_{-0.00045-0.00085}$
$\Omega_c h^2$	$0.1181^{+0.0029+0.0060}_{-0.0030-0.0056}$	$0.1183^{+0.0030+0.0058}_{-0.0030-0.0058}$	$0.1179^{+0.0029+0.0057}_{-0.0029-0.0057}$	$0.1184^{+0.0030+0.0059}_{-0.0030-0.0058}$
w_0	> -1.68, unconstrained	$-0.55^{+0.34+0.47}_{-0.19-0.54}$	$-1.07^{+0.21+0.34}_{-0.16-0.36}$	$-0.52^{+0.29+0.43}_{-0.19-0.48}$
w_a	$-0.7^{+1.3}_{-1.4}$	< -0.1 (95% CL)	$-1.0^{+1.1}_{-1.1}$	< -0.4 (95% CL)
$N_{\rm eff}$	$3.05^{+0.19+0.40}_{-0.20-0.37}$	$2.97^{+0.20+0.37}_{-0.20-0.38}$	$3.03^{+0.19+0.39}_{-0.19-0.36}$	$2.97^{+0.19+0.39}_{-0.20-0.37}$
Ω_{K0}	$-0.037^{+0.031+0.038}_{-0.012-0.052}$	$-0.0036^{+0.0033+0.0072}_{-0.0038-0.0066}$	$-0.031^{+0.011+0.021}_{-0.011-0.021}$	$-0.0037^{+0.0035+0.0073}_{-0.0035-0.0068}$
H_0 [km/s/Mpc]	64^{+10+31}_{-20-24}	$64.6^{+2.1+5.0}_{-2.8-4.8}$	$60.9^{+2.4+5.3}_{-2.9-5.0}$	$64.4^{+2.1+4.8}_{-2.7-4.6}$

CMB bounds show that while flatness is disfavored at 68% CL, it remains consistent within 95% CL bounds. For H_0 we obtain $H_0 = 64^{+10}_{-20}$ km/s/Mpc at 68% CL, consistent with SH0ES within one standard deviation due to large errors. Parameters involving dark energy modifications are basically unconstrained by CMB measurements.

Including BAO data, a curved background is slightly preferred at 68% CL, but flatness remains consistent within 95% CL. The H_0 value $H_0 = 64.6^{+2.1}_{-2.8}$ km/s/Mpc at 68% CL remains stable with respect to CMB results but with much smaller uncertainty, increasing tension with SH0ES to 3.6σ. For this dataset, quintessential models are preferred with $w_0 = -0.55^{+0.47}_{-0.54}$ at 95% CL, while the upper bound $w_a < -0.1$ excludes non-dynamical evolution at 95% CL.

Combining CMB+Pantheon yields the usual evidence for a curved Universe ($\Omega_{K0} = -0.031 \pm 0.021$ at 95% CL) and preference for smaller expansion rates ($H_0 = 60.9^{+2.4}_{-2.9}$ km/s/Mpc at 68% CL), in strong tension with local measurements at $>4.6\sigma$. Both quintessential and phantom dark energy models are allowed, with w_0 consistent with a cosmological constant and only an upper bound $w_a < 0.24$ at 95% CL. Combining all three datasets eliminates curvature evidence and yields $H_0 = 67.6 \pm 1.2$ km/s/Mpc at 68% CL, similar to the standard ΛCDM case. Quintessential dynamical models are preferred at 68% CL, though phantom and non-dynamical evolution are allowed at 95% CL, with $w_0 = -0.89^{+0.22}_{-0.20}$ and $w_a = -0.75 \pm 1.1$ at 95% CL.

Constraints on $N_{\rm eff}$ remain almost unchanged across all curved dark energy models, with additional contributions always constrained to $\Delta N_{\rm eff} \lesssim 0.4$.

4. $\mathbf{w_0}$w CDM + Ω_K + M_ + N_{eff} Simultaneously varying both total neutrino mass and effective relativistic degrees of freedom, results are in

Table XIII and Figure 12 [Figure 12: see original paper].

TABLE XIII. 68% and 95% CL constraints on w$_0$w CDM + Ω_K + M_ + N_{eff}.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02255^{+0.00025+0.00049}_{-0.00025-0.00047}$	$0.02236^{+0.00024+0.00047}_{-0.00045-0.00086}$	$0.02234^{+0.00025+0.00050}_{-0.00047-0.00082}$	$0.02232^{+0.00023+0.00046}_{-0.00048-0.00088}$
$\Omega_c h^2$	$0.1181^{+0.0030+0.0061}_{-0.0030-0.0057}$	$0.1183^{+0.0030+0.0059}_{-0.0030-0.0060}$	$0.1182^{+0.0031+0.0062}_{-0.0030-0.0060}$	$0.1178^{+0.0029+0.0059}_{-0.0029-0.0058}$
w_0	unconstrained	$-0.52^{+0.29+0.43}_{-0.19-0.48}$	$-1.09^{+0.21+0.36}_{-0.17-0.38}$	$-0.90^{+0.10+0.21}_{-0.11-0.20}$
w_a	< 1.6 (95% CL)	< -0.4 (95% CL)	< 0.24 (95% CL)	$-0.72^{+0.63+1.1}_{-0.48-1.1}$
M_ν [eV]	< 0.753 (95% CL)	< 0.231 (95% CL)	< 0.49 (95% CL)	< 0.22 (95% CL)
$N_{\rm eff}$	$3.04^{+0.19+0.40}_{-0.20-0.37}$	$2.97^{+0.20+0.39}_{-0.20-0.39}$	$3.03^{+0.20+0.40}_{-0.20-0.39}$	$2.95^{+0.19+0.37}_{-0.19-0.36}$
Ω_{K0}	$-0.056^{+0.047+0.058}_{-0.019-0.076}$	$-0.0037^{+0.0035+0.0073}_{-0.0035-0.0068}$	$-0.033^{+0.011+0.021}_{-0.011-0.021}$	$-0.0025^{+0.0033+0.0069}_{-0.0033-0.0065}$
H_0 [km/s/Mpc]	57^{+7+31}_{-19-23}	$64.4^{+2.1+4.8}_{-2.7-4.6}$	$59.8^{+2.5+5.7}_{-3.2-5.3}$	$67.6^{+1.2+2.4}_{-1.2-2.4}$

In this largest parameter space, curvature results do not change significantly. CMB data always prefer curved spaces at 68% CL while flatness is allowed within 95% CL ($\Omega_{K0} = -0.056^{+0.058}_{-0.076}$). For H_0, CMB gives $H_0 = 57^{+7}_{-19}$ km/s/Mpc at 68% CL, similar to bounds without $N_{\rm eff}$. The preference for smaller H_0 is driven by combined effects of curvature and massive neutrinos; adding $N_{\rm eff}$ does not reduce tension between datasets.

Including BAO recovers essentially the same results as without varying $N_{\rm eff}$: $\Omega_{K0} = -0.0037^{+0.0073}_{-0.0068}$ at 95% CL (very close to zero but slightly disfavored at 68% CL), and $H_0 = 64.4^{+2.1}_{-2.7}$ km/s/Mpc at 68% CL, producing the same tensions. For $w_0 = -0.52^{+0.43}_{-0.48}$ and $w_a < -0.4$ (both at 95% CL), we observe strong indication for quintessential models with dynamical evolution preferred at >95% CL.

Combining CMB+Pantheon recovers the usual evidence for a closed Universe at >95% CL ($\Omega_{K0} = -0.033 \pm 0.021$ at 95% CL) and lower $H_0 = 59.8^{+2.5}_{-3.2}$ km/s/Mpc at 68% CL, in strong tension with local measurements. Both quintessential and phantom dark energy models are allowed within 68% CL ($w_0 = -1.09^{+0.21}_{-0.17}$ at 68% CL), with only an upper bound $w_a < 0.24$ at 95% CL. Combining all three datasets eliminates curvature evidence and shifts H_0 to higher values ($H_0 = 67.6 \pm 1.2$ km/s/Mpc at 68% CL), similar to standard ΛCDM. Here w_0 is consistent with a cosmological constant within 68% CL, and $w_a = 0$ within 95% CL.

Concerning relic neutrinos, CMB temperature and polarization measurements alone give $M_\nu < 0.753$ eV at 95% CL, while CMB+BAO improves this to $M_\nu < 0.231$ eV at 95% CL. This bound remains almost unchanged including Pantheon data. The same applies to $N_{\rm eff}$: additional contributions are always constrained to $\Delta N_{\rm eff} \lesssim 0.4$ at 95% CL, independent of dark energy parametrization.

D. Non-flat PEDE and its extensions

1. PEDE + Ω_K Table XIV summarizes constraints, with posterior distributions in Figure 13 [Figure 13: see original paper].

TABLE XIV. 68% and 95% CL constraints on PEDE + Ω_K.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02260^{+0.00017+0.00033}_{-0.00017-0.00034}$	$0.02242^{+0.00015+0.00030}_{-0.00033-0.00066}$	$0.02253^{+0.00017+0.00032}_{-0.00031-0.00060}$	$0.02237^{+0.00015+0.00030}_{-0.00032-0.00064}$
$\Omega_c h^2$	$0.1181^{+0.0015+0.0029}_{-0.0015-0.0028}$	$0.1196^{+0.0014+0.0028}_{-0.0014-0.0027}$	$0.1186^{+0.0014+0.0028}_{-0.0014-0.0028}$	$0.1203^{+0.0014+0.0028}_{-0.0014-0.0027}$
Ω_{K0}	$-0.037^{+0.018+0.029}_{-0.013-0.031}$	$-0.0033^{+0.0018+0.0035}_{-0.0019-0.0037}$	$-0.0055^{+0.0064+0.011}_{-0.0057-0.012}$	$-0.0026^{+0.0022+0.0058}_{-0.0024-0.0046}$
H_0 [km/s/Mpc]	$58.0^{+3.8+8.4}_{-4.8-7.8}$	$70.57^{+0.73+1.5}_{-0.81-1.4}$	$62.3^{+2.0+4.0}_{-2.0-3.7}$	$69.94^{+0.70+1.4}_{-0.72-1.4}$

All datasets indicate preference for non-zero spatial curvature. Specifically, Ω_K remains non-zero at >95% CL for CMB, CMB+Pantheon, and CMB+BAO+Pantheon, while for CMB+BAO it remains non-zero at >68% CL. All cases show negative mean values, indicating preference for a closed Universe.

For H_0, CMB alone predicts a very low value ($H_0 = 58^{+3.8}_{-4.8}$ km/s/Mpc at 68% CL), giving 3.5σ tension. Surprisingly, including BAO increases the mean H_0 and decreases errors, yielding $H_0 = 70.57^{+0.73}_{-0.81}$ km/s/Mpc at 68% CL and reducing tension with SH0ES to 2σ. Conversely, adding Pantheon to CMB slightly increases H_0 compared to CMB alone ($H_0 = 62.3 \pm 2$ km/s/Mpc at 68% CL), with tension at 4.8σ. The combined CMB+BAO+Pantheon analysis yields $H_0 = 69.94^{+0.70}_{-0.72}$ km/s/Mpc at 68% CL, slightly higher than Planck's flat ΛCDM value and reducing tension with SH0ES to 2.5σ.

2. PEDE + Ω_K + M_ Adding massive neutrinos, results are in Table XV and Figure 14 [Figure 14: see original paper].

TABLE XV. 68% and 95% CL constraints on PEDE + Ω_K + M_ .

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02254^{+0.00019+0.00035}_{-0.00019-0.00036}$	$0.02240^{+0.00016+0.00029}_{-0.00035-0.00068}$	$0.02253^{+0.00017+0.00032}_{-0.00031-0.00064}$	$0.02235^{+0.00016+0.00031}_{-0.00033-0.00063}$
$\Omega_c h^2$	$0.1183^{+0.0015+0.0030}_{-0.0016-0.0029}$	$0.1197^{+0.0014+0.0028}_{-0.0014-0.0027}$	$0.1184^{+0.0014+0.0028}_{-0.0014-0.0028}$	$0.1203^{+0.0014+0.0028}_{-0.0014-0.0027}$

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
M_ν [eV]	$<$ 0.81 (95% CL)	< 0.095 (95% CL)	< 0.123 (95% CL)	< 0.167 (95% CL)
Ω_{K0}	$-0.064^{+0.041+0.057}_{-0.020-0.069}$	$-0.0029^{+0.0019+0.0043}_{-0.0020-0.0038}$	$-0.0053^{+0.0065+0.011}_{-0.0057-0.012}$	$-0.0029^{+0.0020+0.0039}_{-0.0024-0.0042}$
H_0 [km/s/Mpc]	$51.2^{+6.7+13}_{-7.6-12}$	$70.0^{+0.71+1.5}_{-0.72-1.5}$	$62.3^{+1.9+3.7}_{-1.9-3.6}$	$69.78^{+0.71+1.4}_{-0.73-1.4}$

CMB and CMB+Pantheon data always prefer a closed Universe at >95% CL, while this preference reduces to 68% CL for CMB+BAO and CMB+BAO+Pantheon. Due to degeneracy with massive neutrinos, CMB data alone show preference for lower H_0 values ($H_0 = 51.2^{+6.7}_{-7.6}$ km/s/Mpc at 68% CL), tensioned with SH0ES at 3.2σ despite large error bars. Including BAO measurements recovers the same preference for larger values discussed in the baseline case ($H_0 = 70^{+0.71}_{-0.72}$ km/s/Mpc at 68% CL), reducing tension to 2.5σ. Conversely, CMB+Pantheon gives $H_0 = 62.3 \pm 1.9$ km/s/Mpc at 68% CL, restoring statistically significant tension. Combining all three yields an intermediate value ($H_0 = 69.78^{+0.71}_{-0.73}$ km/s/Mpc at 68% CL) very similar to the standard ΛCDM result. As in previous models, the H_0 tension impacts estimation of matter content, resulting in S_8 tension.

Neutrino mass constraints remain almost unchanged, but we observe improved constraining power from CMB+Pantheon ($M_\nu < 0.26$ eV at 95% CL). Combining CMB+BAO improves this to $M_\nu < 0.22$ eV at 95% CL, while CMB+BAO+Pantheon gives a less tight bound ($M_\nu < 0.37$ eV at 95% CL).

3. PEDE + Ω_K + N_{eff} Replacing massive neutrinos with N_{eff}, results are in Table XVI and Figure 15 [Figure 15: see original paper].

TABLE XVI. 68% and 95% CL constraints on PEDE + Ω_K + N_{eff}.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02259^{+0.00024+0.00050}_{-0.00024-0.00047}$	$0.02233^{+0.00023+0.00046}_{-0.00046-0.00084}$	$0.02253^{+0.00023+0.00046}_{-0.00045-0.00087}$	$0.02219^{+0.00022+0.00045}_{-0.00043-0.00084}$
$\Omega_c h^2$	$0.1178^{+0.0029+0.0058}_{-0.0029-0.0057}$	$0.1182^{+0.0031+0.0063}_{-0.0031-0.0060}$	$0.1182^{+0.0030+0.0060}_{-0.0030-0.0056}$	$0.1176^{+0.0029+0.0059}_{-0.0029-0.0058}$
N_{eff}	$3.03^{+0.19+0.39}_{-0.19-0.38}$	$2.94^{+0.19+0.40}_{-0.20-0.40}$	$3.02^{+0.18+0.38}_{-0.21-0.38}$	$2.86^{+0.18+0.38}_{-0.20-0.38}$
Ω_{K0}	$-0.038^{+0.018+0.030}_{-0.013-0.032}$	$-0.0020^{+0.0021+0.0047}_{-0.0025-0.0046}$	$-0.0061^{+0.0065+0.011}_{-0.0057-0.012}$	$-0.0032^{+0.0024+0.0058}_{-0.0027-0.0052}$
H_0 [km/s/Mpc]	$57.9^{+4.0+8.7}_{-4.6-8.0}$	$70.0^{+1.2+2.5}_{-1.2-2.6}$	$62.3^{+1.8+3.9}_{-2.1-3.5}$	$68.9^{+1.1+2.4}_{-1.2-2.3}$

Planck CMB measurements alone show indication for a closed Universe at >95% CL ($\Omega_{K0} = -0.038^{+0.030}_{-0.032}$ at 95% CL). This indication remains for CMB+Pantheon, while combining with BAO reduces it to 68% CL, with flatness always allowed within 95% CL. From CMB alone we obtain $H_0 = 57.9^{+4.0}_{-4.6}$ km/s/Mpc at 68% CL, tensioned with SH0ES at 3.7σ. Interestingly, CMB+BAO reduces tension to 1.9σ due to the higher value $H_0 = 70.0 \pm 1.2$ km/s/Mpc at 68% CL. CMB+Pantheon again prefers smaller H_0 ($H_0 = 62.3^{+1.8}_{-2.1}$ km/s/Mpc at 68% CL), in contrast with SH0ES at 5.1σ. Combining all three yields results similar to the standard cosmological model, also in tension with local measurements.

Constraints on $N_{\rm eff}$ are almost unchanged across different dark energy parametrizations, with additional contributions always constrained to $\Delta N_{\rm eff} \lesssim 0.4$ at 95% CL. For CMB+BAO+Pantheon, the reference value $N_{\rm eff} = 3.044$ is disfavored at 68% CL with preference for smaller values, similar to what might resolve the "CMB tension" between different CMB datasets [**?**].

4. PEDE + Ω_K + M_ + N_{eff} Finally, simultaneously varying both total neutrino mass and effective relativistic degrees of freedom, results are in Table XVII and Figure 16 [Figure 16: see original paper].

TABLE XVII. 68% and 95% CL constraints on PEDE + Ω_K + M_ + N_{eff}.

Parameter	CMB	CMB+BAO	CMB+Pantheon	CMB+BAO+Pantheon
$\Omega_b h^2$	$0.02254^{+0.00026+0.00052}_{-0.00026-0.00048}$	$0.02233^{+0.00023+0.00046}_{-0.00046-0.00084}$	$0.02251^{+0.00025+0.00051}_{-0.00045-0.00086}$	$0.02218^{+0.00024+0.00045}_{-0.00047-0.00082}$
$\Omega_c h^2$	$0.1182^{+0.0030+0.0059}_{-0.0030-0.0059}$	$0.1182^{+0.0031+0.0063}_{-0.0031-0.0060}$	$0.1182^{+0.0030+0.0060}_{-0.0030-0.0056}$	$0.1176^{+0.0032+0.0060}_{-0.0032-0.0058}$
M_ν [eV]	< 0.84 (95% CL)	< 0.103 (95% CL)	< 0.128 (95% CL)	< 0.172 (95% CL)
$N_{\rm eff}$	$3.05^{+0.20+0.40}_{-0.20-0.40}$	$2.93^{+0.20+0.39}_{-0.19-0.38}$	$3.02^{+0.20+0.40}_{-0.20-0.39}$	$2.86^{+0.18+0.38}_{-0.20-0.38}$
Ω_{K0}	$-0.069^{+0.045+0.066}_{-0.020-0.078}$	$-0.0020^{+0.0021+0.0047}_{-0.0025-0.0046}$	$-0.0061^{+0.0065+0.011}_{-0.0057-0.012}$	$-0.0032^{+0.0024+0.0058}_{-0.0027-0.0052}$
H_0 [km/s/Mpc]	$50.3^{+6.8+14}_{-6.9-13}$	$69.9^{+1.3+2.6}_{-1.3-2.5}$	$62.4^{+1.8+3.9}_{-2.1-3.6}$	$68.9^{+1.1+2.4}_{-1.2-2.3}$

Exploiting only Planck CMB observations in this largest parameter space, we obtain preference for a curved cosmological spacetime at >95% CL with a very small expansion rate $H_0 = 50.3^{+6.8}_{-6.9}$ km/s/Mpc at 95% CL. This small value is due to combined effects of curvature and degeneracy with relic neutrino mass, producing large uncertainty. Combining CMB+BAO reduces the closed Universe preference to slightly more than 68% CL, with flatness consistent at 95% CL, while constraints on H_0 change drastically to prefer higher values ($H_0 = 69.9 \pm 1.3$ km/s/Mpc at 68% CL), reducing tension with SH0ES to 1.9σ. Conversely, CMB+Pantheon finds strong preference for curved spacetime ($\Omega_{K0} = -0.021^{+0.012}_{-0.013}$ at 95% CL) and smaller H_0 ($H_0 = 62.4^{+1.8}_{-2.1}$ km/s/Mpc at 68% CL), increasing tension between different H_0 and S_8 estimations. Interestingly, unlike other baseline extensions, combining CMB+BAO+Pantheon eliminates the closed Universe preference, with flatness consistent within 68% CL. For H_0 we derive results similar to standard ΛCDM: $H_0 = 68.9^{+1.1}_{-1.2}$ km/s/Mpc at 68% CL. Large disagreements among datasets persist, and adding parameters does not significantly reduce tensions.

Concerning the neutrino sector, the CMB bound $M_\nu < 0.84$ eV at 95% CL can be improved to $M_\nu < 0.26$ eV and $M_\nu < 0.24$ eV including Pantheon and BAO measurements respectively. Combining all three gives a less constrained bound ($M_\nu < 0.35$ eV). These results are similar to those without varying N_{eff}, and N_{eff} bounds are also unaffected by neutrino inclusion. In particular, combining CMB+BAO+Pantheon shows the same slight preference for smaller N_{eff} values relative to the standard model prediction, though everything remains consistent within 95% CL and additional contributions are constrained to $\Delta N_{\text{eff}} \lesssim 0.4$ at 95% CL.

V. Summary and Conclusions

Modern cosmology has witnessed remarkable success, with increasingly sensitive astronomical data enabling stringent constraints on cosmological parameters while revealing new challenges. The H_0 and S_8 tensions have emerged as two major challenges for standard ΛCDM cosmology, demanding its revision, and Planck's preference for a closed universe model \cite{8–11,15,16} further suggests that curvature may play a vital role in parameter estimation. Though earlier investigations agreed with spatial flatness, there is no theoretical reason to assume it [**?**], and curvature's effects on cosmological models and key parameters should be investigated. The main theme of this work is the Universe's curvature and its impacts on extended cosmologies, with special focus on cosmological tensions.

To investigate these issues, we considered 16 extended cosmological scenarios emerging from four well-known models (ΛCDM, wCDM, w_0w CDM, PEDE; see Section II) and performed systematic studies using Planck 2018 CMB temperature and polarization spectra combined with BAO and Pantheon SNIa data. These sixteen extended scenarios were constrained using CMB, CMB+BAO, CMB+Pantheon, and CMB+BAO+Pantheon.

A. Summary

1. Curvature Figure 17 [Figure 17: see original paper] summarizes curvature constraints for all cosmological models and data combinations. Planck's indication for a closed Universe observed in baseline ΛCDM persists in all non-flat extended cosmologies, though at different statistical levels due to larger error bars from extended parameter space and/or different dark energy parametrizations. This loss of constraining power is typically due to strong geometrical degeneracies, particularly between the dark energy equation of state and curvature density parameter Ω_{K0}. Such degeneracies can be broken by combining

CMB with complementary datasets like BAO large-scale structure information or Pantheon distance moduli measurements.

For CMB+BAO, the preference for a closed Universe disappears regardless of dark energy model, with curvature constraints shrinking around $\Omega_{K0} = 0$ and pointing toward spatial flatness (within one standard deviation, or slightly more for dynamical dark energy). However, BAO data are generally tensioned with Planck when curvature varies, making this combination not robust. For CMB+Pantheon, Ω_{K0} results become sensitive to dark energy assumptions: fixing $w = -1$ still indicates spatial flatness within one standard deviation, while all other dynamical and non-dynamical parametrizations systematically show preference for a closed geometry.

2. H$_0$ Figure 18 [Figure 18: see original paper] summarizes H_0 constraints, clearly showing how different cosmological models return different H_0 values when curvature is considered. Throughout this work, we find that H_0 has a deep degeneracy with curvature in all proposed models.

In all ΛCDM extensions, CMB alone shows evidence for a closed Universe at >95% CL. Since H_0 and Ω_{K0} are positively correlated (see Figures 1–4), as Ω_{K0} deviates from zero toward negative values, H_0 takes lower values, increasing tension with SH0ES (Figure 18). This increased tension is solely due to the closed Universe evidence from CMB alone. For CMB+BAO and CMB+BAO+Pantheon, estimated H_0 values are almost identical to Planck's flat ΛCDM value, as Ω_{K0} is consistent with zero, except for the PEDE parametrization where H_0 is slightly higher, alleviating tension. For CMB+Pantheon, H_0 values in all four extended scenarios are lower than Planck's flat ΛCDM value. Thus, no effective resolution of the H_0 tension is observed within these ΛCDM extensions.

In all wCDM extensions, CMB alone again returns lower H_0 values compared to Planck's flat ΛCDM estimation due to closed Universe evidence at >68% CL (and >95% CL for wCDM + Ω_K + M_). However, error bars are very large, reducing tension with SH0ES to 1.2σ (wCDM + Ω_K), <2σ (wCDM + Ω_K + N_{eff}), and 2.7σ (wCDM + Ω_K + M_ + N_{eff}). Such alleviation is mainly driven by volume effects of the parameter space. CMB+BAO constraints on H_0 are similar to Planck values, though mean values and errors are slightly larger, reducing tension to 2.3σ (wCDM + Ω_K), 2.4σ (wCDM + Ω_K + M_), 2.2σ (wCDM + Ω_K + N_{eff}), and 2.3σ (wCDM + Ω_K + M_ + N_{eff}). CMB+Pantheon yields relatively lower H_0 values due to >95% CL preference for a closed Universe. CMB+BAO+Pantheon results are almost identical to CMB+BAO for all wCDM extensions.

For w$_0$w CDM + Ω_K and its neutrino extensions, CMB alone returns lower H_0 values with very large error bars, reducing tension with SH0ES to 1σ, 2.5σ, <1σ, and 2.3σ respectively. For CMB+BAO, all extended scenarios show >68% CL preference for a closed Universe, yielding lower H_0 values than Planck's flat

ΛCDM. CMB+Pantheon H_0 estimations are lower than both Planck's value and CMB+BAO estimations due to >95% CL support for a closed Universe. CMB+BAO+Pantheon H_0 values match Planck's flat ΛCDM values.

In extended PEDE scenarios, CMB alone and CMB+Pantheon give lower H_0 values due to >95% CL preference for a closed Universe. For CMB+BAO, despite >68% CL preference for a closed Universe, tension with SH0ES is reduced to many standard deviations: 2σ (PEDE + Ω_K), 2σ (PEDE + Ω_K + M_), 1.9σ (PEDE + Ω_K + N_{eff}), and 1.9σ (PEDE + Ω_K + M_ + N_{eff}). CMB+BAO+Pantheon results are similar to CMB+BAO with hints of a closed Universe at >68% CL.

It is worth noting that low-redshift new physics models (or late-time solutions) cannot solve the H_0 tension once BAO and Pantheon data are considered [**?**, **?**, **?**], unless the dark energy density becomes negative [**?**, **?**].

3. S_8 Figure 19 [Figure 19: see original paper] summarizes S_8 constraints from all extended models using CMB, CMB+BAO, CMB+Pantheon, and CMB+BAO+Pantheon, together with estimated values from Kilo-Degree Survey (KiDS-1000) [**?**], Dark Energy Survey (DES) Year 3 [**?**], and Planck [**?**], all assuming ΛCDM.

For CMB and CMB+Pantheon, estimated S_8 values in all extended scenarios are far from both KiDS-1000 [**?**] and DES-Y3 [**?**] (when they assume vanilla ΛCDM), and also far from Planck's vanilla ΛCDM estimation, except in extended scenarios assuming a cosmological constant where CMB+Pantheon S_8 values are slightly closer. For all other cases with these two datasets, the S_8 tension increases significantly due to indication for a closed Universe when not assuming a cosmological constant.

Conversely, S_8 estimations from CMB+BAO and CMB+BAO+Pantheon are almost identical to Planck's vanilla ΛCDM values, indicating that within such extended scenarios, the S_8 tension increases when there is indication for a closed Universe.

4. M_ From all analyses, we find no evidence for a total neutrino mass. Only in a few cases (ΛCDM + Ω_K + M_ + N_{eff}, wCDM + Ω_K + M_ , w_0w CDM + Ω_K + M_ , PEDE + Ω_K + M_ + N_{eff}) does CMB alone show indication of total neutrino mass at 68% CL, but this disappears when external probes are added. Figure 20 [Figure 20: see original paper] presents a graphical summary of M_ν upper limits at 95% CL for all extended scenarios.

5. N_{eff} Figure 21 [Figure 21: see original paper] summarizes N_{eff} constraints across all extended scenarios for all dataset combinations. Irrespective of cosmological model, estimated N_{eff} values for CMB and CMB+Pantheon are almost identical to the standard value $N_{\mathrm{eff}} = 3.044$, so these datasets show no deviation from standard predictions. The combination CMB+BAO+Pantheon shows slight preference for smaller N_{eff} values, though everything remains consistent with the standard model within 95% CL, and additional contributions are always constrained to $\Delta N_{\text{eff}} \lesssim 0.4$ at 95% CL.

B. Concluding Remarks and the Road Ahead

This comprehensive analysis of curvature effects in extended cosmological models reveals several key findings. The indication for a closed Universe from Planck 2018 CMB data persists across most extended models, though its statistical significance varies with parameter space volume and dark energy parametrization. The degeneracy between curvature and other cosmological parameters, particularly H_0, means that allowing curvature can significantly alter constraints on the expansion rate and other key parameters.

Importantly, we find that the assumption of spatial flatness can bias constraints on both dark energy properties and neutrino parameters. The tension between early-universe (CMB) and late-universe (SH0ES, BAO, SNIa) measurements is not resolved by simply adding curvature or extending the neutrino sector—indeed, in many cases these extensions exacerbate existing tensions.

The PEDE model shows unique behavior in that its combination with BAO data can alleviate the H_0 tension more effectively than other dark energy parametrizations, though this comes at the cost of increased S_8 tension. This suggests that solutions to cosmological tensions may require more radical modifications to the standard cosmological model than simply allowing curvature or varying neutrino properties.

Future work should explore additional combinations of datasets, consider alternative curvature parametrizations, and investigate whether the curvature indication persists with next-generation CMB experiments such as CMB-S4 and the Simons Observatory. The interplay between curvature, dark energy dynamics, and neutrino properties remains a rich area for theoretical and observational investigation, with important implications for both cosmology and particle physics.

Footnotes:

[1] Apart from these dark energy parametrizations, many works aiming to probe the dynamical nature of dark energy are available; see for instance Refs. \cite{88–102}.

[2] Unless otherwise specified, "0" attached to any quantity refers to its present value; for example, Ω_{K0} is the present value of the curvature density parameter.

[3] From now on we use w instead of w_{DE} and label the corresponding cosmological model as wCDM (instead of wDECDM) as standard in the literature.

[4] The corresponding cosmological model is labeled as w_0w CDM.

[5] While this bound $\Delta N_{\text{eff}} \lesssim 0.4$ is tight enough to severely constrain exotic scenarios like additional neutrino species, it is too large for probing other standard model extensions involving extra relativistic species decoupled at high temperatures (order of top quark annihilation), whose contribution $\Delta N_{\text{eff}} \sim 0.027$ is much smaller than current constraining power.

[6] In Figure 19 we have summarized constraints on the S_8 parameter from all extended models using CMB, CMB+BAO, CMB+Pantheon, and CMB+BAO+Pantheon, together with estimated S_8 values from KiDS-1000 [**?**], DES-Y3 [**?**], and Planck [**?**], all obtained assuming ΛCDM.

6. Extended Scenarios and Parameter Constraints

Both the KiDS-100 and DES collaborations have explored extended scenarios in Refs. [72, 74]; however, since the specific parameter combinations investigated in our study were not considered in those works, a direct comparison cannot be made safely.

Figure 18 presents whisker plots showing 68% confidence level constraints on the Hubble constant H_0 for various cosmological scenarios in the presence of non-zero cosmic curvature, considering four dataset combinations: CMB alone, CMB+BAO, CMB+Pantheon, and CMB+BAO+Pantheon. The yellow vertical dotted line corresponds to the H_0 value from the Planck 2018 team [8] assuming a ΛCDM model, while the magenta vertical band represents the H_0 measurement from the SH0ES team ($H_0 = 73.04 \pm 0.99$ km/s/Mpc at 68% CL) [32]. Our estimations are consistent with this standard value.

For the remaining two cases—CMB+BAO and CMB+BAO+Pantheon—the values of N_{eff} are slightly lower ($N_{\text{eff}} < 3$) but remain consistent with the standard value within 68% CL.

6.1 Dark Energy Equation of State Parameters

Here we summarize our main results on the dark energy equation of state parameter w of the wCDM model and the (w_0, w_a) parameters of the w_0w_aCDM model.

Figure 19 displays whisker plots with 68% CL constraints on the S_8 $(= \sigma_8\sqrt{\Omega_{m0}/0.3})$ parameter obtained across various cosmological scenarios for several observational datasets: CMB, CMB+BAO, CMB+Pantheon, and CMB+BAO+Pantheon. The magenta vertical band corresponds to the estimate from the Kilo-Degree Survey (KiDS-1000), yielding $S_8 = 0.766^{+0.020}_{-0.014}$ [71], while the green vertical band shows the value provided by the Planck collaboration, $S_8 = 0.834 \pm 0.016$ [8], both obtained assuming a ΛCDM scenario.

Figure 20 shows whisker plots with 95% CL upper bounds on the sum of neutrino masses M_ν [eV] obtained in various cosmological scenarios for the same four dataset combinations.

Figure 21 presents whisker plots with 68% CL constraints on $N_{\rm eff}$ obtained across various cosmological scenarios. The magenta vertical line corresponds to the standard value $N_{\rm eff} = 3.044$.

Figure 22 displays whisker plots with 68% CL constraints and 95% CL upper limits on w_0 and w_a for the wCDM and w_0w_aCDM models across all observational datasets. The magenta vertical lines in the two panels represent the standard cosmological constant value $w_0 = -1$ and the non-dynamical $w_a = 0$ limit, respectively. Missing blue lines (Planck-only results) in the left panel indicate that w is unconstrained for those models.

In the wCDM model, only the CMB+Pantheon dataset provides strong indication of phantom dark energy (i.e., $w < -1$) at more than 95% CL, and this occurs only when the equation of state is constant and non-dynamical. For all other cases, $w = -1$ is allowed within 68% CL.

However, for the w_0w_aCDM model—where the dark energy equation of state is dynamical—CMB+BAO data suggest quintessence dark energy (i.e., $w_0 > -1$), while all other dataset combinations agree with a cosmological constant. Regarding the dynamical nature of dark energy quantified through the parameter w_a, we find that CMB alone and CMB+Pantheon are very consistent with $w_a = 0$ (except in the model w_0w_aCDM + Ω_K, where $w_a \neq 0$ at more than 68% CL). However, when BAO data are added to CMB, an indication for $w_a < 0$ at more than 2σ emerges. This evidence is mildly diluted but remains true at slightly more than 68% CL when the full CMB+BAO+Pantheon combination is explored. In other words, for this w_0w_aCDM model, CMB+BAO data prefer a quintessence dynamical dark energy at more than 95% CL.

Note that w_0 remains unconstrained in the scenarios w_0w_aCDM + $\Omega_K + M_\nu$, w_0w_aCDM + $\Omega_K + M_\nu + N_{\rm eff}$, and w_0w_aCDM + $\Omega_K + N_{\rm eff}$, as even though the 68% CL upper bound on w_0 is available, it becomes unconstrained at 95% CL.

B. Concluding Remarks and Future Prospects

There is no doubt that modern cosmology is incomplete without observational data. From the detection of cosmic microwave background radiation to the discovery of late-time cosmic acceleration, we have witnessed how observational data have been crucially important for understanding our Universe. Whenever we discuss cosmology, the six-parameter ΛCDM model—under the assumptions of a positive cosmological constant, pressureless dark matter, and a flat Universe—naturally enters the picture because of its remarkable agreement with a wide range of astronomical data. However, the true nature of dark energy and dark matter within this simplest cosmological scenario remains one of the longstanding debates in cosmology.

The assumption of a flat Universe has created further debate within the scientific community. A series of recent articles have argued that present observational data hint at evidence for a closed Universe [8–11, 18]. This evidence has sparked significant interest because any indication of a closed Universe increases tensions in key cosmological parameters, further challenging the flat ΛCDM cosmology. Generally, the curvature of the Universe is "assumed" to be zero when cosmological models are analyzed, motivated by three factors: (i) the inflationary paradigm has been tremendously successful in predicting a spatially flat Universe, (ii) past observational data preferred a curvature very close to zero, and (iii) the inclusion of extra parameters such as the curvature parameter may increase degeneracies in the parameter space. However, over the last several years, the sensitivity of astronomical data has improved substantially, and given such improvements, it is natural to allow observational data to determine the nature of the curvature parameter. This means that assuming a flat Universe (i.e., $\Omega_K = 0$) when analyzing cosmological models is not a realistic approach. We further note that forcing $\Omega_K = 0$ could bias the results, potentially hiding the intrinsic nature of cosmological models forever [19]. At this stage, where we have abundant astronomical data from various sources, imposing $\Omega_K = 0$ is no longer justified.

Thus, in the present article we investigated the effects of curvature on cosmological models and their associated free and derived parameters. We considered several classical and well-known cosmological models and their extensions, including variations in the dark energy and neutrino sectors, in the presence of cosmic curvature. Our analyses clearly indicate that it is preferable not to fix $\Omega_K = 0$ when analyzing cosmological models. We anticipate that upcoming observational data from various astronomical surveys [135–138] will be highly promising in revealing the intrinsic nature of the dark sector as well as the curvature of the Universe.

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