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Full Text

Preamble

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UNIFORM EXPONENTIAL STABILITY AND CONTROL CONVERGENCE OF SEMI-DISCRETE SCHEME FOR A TIMOSHENKO BEAM

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Abstract. This paper considers numerical approximations of a Timoshenko beam under boundary control. The continuous system under boundary feedback is known to be exponentially stable. Firstly, the continuous system is transformed into an equivalent first-order port-Hamiltonian formulation. A basically order reduction finite difference scheme is applied to derive a family of semi-discretized systems. Secondly, a completely new method which is based on a mixed discrete observability inequality involving final state observability and exact observability is developed to prove the uniform exponential stability of the discrete systems. More interestingly, the proof for the uniformly exponential stability of discrete systems is almost parallel to that of the continuous counterpart. Thirdly, the solutions of the semi-discretized systems are shown to be strongly convergent to the solution of the original system through Trotter-Kato theorem. Finally, both exact controllability of continuous system and the discrete systems are proved in light of Russell's "controllability via stability" principle and the explicit controls are derived. Moreover, the discrete controls are shown for the first time to be convergent to the continuous control by the proposed approach.

Key words. Timoshenko beam; exponential stability; exact controllability; finite-difference; semi-discretization.

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1. Introduction

A major concern in control of systems described by partial differential equations (PDEs) is finite-dimensional approximation. This is because most feedback controls like observer-based feedback controls for PDEs are usually infinite-dimensional. In order to apply the designed PDE feedback control, discretization is indispensable in most situations. For this purpose, a first natural step is to discretize the spatial variable of PDEs while keeping time continuous. This process is called semi-discretization. The benefits of such semi-discretization are obvious. On the one hand, the semi-discretized models are finite-dimensional and can be handled by most control engineers. On the other hand, if the semi-discretized modes preserve as many physical properties of the original PDEs as possible, the discrete modes themselves can be regarded as physical modes instead of infinite-dimensional PDEs. Unfortunately, this is not always possible because spurious high frequency modes might be produced during discretization, which destroys such preservation.

The papers [2] and [14] first pointed out independently that classical finite-difference or finite element discretizations for a 1-D wave equation cannot preserve uniform exponential stability. This has been confirmed by many researchers in the subsequent two decades, which includes not only uniform exponential stability but also uniform controllability and observability [31, 38]. This seriously restricts the applications of PDE feedback controls.

Several remedies have been introduced to overcome this difficulty, such as Tychonoff regularization [14], mixed finite elements [2, 4], filtering of high frequencies [3, 15], two-grid algorithms [23, 27], non-uniform numerical meshes [8], and the vanishing viscosity approach [25, 26, 31]. Among these aforementioned methods, the vanishing viscosity approach is a popular one, which introduces a vanishing damping term with step size in discrete models. However, these approaches either lose the popularity of the finite difference scheme or add artificial numerical viscosity which differs for different systems. In addition, the artificial numerical viscosity method is most often confined to special boundary conditions because the proofs for uniform exponential stability and controllability rely heavily on the analytic forms of the eigen-pairs.

Very recently, a natural finite-difference scheme based on order reduction method was introduced in [22] for wave equation with boundary damping. It has demonstrated that the scheme not only preserves uniform exponential stability but also that the proof of uniform exponential stability is completely parallel to the continuous counterpart, which significantly simplifies the mathematical analysis. The approach is somehow universal, which has been applied successfully to uniform exponential stability for wave equation with local viscosity damping ([13, 37]), uniform exponential stability of Schrödinger equation ([12, 20]), uniform exponential stability and observability of Euler-Bernoulli equation ([24]), and even observer-based uniform exponential stability of wave equation which is actually a coupled PDEs ([30]). The remaining problem, however, is the convergence of the discrete controls, which has been outstanding for quite a while. The uniform controllability of the semi-discretization for beam equation has been studied in [6, 7, 21] by numerical viscosity approach.

Motivated by these aforementioned works, in this paper we study a classical Timoshenko beam which has been extensively investigated in the past decades due to its wide applications in engineering, both from PDEs and numerical approximation points of view. The Timoshenko beam model is a variant of the Euler-Bernoulli beam model, which takes the effect of shear and rotation into account. In this paper, we consider a Timoshenko beam model described by the following equations:

$$\begin{cases} \rho w_{tt}(x,t) - K(w_x(x,t) - \phi(x,t))_x = 0, & x \in (0,1), t > 0 \\ I_\rho \phi_{tt}(x,t) - EI\phi_{xx}(x,t) - K(w_x(x,t) - \phi(x,t)) = 0, & x \in (0,1), t > 0 \\ w(0,t) = \phi(0,t) = 0, & t > 0 \\ w_x(1,t) - \phi(1,t) = u_2(t), & t > 0 \\ \phi_x(1,t) = u_1(t), & t > 0 \end{cases}$$

where $w(x,t)$ is the transverse displacement and $\phi(x,t)$ the rotation angle of a filament of the beam, both at time t and location x. The constant coefficients ρ, I_ρ, EI and K are the mass per unit length, the rotary moment of inertia of a cross section, the product of Young's modulus of elasticity and the moment of inertia of a cross section, and the shear modulus, respectively. $u_1(t)$ and $u_2(t)$ are boundary controls that monitor $w_x(x,t)$ and $\phi_t(x,t)$ at $x=1$ and transform them into the lateral force and moment applied at $x=1$, respectively. For notational simplicity, hereafter we omit all obvious space and time domains in the rest of the paper.

Stabilization for Timoshenko beam has been studied by many researchers through designing boundary or interior damping in the past several decades. Kim and Renardy proposed the following boundary feedback controls in paper [18]:

$$u_1(t) = \alpha_1 \rho w_t(1,t), \quad u_2(t) = \alpha_2 I_\rho \phi_t(1,t).$$

They proved that the energy of system (1.1) under boundary controls (1.2) decays exponentially by the C_0-semigroup theory and the Lyapunov functional method. In [34], Xu and Feng presented detailed spectral analysis and Riesz basis property of the generalized eigenfunctions of the beam (1.1)-(1.2). Gorrec, Jacob, and Zwart et al. obtained the exponential stability of (1.1)-(1.2) by the stability theory of port-Hamiltonian systems on infinite-dimensional space in [17, 33]. Raposo et al. added interior frictional dissipative terms $w_t(x,t)$ and $\phi_t(x,t)$ into (1.1) to get

$$\begin{cases} \rho w_{tt}(x,t) - K(w_x(x,t) - \phi(x,t))_x + w_t(x,t) = 0, \\ I_\rho \phi_{tt}(x,t) - EI\phi_{xx}(x,t) - K(w_x(x,t) - \phi(x,t)) + \phi_t(x,t) = 0, \\ w(0,t) = \phi(0,t) = 0, \quad w_x(1,t) - \phi(1,t) = \phi_x(1,t) = 0. \end{cases}$$

and obtained in [28] the exponential stability of system (1.3) by means of the frequency domain approach of C_0-semigroups. Yan et al. studied in [35] the stabilization problem of Timoshenko beam (1.1) under linear dissipative boundary feedback controls and presented various necessary and sufficient conditions for the system to be asymptotically stable, where the equivalence between the exponential stability and the asymptotic stability for the closed-loop system was also given by the frequency domain analysis.

Recently, Rivera and Naso proposed boundary dissipation only on one side of the bending moment, i.e., $EI\phi_x(0,t) = \alpha_2 \phi_t(0,t)$ with other three Dirichlet boundary conditions to (1.1). They proved the exponential stability under the condition that the wave speeds of the system are equal to each other [29]. Almeida Júnior, Ramos and Freitas found some new facts related to the classical Timoshenko system. Precisely, they proved the damped shear beam model, which corresponds to a part of the classical Timoshenko beam model, possesses an energy exponential decay for any coefficients of the system under the Dirichlet-Neumann boundary conditions [1].

In this paper, we focus on system (1.1)-(1.2) and apply Russell's principle to construct controls which steer any initial state to rest both in continuous and discrete cases. However, it is well known that either the exponential stability or uniform exponential stability of some systems are needed when Russell's principle is applied. We hence start from the uniform exponential stability of discrete systems since the exponential stability of continuous system has been well established. The most difficult part of this paper is to test uniform exponential stability of discretized Timoshenko beam, for which we have encountered difficulty in applying existing approaches. The main reason is that it is difficult to find a suitable Lyapunov function or a frequency domain energy multiplier for this coupled system. In this paper, we propose a completely new proof for the uniform exponential stability, which is quite different from previous works [12, 25]. The convergence analysis, i.e., the solutions of the semi-discretized systems strongly converge to the solution of the original system, is also developed. This is realized through Trotter-Kato theorem and does not utilize the methods of [20, 31] because the results there cannot be applied directly to the discrete approximations of the controls. Based on the uniform exponential stability result and convergence analysis, we derive that the discrete controls are convergent to the continuous system control when the initial values of the continuous system and discrete systems satisfy some priori convergence conditions.

The rest of this paper is organized as follows. In Section 2, we restate the exponential stability of an equivalent order reduced continuous system and discretize its spatial variable by the method of [22, 20]. Uniform exponential stability is also presented in this section. In Section 3, convergence of the numerical approximate solutions to the continuous one is analyzed by using the Trotter-Kato theorem. In Section 4, we obtain the controls that steer the initial values to rest by Russell's principle both for continuous and discrete systems. Moreover, we show that the discrete controls are strongly convergent to the continuous system counterpart, followed by concluding remarks in Section 5.

2. Preliminary results on systems (1.1)-(1.2)

In this section, we discuss uniform exponential stability for Timoshenko beam (1.1) under boundary feedbacks (1.2). Subsection 2.1 gives exponential stability for continuous system and Subsection 2.2 presents uniform exponential stability for semi-discrete models.

2.1. Exponential stability of continuous system (1.1)-(1.2)

To have an almost one-to-one correspondence from the exponential stability of the continuous PDE system to the uniform exponential stability of the discrete ODE counterparts discussed in the next subsection, we introduce order reduction transforms:

$$\begin{cases} y_1(x,t) = w_x(x,t) - \phi(x,t), \\ y_2(x,t) = \rho w_t(x,t), \\ y_3(x,t) = \phi_x(x,t), \\ y_4(x,t) = I_\rho \phi_t(x,t), \end{cases}$$

and obtain an equivalent system of (1.1)-(1.2) as follows:

$$\begin{cases} \dot{y}_1(x,t) = \beta_2 y_2'(x,t) - \beta_4 y_4(x,t), \\ \dot{y}_2(x,t) = \beta_1 y_1'(x,t), \\ \dot{y}_3(x,t) = \beta_4 y_4'(x,t), \\ \dot{y}_4(x,t) = \beta_3 y_3'(x,t) + \beta_1 y_1(x,t), \\ y_1(1,t) = \alpha_1 y_2(1,t), \quad y_2(0,t) = 0, \\ y_3(1,t) = \alpha_2 y_4(1,t), \quad y_4(0,t) = 0, \end{cases}$$

in which $\beta_1 = K$, $\beta_2 = \rho^{-1}$, $\beta_3 = EI$ and $\beta_4 = I_\rho^{-1}$, and dot $\dot{}$ stands for the derivative in t and prime $'$ for the derivative with respect to x. It seems that in system (2.2) all boundary derivatives with respect to x disappear, which is the biggest merit of the order reduction method. Set $Y(x,t) = (y_1(x,t), y_2(x,t), y_3(x,t), y_4(x,t))^\top$. Then the first four equations of (2.2) can be written as a port-Hamiltonian system

$$\dot{Y}(x,t) = P_1[Y(x,t)]' + P_0 H Y(x,t),$$

where

$$H = \mathrm{diag}(\beta_1, \beta_2, \beta_3, \beta_4), \quad P_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We consider system (2.2) or (2.3) in the state space $X := L^2((0,1);\mathbb{C}^4)$ with the inner product

$$\langle Y, Z \rangle_X = \int_0^1 Y(x)^* H Z(x)\, dx.$$

The rest of this subsection has been employed in [33]. To formulate the boundary conditions of (2.2) or (2.3) in a better manner, we introduce the boundary effort and flow defined respectively as $e_\partial = \frac{\sqrt{2}}{2}[Y(1) + Y(0)]$ and $f_\partial = \frac{\sqrt{2}}{2} P_1 H [Y(1) - Y(0)]$. We therefore consider the operator

$$\begin{cases} AY(x) = P_1(Y(x))' + P_0 HY(x), \\ D(A) = \{Y \in H^1((0,1);\mathbb{C}^4),\, W_B HY(1) = 0\}, \end{cases}$$

where $H^1((0,1);\mathbb{C}^4)$ is the Sobolev space and

$$W_B = \begin{pmatrix} \beta_2\beta_1^{-1} & -\alpha_1 & 0 & 0 \\ 0 & 0 & \beta_4\beta_3^{-1} & -\alpha_2 \\ \beta_2\beta_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & \beta_4\beta_3^{-1} & 0 \end{pmatrix}.$$

In this way, system (2.2) is formulated as an evolution equation in X:

$$\dot{Y}(\cdot,t) = AY(\cdot,t).$$

The following Lemma gives basic property of the matrix W_B. Because it involves simple calculation, we omit the proof.

Lemma 2.1. Let I_4 be the identity operator on $\mathbb{C}^4$ and $\Sigma = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}$. Then the matrix W_B has full rank and satisfies $W_B \Sigma W_B^* \geq 0$.

Lemma 2.2 was brought from [17, Lemma 7.2.1].

Lemma 2.2. For any $Y \in D(A)$, there holds

$$\mathrm{Re}\langle AY, Y\rangle_X = -\alpha_1\beta_1\beta_2|y_2(1)|^2 - \alpha_2\beta_3\beta_4|y_4(1)|^2 \leq 0.$$

Now we can show that (2.7) or (2.2) is exponentially stable.

Theorem 2.3. The operator A generates a C_0-semigroup of contractions $T(t)$ which is exponentially stable, i.e., there exist two constants $K, \omega > 0$ such that $\|T(t)\| \leq Ke^{-\omega t}$.

Proof. The first claim follows from [17, Theorem 7.2.4] and Lemma 2.1 and the second claim follows from [17] and Lemma 2.2.

Remark 2.1. By basic property of C_0-semigroups, the operator A^*, the dual operator of A, also generates a C_0-semigroup of contractions $T^*(t)$ which is also exponentially stable. Moreover, the dual system of (2.2) is

$$\begin{cases}\dot y_1(x,t)=\beta_2 y_2'(x,t)+\beta_4 y_4(x,t),\\ \dot y_2(x,t)=\beta_1 y_1'(x,t),\\ \dot y_3(x,t)=\beta_4 y_4'(x,t)+\beta_1 y_1(x,t),\\ \dot y_4(x,t)=\beta_3 y_3'(x,t),\\ y_1(1,t)=\alpha_1 y_2(1,t),\quad y_2(0,t)=0,\\ y_3(1,t)=\alpha_2 y_4(1,t),\quad y_4(0,t)=0,\\ (y_1(x,0),y_2(x,0),y_3(x,0),y_4(x,0))=Y(x,0).\end{cases}$$

2.2. Uniform exponential stability for semi-discrete systems of (2.2)

In this section we discuss uniform exponential stability for a semi-discrete scheme of Timoshenko beam (1.1)-(1.2). The process is the same as [20, 22]. For integer N, let $0=x_0<x_1<\cdots<x_N<x_{N+1}=1$, $x_i=ih$, $i=0,1,\cdots,N+1$, and $h=\frac{1}{N+1}$. Let $v_j:=(j+\frac{1}{2})h$ and f_j be the value of $f(x)$ at node x_i. Define $\delta_x f_{j+\frac{1}{2}}=\frac{f_{j+1}-f_j}{h}$ and $f_{j+\frac{1}{2}}=\frac{f_{j+1}+f_j}{2}$ to denote the central divided difference operator of first-order derivative $f_x(v_j)$ and the average operator at $f(v_j)$, respectively. Set $y_{l,j}(t)=y_l(x_j,t)$ to be grid functions at grids for $l=1,2,3,4$ and $j=0,1,\cdots,N$. The first four equations of system (2.2) are valid at v_j, i.e.,

$$\begin{cases}\dot y_1(v_j,t)=\beta_2 y_2'(v_j,t)-\beta_4 y_4(v_j,t),\\ \dot y_2(v_j,t)=\beta_1 y_1'(v_j,t),\\ \dot y_3(v_j,t)=\beta_4 y_4'(v_j,t),\\ \dot y_4(v_j,t)=\beta_3 y_3'(v_j,t)+\beta_1 y_1(v_j,t).\end{cases}$$

Now replacing $y_l(v_j,t)$ and $y_l'(v_j,t)$ by $y_{l,j+\frac{1}{2}}(t)$ and $\delta_x y_{l,j+\frac{1}{2}}(t)$ in (2.10) respectively, we derive a semi-discretized finite difference scheme of system (2.2) as follows:

$$\begin{cases}\dot y_{1,j+\frac{1}{2}}(t)=\beta_2\delta_x y_{2,j+\frac{1}{2}}(t)-\beta_4 y_{4,j+\frac{1}{2}}(t),\\ \dot y_{2,j+\frac{1}{2}}(t)=\beta_1\delta_x y_{1,j+\frac{1}{2}}(t),\\ \dot y_{3,j+\frac{1}{2}}(t)=\beta_4\delta_x y_{4,j+\frac{1}{2}}(t),\\ \dot y_{4,j+\frac{1}{2}}(t)=\beta_3\delta_x y_{3,j+\frac{1}{2}}(t)+\beta_1 y_{1,j+\frac{1}{2}}(t),\quad j=0,1,\cdots,N,\\ y_{1,N+1}(t)=\alpha_1 y_{2,N+1}(t),\quad y_{2,0}(t)=0,\\ y_{3,N+1}(t)=\alpha_2 y_{4,N+1}(t),\quad y_{4,0}(t)=0.\end{cases}$$

For more information, one refers to [22] and [24]. In the state space $\mathbb{C}^{4(N+1)}$, define

$$X_h=\{Z_h\in\mathbb{C}^{4(N+1)}: z_{2,0}=z_{4,0}=0,\ z_{1,N+1}=\alpha_1 z_{2,N+1},\ z_{3,N+1}=\alpha_2 z_{4,N+1}\},$$

with the inner product

$$\langle Y_h, Z_h \rangle_h = h \sum_{j=0}^{N} \sum_{l=1}^{4} \beta_l y_{l,j+\frac{1}{2}} \overline{z_{l,j+\frac{1}{2}}}.$$

To formulate (2.11) into vectorial form, we introduce some matrices. Let A_h, B_h, C_h, and D_h be square matrices of order $N+1$:

$$A_h = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad B_h = \begin{pmatrix} 0 & \cdots & 0 & \alpha_1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$C_h = \mathrm{diag}(0, \cdots, 0, \alpha_2), \quad D_h = \mathrm{diag}(0, \cdots, 0, 0).$$

Set

$$\Phi_h = \begin{pmatrix} \frac{1}{2} I_{N+1} & \frac{1}{2} A_h & 0 & 0 \\ \frac{1}{2} A_h^\top & \frac{1}{2} I_{N+1} & 0 & 0 \\ 0 & 0 & \frac{1}{2} I_{N+1} & \frac{1}{2} A_h \\ 0 & 0 & \frac{1}{2} A_h^\top & \frac{1}{2} I_{N+1} \end{pmatrix},$$

$$\Psi_h = \begin{pmatrix} 0 & \beta_2 h^{-1} A_h & 0 & 0 \\ \beta_1 h^{-1} A_h^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_4 h^{-1} A_h \\ 0 & 0 & \beta_3 h^{-1} A_h^\top & 0 \end{pmatrix},$$

$$\Omega_h = \begin{pmatrix} 0 & 0 & \frac{1}{2}\beta_1 A_h & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}\beta_4 A_h^\top & 0 & 0 & 0 \end{pmatrix}.$$

The state variable of (2.11) is $Y_h(t) = (y_{1h}(t), y_{2h}(t), y_{3h}(t), y_{4h}(t))^\top$ with

$$y_{1h}(t) = (y_{1,0}(t), \cdots, y_{1,N}(t))^\top, \quad y_{2h}(t) = (y_{2,1}(t), \cdots, y_{2,N+1}(t))^\top,$$

$$y_{3h}(t) = (y_{3,0}(t), \cdots, y_{3,N}(t))^\top, \quad y_{4h}(t) = (y_{4,1}(t), \cdots, y_{4,N+1}(t))^\top.$$

Then (2.11) is abstractly written as

$$\begin{cases} \dot{Y}_h(t) = \mathcal{A}_h Y_h(t), \\ Y_h(0) = (y_{1h}^0, y_{2h}^0, y_{3h}^0, y_{4h}^0)^\top, \end{cases}$$

where $\mathcal{A}_h = \Phi_h^{-1}(\Psi_h + \Omega_h)$ because Φ_h is evidently invertible.

The classical semi-discrete scheme is similar to (2.14) where the average operator $\Phi_h = \text{diag}\{I_{N+1}, I_{N+1}, I_{N+1}, I_{N+1}\}$, i.e.,

$$\begin{cases} \dot{Y}_h(t) = \mathcal{A}_h Y_h(t), \\ Y_h(0) = (y_{1h}^0, y_{2h}^0, y_{3h}^0, y_{4h}^0)^\top, \end{cases}$$

where $\mathcal{A}_h = \Psi_h + \Omega_h$.

Here we explain the significance of the discrete scheme (2.14). We plot Figures 1 and 2, respectively. Figure 1 [Figure 1: see original paper] depicts the maximal real parts of the eigenvalues of the classical semi-discrete scheme (2.15) and the semi-discrete scheme (2.14) with step size h. Figure 2 [Figure 2: see original paper] depicts the distributions of the eigenvalues of (2.14) and (2.15) in which $N = 100$. From Figures 1 and 2 we see that the real parts of the eigenvalues of (2.14) approach a negative number and those of (2.15) approach zero. In both figures, we take $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \alpha_1 = \alpha_2 = 1$.

Furthermore, the inner product in X_h can be rewritten as

$$\langle Y_h, Z_h \rangle_h = h(\Phi_h Z_h)^* H_h Y_h,$$

where $H_h = \text{diag}(\beta_1 I_{N+1}, \beta_2 I_{N+1}, \beta_3 I_{N+1}, \beta_4 I_{N+1})$. The energy $E_h(t)$ of (2.11) is

$$E_h(t) = \langle Y_h(t), Y_h(t) \rangle_h = h \sum_{l=1}^{4} \sum_{j=0}^{N} \beta_l |y_{l,j+\frac{1}{2}}|^2.$$

The following Lemma 2.4 is the discrete counterpart of Lemma 2.2.

Lemma 2.4. The operator $\mathcal{A}_h$ defined in (2.14) is dissipative on X_h for all $h \in (0,1)$. In other words, the solution $Y_h(t)$ to (2.11) satisfies

$$\dot{E}_h(t) = -\alpha_1\beta_1\beta_2|y_{2,N+1}|^2 - \alpha_2\beta_3\beta_4|y_{4,N+1}|^2 \le 0.$$

Proof. It suffices to consider only real solutions. Multiplying the first four equations of (2.11) by $h\beta_l y_{l,j+\frac{1}{2}}$ ($l = 1, 2, 3, 4$) and adding up for l and j, we obtain

$$\dot{E}_h(t) = h\beta_1\beta_2 \sum_{j=0}^{N} [y_{1,j+\frac{1}{2}}\delta_x y_{2,j+\frac{1}{2}} + y_{2,j+\frac{1}{2}}\delta_x y_{1,j+\frac{1}{2}}] + h\beta_3\beta_4 \sum_{j=0}^{N} [y_{3,j+\frac{1}{2}}\delta_x y_{4,j+\frac{1}{2}} + y_{4,j+\frac{1}{2}}\delta_x y_{3,j+\frac{1}{2}}].$$

A simple calculation shows that

$$\sum_{j=0}^{N} y_{1,j+\frac{1}{2}} \delta_x y_{2,j+\frac{1}{2}} = y_{1,N+1} y_{2,N+1} - y_{1,0} y_{2,0},$$

$$\sum_{j=0}^{N} y_{3,j+\frac{1}{2}} \delta_x y_{4,j+\frac{1}{2}} = y_{3,N+1} y_{4,N+1} - y_{3,0} y_{4,0}.$$

The above two identities, together with $y_{1,N+1} = \alpha_1 y_{2,N+1}$ and $y_{3,N+1} = \alpha_2 y_{4,N+1}$, give equality (2.17).

Next we show a mixed observability inequality involving the final state observability (see, e.g., [32, Definition 6.1.1]) and the exact observability, which is the discrete counterpart of [17, Lemma 9.1.2] or [33, Lemma III.1] and plays an important role in verifying the uniform exponential stability of (2.11).

Lemma 2.5. Set $m = 2^{-1} \max\{\beta_1, \beta_4\}$, $\eta = \frac{1-mh}{1+mh}$. Let M' and M be positive constants and assume that $0 < h < h_1$, $h_1 = \min\{2, \beta_1\beta_2^{-1}, \beta_4\beta_3^{-1}, m^{-1}\}$ and $M' I_4 \leq H \leq M I_4$. Let the output $O(t)$ for (2.11) be defined by

$$O(t) = Y_{N+1}(t), \quad Y_{N+1}(t) = (y_{1,N+1}(t), y_{2,N+1}(t), y_{3,N+1}(t), y_{4,N+1}(t))^\top.$$

Then for any solution $Y_h(t)$ to (2.11), there exist a positive constant τ such that

$$\int_0^\tau \|O(t)\|_{\mathbb{C}^4}^2 \, dt + C(h) E_h(0) \geq c(h) E_h(\tau),$$

where the positive functions $c(h)$ and $C(h)$ are given by

$$c(h) = \frac{4mM\tau\eta^{N+1}}{(1-\eta^{N+2})(1+mh)} - 1, \quad C(h) = \frac{1+mh}{4mM\tau\eta^{N+1}}.$$

Proof. Firstly, for $j = 0, 1, \cdots, N+1$ we set $Y_j(t) = (y_{1,j}(t), y_{2,j}(t), y_{3,j}(t), y_{4,j}(t))^\top$, and define the auxiliary functions $f_j(t)$ by

$$f_j(t) = h Y_j^\top(t) H_h Y_j(t).$$

A simple calculation gives

$$F_j(\tau) = \int_0^\tau f_j(t) \, dt,$$

which is written simply as F_j. A simple calculation gives

$$F_{j+1} - F_j = \int_0^\tau [f_{j+1}(t) - f_j(t)]\, dt = 2h \int_0^\tau \sum_{l=1}^{4} \beta_l \delta_x y_{l,j+\frac{1}{2}}(t) y_{l,j+\frac{1}{2}}(t)\, dt.$$

It follows from (2.11) that

$$\begin{cases} \beta_1 \delta_x y_{1,j+\frac{1}{2}}(t) = \dot{y}_{2,j+\frac{1}{2}}(t), \\ \beta_2 \delta_x y_{2,j+\frac{1}{2}}(t) = \dot{y}_{1,j+\frac{1}{2}}(t) + \beta_4 y_{4,j+\frac{1}{2}}(t), \\ \beta_3 \delta_x y_{3,j+\frac{1}{2}}(t) = \dot{y}_{4,j+\frac{1}{2}}(t), \\ \beta_4 \delta_x y_{4,j+\frac{1}{2}}(t) = \dot{y}_{3,j+\frac{1}{2}}(t) + \beta_1 y_{1,j+\frac{1}{2}}(t), \end{cases}$$

which means that

$$\sum_{l=1}^{4} \beta_l \delta_x y_{l,j+\frac{1}{2}}(t) y_{l,j+\frac{1}{2}}(t) = \frac{1}{2}\frac{d}{dt}[y_{1,j+\frac{1}{2}}(t) y_{2,j+\frac{1}{2}}(t) + y_{3,j+\frac{1}{2}}(t) y_{4,j+\frac{1}{2}}(t)] + \beta_1 y_{1,j+\frac{1}{2}}(t) y_{3,j+\frac{1}{2}}(t) + \beta_4 y_{4,j+\frac{1}{2}}(t) y_{2,j+\frac{1}{2}}($$

Integrating this identity from 0 to τ gives

$$\int_0^\tau \sum_{l=1}^{4} \beta_l \delta_x y_{l,j+\frac{1}{2}}(t) y_{l,j+\frac{1}{2}}(t)\, dt = \frac{1}{2}[y_{1,j+\frac{1}{2}}(t) y_{2,j+\frac{1}{2}}(t) + y_{3,j+\frac{1}{2}}(t) y_{4,j+\frac{1}{2}}(t)]_0^\tau + \int_0^\tau [\beta_1 y_{1,j+\frac{1}{2}}(t) y_{3,j+\frac{1}{2}}(t) + \beta_4 y_{4,j+\frac{1}{2}}($$

Combining (2.22) and (2.24), we arrive at

$$F_{j+1} - F_j = h[y_{1,j+\frac{1}{2}}(t) y_{2,j+\frac{1}{2}}(t) + y_{3,j+\frac{1}{2}}(t) y_{4,j+\frac{1}{2}}(t)]_0^\tau + 2h \int_0^\tau [\beta_1 y_{1,j+\frac{1}{2}}(t) y_{3,j+\frac{1}{2}}(t) + \beta_4 y_{4,j+\frac{1}{2}}(t) y_{2,j+\frac{1}{2}}(t)]\, dt.$$

Secondly, for the second term of the right-hand side of (2.25), we apply the Cauchy-Schwartz inequality to obtain

$$\int_0^\tau [\beta_1 y_{1,j+\frac{1}{2}}(t) y_{3,j+\frac{1}{2}}(t) + \beta_4 y_{4,j+\frac{1}{2}}(t) y_{2,j+\frac{1}{2}}(t)]\, dt \le \frac{m}{2}(F_{j+1} + F_j).$$

Substitution of this into (2.25), for $j = 0, 1, \cdots, N$, we obtain

$$F_{j+1} \ge \eta F_j + 2h\delta [y_{1,j+\frac{1}{2}}(t) y_{2,j+\frac{1}{2}}(t) + y_{3,j+\frac{1}{2}}(t) y_{4,j+\frac{1}{2}}(t)]_0^\tau,$$

where $\eta = \frac{1-mh}{1+mh}$ and $\delta = \frac{h}{1+mh}$. By deduction argument, we obtain

$$F_j(\tau) \geq \eta^{-(N+1-j)} F_{N+1}(\tau) - 2h\delta \sum_{k=j}^{N} \eta^{-(k+1-j)} [y_{1,k+\frac{1}{2}}(t) y_{2,k+\frac{1}{2}}(t) + y_{3,k+\frac{1}{2}}(t) y_{4,k+\frac{1}{2}}(t)]_0^\tau.$$

Finally, since from Lemma 2.4, $E_h(t_2) \leq E_h(t_1)$ for any $t_1 \leq t_2$, we deduce that $E_h(t) \geq E_h(\tau)$ for $t \in [0, \tau]$. This, invoking the definition of F_j and $E_h(t)$, and the estimate (2.27) leads to

$$\int_0^\tau E_h(t)\, dt = F_{N+1}(\tau) \geq \frac{2\tau E_h(\tau)}{\eta^{-(N+1)} - 1} - \frac{2h\delta}{\eta^{-(N+1)} - 1} \sum_{j=0}^{N} \eta^{-(j+1)} [y_{1,j+\frac{1}{2}}(t) y_{2,j+\frac{1}{2}}(t) + y_{3,j+\frac{1}{2}}(t) y_{4,j+\frac{1}{2}}(t)]_0^\tau.$$

Together with the following inequality

$$F_{N+1}(\tau) \leq M \int_0^\tau \|Y_{N+1}(t)\|_{\mathbb{C}^4}^2\, dt,$$

we obtain (2.19).

Remark 2.2. By the definition of η and $(N+1)h = 1$ in Lemma 2.5, it is easy to see that

$$\lim_{h \to 0} \eta^{N+2} = \lim_{h \to 0} \left(\frac{1 - mh}{1 + mh} \right)^{N+2} = e^{-2m}.$$

This means that $\frac{4mM\tau}{(1-\eta^{N+2})(1+mh)}$ is uniformly larger than η^{N+1} whenever we choose τ large enough. Moreover, since $\eta^{N+1} \leq 1$, we can choose τ to ensure that $c(h)$ and $C(h)$ defined by (2.20) satisfy $\sup_{h \in (0, h_2)} c(h) < 1$ for some $h_2 < 1$.

Now we are in a position to state the main result of this section.

Theorem 2.6. Let $h^* = \min\{h_1, h_2\}$ where h_1 is specified by Lemma 2.5, and let $T_h(t)$ be the semigroup generated by $\mathcal{A}_h$ defined by (2.14). Then $T_h(t)$ is uniformly exponentially stable with respect to $h \in (0, h^*)$, i.e., there are two constants K and ω independent of h such that

$$\|T_h(t) Y_h(0)\|_{X_h} \leq K e^{-\omega t} \|Y_h(0)\|_{X_h}.$$

Proof. By (2.17), $y_{1,N+1}(t) = \alpha_1 y_{2,N+1}(t)$ and $y_{3,N+1}(t) = \alpha_2 y_{4,N+1}(t)$, it follows that

$$E_h(0) = E_h(\tau) - \int_0^\tau \dot{E}_h(t)\,dt = E_h(\tau) + \alpha_1\beta_1\beta_2 \int_0^\tau |y_{2,N+1}(t)|^2\,dt + \alpha_2\beta_3\beta_4 \int_0^\tau |y_{4,N+1}(t)|^2\,dt \ge E_h(\tau) + k\int_0^\tau \|Y$$

where $k = \min\{\alpha_1\beta_1, 2\beta_1\alpha_1, \alpha_2\beta_3, 2\beta_3\alpha_2\}$. Plugging (2.19) into (2.29), we obtain

$$E_h(\tau) \le \frac{1 + kC(h)}{1 + kc(h)} E_h(0).$$

Thus $\sup_{h\in(0,h^*)} \frac{1+kC(h)}{1+kc(h)} < 1$ for some τ by virtue of Remark 2.2. By C_0-semigroup property, this implies that the energy $E_h(t)$ of system (2.11) decays uniformly exponentially to zero as $t \to \infty$. In terms of the X_h norm and energy $E_h(t)$, we have finally obtained the relationship (2.28).

Remark 2.3. Similar to Remark 2.1, the dual system

$$\begin{cases} \dot{y}_{1,j+\frac{1}{2}}(t) = \beta_2\delta_x y_{2,j+\frac{1}{2}}(t) + \beta_4 y_{4,j+\frac{1}{2}}(t), \\ \dot{y}_{2,j+\frac{1}{2}}(t) = \beta_1\delta_x y_{1,j+\frac{1}{2}}(t), \\ \dot{y}_{3,j+\frac{1}{2}}(t) = \beta_4\delta_x y_{4,j+\frac{1}{2}}(t) + \beta_1 y_{1,j+\frac{1}{2}}(t), \\ \dot{y}_{4,j+\frac{1}{2}}(t) = \beta_3\delta_x y_{3,j+\frac{1}{2}}(t), \quad j = 0, 1, \cdots, N, \\ y_{1,N+1}(t) = \alpha_1 y_{2,N+1}(t), \quad y_{2,0}(t) = 0, \\ y_{3,N+1}(t) = \alpha_2 y_{4,N+1}(t), \quad y_{4,0}(t) = 0, \\ Y_h(0) = (y_{1h}^0, y_{2h}^0, y_{3h}^0, y_{4h}^0)^\top, \end{cases}$$

is also uniformly exponentially stable.

3. Convergence of solutions of (2.11) to (2.2)

The objective of this subsection is to show that the C_0-semigroups $T_h(t)$ generated by $\mathcal{A}_h$ defined by (2.14) approximate $T(t)$ generated by A defined by (2.5) in some sense.

Theorem 3.1. (Trotter-Kato [16]) For every $h \in (0, 1)$, assume that there exist bounded linear operators $P_h : X \to X_h$ and $F_h : X_h \to X$ satisfying:

(A1) There exist two positive constants M_1 and M_2 such that $\|P_h\| \le M_1$ and $\|F_h\| \le M_2$ for all $h \in (0, 1)$;

(A2) $\|F_h P_h Y - Y\|_X \to 0$ as $h \to 0$ for all $Y \in X$;

(A3) $F_h P_h = I_h$.

If (A1) and (A3) are fulfilled, then the following two statements are equivalent:

(a) There exists a $\lambda_0 \in \bigcap_{n=1}^{\infty} \rho(A_h)$ such that for all $Y \in X$, $\|F_h(\lambda_0 I_h - A_h)^{-1} P_h Y - (\lambda_0 I - A)^{-1} Y\|_X \to 0$ as $h \to 0$;

(b) For every $Y \in X$ and $t \geq 0$,

$$\|F_h T_h(t) P_h Y - T(t)Y\|_X \to 0 \quad \text{as } h \to 0$$

uniformly on bounded time t-intervals.

In order to apply Theorem 3.1, one always replaces property (a) by a condition involving convergence of the operators A_h to A in some sense [16].

Proposition 3.2. Let the assumptions of Theorem 3.1 be fulfilled. Then statement (a) of Theorem 3.1 is equivalent to assumption (A2) of Theorem 3.1 with additionally the following two conditions:

(C1) There exists a subset $D \subset D(A)$ such that $\overline{D} = X$ and $(\lambda_0 I - A)^{-1} D = D(A)$ for a $\lambda_0 > \omega$;

(C2) For all $Y \in D$, there exists a sequence $Y_h \in D(A_h)$ such that $F_h Y_h = Y$ and $\lim_{h \to 0} \|F_h A_h Y_h - AY\|_X = 0$.

To apply Theorem 3.1 and Proposition 3.2 to the convergence analysis in this section, we firstly introduce the operators $F_h : X_h \to X$ and $P_h : X \to X_h$. Let χ_S be the characteristic function of a set S. Then F_h is defined by

$$F_h Y_h = \begin{pmatrix} \sum_{i=0}^{N} y_{1,i+\frac{1}{2}} \chi_{(x_i, x_{i+1}]} \\ \sum_{i=0}^{N} y_{2,i+\frac{1}{2}} \chi_{(x_i, x_{i+1}]} \\ \sum_{i=0}^{N} y_{3,i+\frac{1}{2}} \chi_{(x_i, x_{i+1}]} \\ \sum_{i=0}^{N} y_{4,i+\frac{1}{2}} \chi_{(x_i, x_{i+1}]} \end{pmatrix},$$

where $y_{2,0} = y_{4,0} = 0$, $y_{1,N+1} = \alpha_1 y_{2,N+1}$, $y_{3,N+1} = \alpha_2 y_{4,N+1}$ were assigned to unify the notation of $y_{l,j+\frac{1}{2}}$. On the other hand, P_h is defined as, for $Z = (z_1(x), z_2(x), z_3(x), z_4(x))^\top$,

$$P_h Z = \begin{pmatrix} (I_1 z_1(x), I_3 z_3(x))^\top \\ (I_2 z_2(x), I_4 z_4(x))^\top \end{pmatrix},$$

where

$$I_l z_l(x) = 2h^{-1} \begin{pmatrix} \int_{x_0}^{x_1} z_l(x)\, dx \\ \vdots \\ \int_{x_N}^{x_{N+1}} z_l(x)\, dx \end{pmatrix}, \quad l = 1, 3,$$

$$I_k z_k(x) = 2h^{-1} \begin{pmatrix} \int_{x_0}^{x_1} z_k(x)\,dx \\ \vdots \\ \int_{x_N}^{x_{N+1}} z_k(x)\,dx \end{pmatrix}, \quad k = 2, 4,$$

with

$$\int_{x_j}^{x_{j+1}} z_1(x)\,dx + \alpha_1 I_2^{N+1} z_2(x), \quad I_1^{N+1} z_1(x) = \int_{x_N}^{x_{N+1}} z_1(x)\,dx + \alpha_1 I_2^{N+1} z_2(x),$$

$$I_2^i z_2(x) = \int_{x_{i-1}}^{x_i} z_2(x)\,dx, \quad i = 2, 3, \cdots, N+1,$$

and similar definitions for I_3 and I_4.

Lemma 3.3. The operators F_h defined by (3.3) and P_h defined by (3.4) satisfy assumptions (A1)-(A3) in Theorem 3.1 with $M_1 = M_2 = 1$.

Proof. By the definitions of the inner products of X and X_h, it is easy to see that

$$\|F_h Y_h\|_X^2 = h \sum_{l=1}^{4} \sum_{j=0}^{N} \beta_l |y_{l,j+\frac{1}{2}}|^2 = \|Y_h\|_{X_h}^2,$$

which implies that $\|F_h\| = 1$. Moreover, by setting $I_2^0 = 0$, $I_4^0 = 0$, $I_1^{N+1} = \alpha_1 I_2^{N+1}$ and $I_3^{N+1} = \alpha_2 I_4^{N+1}$, we have

$$\|P_h Z\|_{X_h}^2 = h \sum_{l=1}^{4} \sum_{j=0}^{N} \beta_l \left| \frac{1}{h} \int_{x_j}^{x_{j+1}} z_l(x)\,dx \right|^2 \le \int_0^1 \sum_{l=1}^{4} \beta_l |z_l(x)|^2\,dx = \|Z\|_X^2,$$

and hence assumption (A1) holds. To prove assumption (A2), set $D = [C^1(0,1)]^4$. It suffices to show that $F_h P_h Y \to Y$ as $h \to 0$ for all $Y \in D$ since F_h and P_h are bounded linear operators and D is dense in X. In fact, for $Z \in D$,

$$F_h[P_h Z] = \begin{pmatrix} \sum_{i=0}^{N} \left(\frac{1}{h} \int_{x_i}^{x_{i+1}} z_1(x)\,dx \right) \chi_{(x_i, x_{i+1}]} \\ \vdots \\ \sum_{i=0}^{N} \left(\frac{1}{h} \int_{x_i}^{x_{i+1}} z_4(x)\,dx \right) \chi_{(x_i, x_{i+1}]} \end{pmatrix} \to Z \quad \text{in } X \text{ as } h \to 0.$$

By the same idea of [16, Section 4, p.34], we obtain that (A2) holds. Finally, using $y_{2,0} = y_{4,0} = 0$, we have

$$\int_{x_{i-1}}^{x_i} \sum_{j=0}^{N} y_{k,j+\frac{1}{2}} \chi_{(x_j,x_{j+1}]}\, dx = h y_{k,i+\frac{1}{2}}, \quad k = 2,4, \quad i = 1,2,\cdots,N+1.$$

Substituting this into $I_l\left(\sum_{i=0}^{N} y_{l,i+\frac{1}{2}} \chi_{(x_i,x_{i+1}]}\right)$ for $l = 1,3$, we obtain

$$\int_{x_i}^{x_{i+1}} \sum_{j=0}^{N} y_{l,j+\frac{1}{2}} \chi_{(x_j,x_{j+1}]}\, dx = h y_{l,i+\frac{1}{2}}, \quad l = 1,3, \quad i = 0,1,\cdots,N,$$

where $y_{1,N+1} = \alpha_1 y_{2,N+1}$ and $y_{3,N+1} = \alpha_2 y_{4,N+1}$ were used. These identities imply (A3).

Theorem 3.4. Let $T(t)$ and $T_h(t)$ be the C_0-semigroups generated by the operators A and $\mathcal{A}_h$ which are defined by (2.5) and (2.14), respectively. Let $Y(0)$ be the initial value of (2.2) and set $Y_h^0 := P_h Y(0)$ to be the initial data of (2.11). Then

$$\|F_h Y_h^0 - Y(0)\|_X \to 0 \quad \text{and} \quad \|F_h T_h(t) Y_h^0 - T(t) Y(0)\|_X \to 0 \quad \text{as } h \to 0$$

uniformly on bounded time t intervals.

Proof. The condition (C1) of Proposition 3.2 is obviously valid whenever we choose $D = [C^1(0,1)]^4$ and $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}\lambda_0 > 0$. It is therefore sufficient to prove (C2) of Proposition 3.2. Indeed, for $Y = (y_1(x), y_2(x), y_3(x), y_4(x))^\top \in D(A)$, we introduce $Y_h \in D(\mathcal{A}_h)$ through

$$y_{1h} = (y_1(x_0), y_1(x_1), \cdots, y_1(x_N))^\top, \quad y_{2h} = (y_2(x_1), y_2(x_2), \cdots, y_2(x_{N+1}))^\top,$$

$$y_{3h} = (y_3(x_0), y_3(x_1), \cdots, y_3(x_N))^\top, \quad y_{4h} = (y_4(x_1), y_4(x_2), \cdots, y_4(x_{N+1}))^\top,$$

and $Y_h = (y_{1h}^\top, y_{2h}^\top, y_{3h}^\top, y_{4h}^\top)^\top$. We thus have $F_h Y_h \to Y$ as $h \to 0$. Set $Z_h = \mathcal{A}_h Y_h$ and we obtain $\Phi_h Z_h = (\Psi_h + \Omega_h) Y_h$. By the expressions of Φ_h, Ψ_h, and Ω_h, we have

$$\begin{cases} z_{1,j+\frac{1}{2}} = \beta_2 \frac{y_2(x_{j+1}) - y_2(x_j)}{h}, \\ z_{2,j+\frac{1}{2}} = \beta_1 \frac{y_1(x_{j+1}) - y_1(x_j)}{h} + \beta_4 \frac{y_4(x_{j+1}) + y_4(x_j)}{2}, \\ z_{3,j+\frac{1}{2}} = \beta_4 \frac{y_4(x_{j+1}) - y_4(x_j)}{h}, \\ z_{4,j+\frac{1}{2}} = \beta_3 \frac{y_3(x_{j+1}) - y_3(x_j)}{h} + \beta_1 \frac{y_1(x_{j+1}) + y_1(x_j)}{2}, \quad j = 0,1,\cdots,N, \end{cases}$$

where we set artificially that $z_{2,0} = z_{4,0} = 0$, $z_{1,N+1} = \alpha_1 z_{2,N+1}$ and $z_{3,N+1} = \alpha_2 z_{4,N+1}$ to unify the notations $z_{l,j+\frac{1}{2}}$ for $j = 0, 1, \cdots, N$ and $l = 1, 2, 3, 4$. Plugging (3.8) into the operator F_h and using the same techniques as above, it is easy to show that $\|F_h \mathcal{A}_h Y_h - AY\|_X \to 0$ as $h \to 0$. For the consideration of the page limit, we omit the details.

Remark 3.1. For the C_0-semigroups $T_h^*(t)$ and $T^*(t)$ given in Remarks 2.1 and 2.3 we have the convergence

$$\|F_h T_h^*(t) Y_h^0 - T^*(t) Y(0)\|_X \to 0 \quad \text{as } h \to 0$$

uniformly on bounded time t-intervals, where $Y(0) \in X$ is the initial value of (2.9) and $Y_h^0 = P_h Y(0)$ is the initial data of (2.30).

4. Convergence of the controls

In this section, we give exact controllability of (1.1), in particular the analytic form of the control that drives the initial state to rest. Same as (2.1), we introduce the transforms

$$q_1(x,t) = w_x(x,t) - \phi(x,t), \quad q_2(x,t) = \rho w_t(x,t), \quad q_3(x,t) = \phi_x(x,t), \quad q_4(x,t) = I_\rho \phi_t(x,t),$$

and obtain the equivalent form of (1.1)

$$\begin{cases} \dot{q}_1(x,t) = \beta_2 q_2'(x,t) - \beta_4 q_4(x,t), \\ \dot{q}_2(x,t) = \beta_1 q_1'(x,t), \\ \dot{q}_3(x,t) = \beta_4 q_4'(x,t), \\ \dot{q}_4(x,t) = \beta_3 q_3'(x,t) + \beta_1 q_1(x,t), \\ q_2(0,t) = 0, \quad q_4(0,t) = 0, \\ q_1(1,t) = u_1(t), \quad q_3(1,t) = u_2(t), \\ q_l(x,0) = q_l^0(x), \quad l = 1, 2, 3, 4, \end{cases}$$

where β_l $(l = 1, 2, 3, 4)$ are explained after (2.2) and $q_1^0(x) = w_0'(x) - \phi_0(x)$, $q_2^0(x) = w_1(x)$, $q_3^0(x) = \phi_0'(x)$ and $q_4^0(x) = \phi_1(x)$. The aim of this section is twofold. The first one is to design controls $u_1(t)$ and $u_2(t)$ which drive the solution of (4.1) from the initial state to rest at finite time τ. The second one is to construct a family of discrete controls which converge to the continuous one. We split this section into two subsections.

4.1. Exact controllability of (4.1)

The state space of (4.1) is X also. Set α_1 and α_2 to be zero in W_B defined by (2.6):

$$W_B^0 = \begin{pmatrix} \beta_2\beta_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & \beta_4\beta_3^{-1} & 0 \\ \beta_2\beta_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & \beta_4\beta_3^{-1} & 0 \end{pmatrix}.$$

The corresponding operator A defined by (2.5) is denoted by A_0 in this case:

$$\begin{cases} A_0 Y(x) = P_1(Y(x))' + P_0 HY(x), \\ D(A_0) = \{Y \in H^1((0,1);\mathbb{C}^4),\ W_B^0 HY(1) = 0\}. \end{cases}$$

It follows from Lemma 2.2 that the operator A_0 is skew-adjoint. Now we consider the dual system of (4.1), which corresponds to (A_0, B^*), i.e.,

$$\begin{cases} \dot{P}(x,t) = A_0 P(x,t), \\ P(x,0) = P^0(x) = (p_1^0(x), p_2^0(x), p_3^0(x), p_4^0(x))^\top, \\ O_1(t) = B^* P(x,t) = (\beta_1\beta_2 p_2(1,t), \beta_3\beta_4 p_4(1,t))^\top, \end{cases}$$

where $O_1(t)$ is the output of adjoint system of (4.1) without controls, and $B^* \in L(X_1, \mathbb{C}^2)$ is the observation operator. The boundary control system (4.1) can be reformulated as follows

$$\begin{cases} \dot{Q}(x,t) = A_0 Q(x,t) + Bu(t), \\ Q(x,0) = Q^0(x) = (q_1(x), q_2(x), q_3(x), q_4(x))^\top, \end{cases}$$

where $Q(x,t) = (q_1(x,t), q_2(x,t), q_3(x,t), q_4(x,t))^\top$ is the state of system, $u(t) = (u_1(t), u_2(t))^\top$ is the control, and $B \in L(\mathbb{C}^2, X_{-1})$ is the input operator. To apply Theorem 2 of [11] to obtain the exact controllability of (4.1) or (4.6), we first present an admissibility result.

Theorem 4.1. Let $T_0(t)$ be the unitary C_0-group generated by A_0 defined by (4.3). Then the control operator B defined in (4.6) is admissible for $T_0(t)$.

Proof. It suffices to consider the real part of the solution. In light of duality, we only need to show that the observation operator B^* is admissible for the semi-group $T_0^*(t)$ since (A_0, B) and (A_0, B^*) are a pair of dual systems. Now, for every $P^0 \in D(A_0)$ and the state trajectory $P(x,t) = T_0^*(t)P^0(x)$ of (4.4), we introduce the auxiliary function

$$\mathcal{E}(t) = \int_0^1 [x p_1(x,t) p_2(x,t) + x p_3(x,t) p_4(x,t)]\, dx.$$

Differentiating $\mathcal{E}(t)$ gives

$$\dot{\mathcal{E}}(t) = \int_0^1 [x\dot{p}_1(x,t)p_2(x,t)+xp_1(x,t)\dot{p}_2(x,t)+x\dot{p}_3(x,t)p_4(x,t)+xp_3(x,t)\dot{p}_4(x,t)]\,dx$$

$$= \int_0^1 \sum_{l=1}^{4} x\beta_l p_l'(x,t)p_l(x,t)\,dx - \int_0^1 [\beta_4 xp_2(x,t)p_4(x,t) + \beta_1 xp_1(x,t)p_3(x,t)]\,dx.$$

Performing integration by parts gives

$$\int_0^1 \sum_{l=1}^{4} x\beta_l p_l'(x,t)p_l(x,t)\,dx = \frac{1}{2}[\beta_1|p_1(1,t)|^2+\beta_2|p_2(1,t)|^2+\beta_3|p_3(1,t)|^2+\beta_4|p_4(1,t)|^2]-\frac{1}{2}E_P(t),$$

in which

$$E_P(t) = \int_0^1 \sum_{l=1}^{4} \beta_l |p_l(x,t)|^2\,dx$$

is the energy of (4.4) and $m_1 = \max\{\beta_1, \beta_4\}$. Combining equations (4.7)-(4.9), we derive

$$\dot{\mathcal{E}}(t) \geq \frac{1}{2}[\beta_2|p_2(1,t)|^2 + \beta_4|p_4(1,t)|^2] - (1+m_1)E_P(t).$$

Similarly,

$$|\mathcal{E}(t)| \leq m_2 E_P(t), \quad \text{with } m_2 = \max\{\beta_1^{-1}, \beta_3^{-1}\}.$$

Integrating (4.11) from 0 to τ and using (4.12) yields

$$\int_0^\tau [\beta_2|p_2(1,t)|^2+\beta_4|p_4(1,t)|^2]\,dt \leq 2(1+m_1)\int_0^\tau E_P(t)\,dt+2m_2[E_P(\tau)+E_P(0)].$$

Finally, using the expression of $O_1(t)$ and the conservation of the energy $E_P(t)$, we obtain

$$\int_0^\tau \|O_1(t)\|_{\mathbb{C}^2}^2\,dt \leq k_\tau E_P(0)$$

with $k_\tau = 2(\tau(1+m_1)+2m_2)\min\{\beta_1^{-2},\beta_4^{-2}\}$. This means that B^* is admissible for $T_0^*(t)$.

In light of the infinite dimensional version of Russell's "controllability via stability" principle, we get the control which steers the initial state of (4.6) or (4.1) to rest in some time τ.

Theorem 4.2. Let time τ be chosen so that $\|L_\tau\| < 1$. A control $u \in C([0,\tau];\mathbb{C}^2)$ for (4.6) or (4.1) driving $Q^0 \in X$ to rest in time τ is given by

$$u(t) = B^*[W_f(\cdot,t) + W_b(\cdot,t)],$$

where $W_f = (w_1, w_2, w_3, w_4)^\top$ and $W_b = (w_{b,1}, w_{b,2}, w_{b,3}, w_{b,4})^\top$ are the solutions of a forward system and a backward system

$$\begin{cases} \dot{w}_1(x,t) = \beta_2 w_2'(x,t) - \beta_4 w_4(x,t), \\ \dot{w}_2(x,t) = \beta_1 w_1'(x,t), \\ \dot{w}_3(x,t) = \beta_4 w_4'(x,t), \\ \dot{w}_4(x,t) = \beta_3 w_3'(x,t) + \beta_1 w_1(x,t), \\ w_2(0,t) = 0, \quad w_4(0,t) = 0, \\ w_1(1,t) = \beta_1\beta_2 w_2(1,t), \quad w_3(1,t) = \beta_3\beta_4 w_4(1,t), \\ w_l(x,0) = w_l^0(x), \quad l = 1,2,3,4, \end{cases}$$

and

$$\begin{cases} \dot{w}_{b,1}(x,t) = \beta_2 w_{b,2}'(x,t) - \beta_4 w_{b,4}(x,t), \\ \dot{w}_{b,2}(x,t) = \beta_1 w_{b,1}'(x,t), \\ \dot{w}_{b,3}(x,t) = \beta_4 w_{b,4}'(x,t), \\ \dot{w}_{b,4}(x,t) = \beta_3 w_{b,3}'(x,t) + \beta_1 w_{b,1}(x,t), \\ w_{b,2}(0,t) = 0, \quad w_{b,4}(0,t) = 0, \\ w_{b,1}(1,t) = \beta_1\beta_2 w_{b,2}(1,t), \quad w_{b,3}(1,t) = \beta_3\beta_4 w_{b,4}(1,t), \\ w_{b,l}(x,\tau) = w_l(x,\tau), \quad l = 1,2,3,4, \end{cases}$$

respectively, with $W_f(x,0) = W_f^0(x) = (I - L_\tau)^{-1} Q^0(x)$ and $L_\tau = T^*(\tau)T(\tau)$.

Proof. The proof was actually given in Proposition 2.2 of [5] and Theorem 2 of [11] which relies heavily on the admissibility result Theorem 4.1 and the exponential stability of the semigroup $T(t)$. Here we just give a sketch of the proof for completeness.

Firstly, we notice that the operator $A_0 - BB^*$ corresponds to the operator A with $\alpha_1 = \beta_1\beta_2$ and $\alpha_2 = \beta_3\beta_4$. Furthermore, Theorem 2.3 tells us that $T(t)$ is exponentially stable. Thus system (4.16) can be written as an abstract evolution equation

$$\dot{W}_f(\cdot,t) = (A_0 - BB^*)W_f(\cdot,t), \quad W_f(x,0) = W_f^0(x).$$

Similarly, (4.17) can be written as an abstract evolutionary equation

$$\dot{W}_b(\cdot,t) = (A_0 + BB^*)W_b(\cdot,t), \quad W_b(x,\tau) = W_f(x,\tau).$$

Let $Q(x,t) = W_f(x,t) - W_b(x,t)$ for $t \in [0,\tau]$. Clearly, $Q(x,t)$ and $u(t)$ satisfy (4.1).

Secondly, let $W_f(x,t) = S(t)W_f^0(x)$ with $S(t)$ being the C_0-semigroup corresponding to (4.17) and $W_b^0(x) = W_b(x,0)$ the initial value. It then follows from (4.16)-(4.17) and $W_b(x,\tau) = W_f(x,\tau)$ that $W_b(x,t) = S(\tau - t)W_b(x,\tau) = S(\tau - t)S(\tau)W_f^0(x)$. The proof will be accomplished if we can show that $S(t) = T^*(t)$. Indeed, on the one hand, it has

$$\dot{W}_b(x,t) = A_X S(t) W_f^0(x)$$

with the infinitesimal generator A_X of the semigroup $S(t)$. On the other hand, from (4.17), we have

$$\dot{W}_b(x,t) = (A_0+BB^*)S(\tau-t)W_b(x,\tau) = (A_0+BB^*)S(t)W_b^0(x) = (A_0-BB^*)^*S(t)W_b^0(x).$$

Identities (4.18) and (4.19) imply that $A_X = (A_0 - BB^*)^*$ and hence $S(t) = T^*(t)$. Since $T(t)$ is exponentially stable and so is $T^*(t)$ (Remark 2.1), there exists τ such that $\|L_\tau\| < 1$ and therefore the operator $I - L_\tau$ is invertible.

Finally, since $W_b(x,\tau) = W_f(x,\tau)$, $Q(x,\tau) = 0$ is trivially valid.

4.2. Uniform controllability and convergence of discrete controls

We begin with (4.1) and discretize it using the same discrete scheme as that of Subsection 2.2. We therefore obtain a family of finite-dimensional control systems:

$$\begin{cases} \dot{q}_{1,j+\frac{1}{2}}(t) = \beta_2 \delta_x q_{2,j+\frac{1}{2}}(t) - \beta_4 q_{4,j+\frac{1}{2}}(t), \\ \dot{q}_{2,j+\frac{1}{2}}(t) = \beta_1 \delta_x q_{1,j+\frac{1}{2}}(t), \\ \dot{q}_{3,j+\frac{1}{2}}(t) = \beta_4 \delta_x q_{4,j+\frac{1}{2}}(t), \\ \dot{q}_{4,j+\frac{1}{2}}(t) = \beta_3 \delta_x q_{3,j+\frac{1}{2}}(t) + \beta_1 q_{1,j+\frac{1}{2}}(t), \quad j = 0,1,\cdots,N, \\ q_{2,0}(t) = 0, \quad q_{4,0}(t) = 0, \\ q_{1,N+1}(t) = u_{1h}(t), \quad q_{3,N+1}(t) = u_{2h}(t), \\ q_l(ih,0) = q_l^0(ih), \quad l = 1,2,3,4, \quad i = 0,1,\cdots,N+1. \end{cases}$$

The energy of system (4.20) is defined by

$$E_{C,h}(t) = h\sum_{l=1}^{4}\sum_{j=0}^{N}\beta_l|q_{l,j+\frac{1}{2}}|^2.$$

It follows from Lemma 2.4 that $E_{C,h}(t)$ satisfies

$$\dot{E}_{C,h}(t) = \beta_1\beta_2 u_{1h}(t)q_{2,N+1}(t) + \beta_3\beta_4 u_{2h}(t)q_{4,N+1}(t).$$

This motivates us to introduce the output for (4.20) as follows:

$$O_h(t) = B_h^* Q_h(t) := (\beta_1\beta_2 q_{2,N+1}(t), \beta_3\beta_4 q_{4,N+1}(t))^\top,$$

where the unknown quantities are $Q_h(t) = (q_{1h}(t), q_{2h}(t), q_{3h}(t), q_{4h}(t))^\top$ with

$$q_{1h}(t) = (q_{1,0}(t), \cdots, q_{1,N}(t))^\top, \quad q_{2h}(t) = (q_{2,1}(t), \cdots, q_{2,N+1}(t))^\top,$$

$$q_{3h}(t) = (q_{3,0}(t), \cdots, q_{3,N}(t))^\top, \quad q_{4h}(t) = (q_{4,1}(t), \cdots, q_{4,N+1}(t))^\top.$$

This makes system (4.20) be passive, i.e., $\dot{E}_{C,h}(t) = O_h^\top(t)u_h(t)$ and $u_h(t) = (u_{1h}(t), u_{2h}(t))^\top$. Theorem 2.6 tells us that the resulting closed-loop system under the negative proportional feedback $u_h(t) = -O_h(t)$ is uniformly exponentially stable. Using these basic facts, we construct controls $u_h(t)$ which drive the solutions of (4.1) from the initial data to rest in some time τ. We still apply Russell's "controllability via stability" principle.

Now, consider the forward system

$$\begin{cases}\dot{w}_{1,j+\frac{1}{2}}(t) = \beta_2\delta_x w_{2,j+\frac{1}{2}}(t) - \beta_4 w_{4,j+\frac{1}{2}}(t) + \beta_1 w_{1,j+\frac{1}{2}}(t),\\ \dot{w}_{2,j+\frac{1}{2}}(t) = \beta_1\delta_x w_{1,j+\frac{1}{2}}(t),\\ \dot{w}_{3,j+\frac{1}{2}}(t) = \beta_4\delta_x w_{4,j+\frac{1}{2}}(t),\\ \dot{w}_{4,j+\frac{1}{2}}(t) = \beta_3\delta_x w_{3,j+\frac{1}{2}}(t) + \beta_1 w_{1,j+\frac{1}{2}}(t), \quad j = 0, 1, \cdots, N,\\ w_{1,N+1}(t) = \alpha_1 w_{2,N+1}(t), \quad w_{2,0}(t) = 0,\\ w_{3,N+1}(t) = \alpha_2 w_{4,N+1}(t), \quad w_{4,0}(t) = 0,\\ w_{lh}(0) = (w_{l,1}, \cdots, w_{l,N+1})^\top \in \mathbb{C}^{N+1}, \quad l = 2, 4,\\ w_{kh}(0) = (w_{l,0}, \cdots, w_{l,N})^\top \in \mathbb{C}^{N+1}, \quad k = 1, 3,\end{cases}$$

and the backward system

$$\begin{cases}\dot{w}_{b,1,j+\frac{1}{2}}(t) = \beta_2\delta_x w_{b,2,j+\frac{1}{2}}(t) - \beta_4 w_{b,4,j+\frac{1}{2}}(t) + \beta_1 w_{b,1,j+\frac{1}{2}}(t),\\ \dot{w}_{b,2,j+\frac{1}{2}}(t) = \beta_1\delta_x w_{b,1,j+\frac{1}{2}}(t),\\ \dot{w}_{b,3,j+\frac{1}{2}}(t) = \beta_4\delta_x w_{b,4,j+\frac{1}{2}}(t),\\ \dot{w}_{b,4,j+\frac{1}{2}}(t) = \beta_3\delta_x w_{b,3,j+\frac{1}{2}}(t) + \beta_1 w_{b,1,j+\frac{1}{2}}(t), \quad j = 0, 1, \cdots, N,\\ w_{b,1,N+1}(t) = \alpha_1 w_{b,2,N+1}(t), \quad w_{b,2,0}(t) = 0,\\ w_{b,3,N+1}(t) = \alpha_2 w_{b,4,N+1}(t), \quad w_{b,4,0}(t) = 0,\\ w_{b,lh}(\tau) = w_{lh}(\tau), \quad l = 1, 2, 3, 4,\end{cases}$$

where $\alpha_1 = \beta_1\beta_2$ and $\alpha_2 = \beta_3\beta_4$. The unknowns W_{fh} and W_{bh} of (4.23) and (4.24) are defined similarly as Q_h, respectively. Hence the desired control $u_h(t)$ is given by

$$u_h(t) = B_h^*[W_{fh}(t) + W_{bh}(t)],$$

provided that we choose the initial value of $W_{fh}(0)$ as $W_{fh}(0) = (I_h - L_{\tau,h})^{-1}Q_h(0)$ for $h \in [0, h^*]$. Here $T_h(t)$ is the C_0-semigroup defined in Theorem 2.6. For simplicity, we use the notations $L_{\tau,h} := T_h^*(\tau)T_h(\tau)$ and $L_\tau := T^*(\tau)T(\tau)$.

Now we present the main result of this section, i.e., the discrete controls $u_h(t)$ given by (4.25) are convergent to $u(t)$ constructed by (4.15) if some convergence conditions are satisfied for the initial data.

Theorem 4.3. Choose τ so that $\|L_{\tau,h}\|_{X_h} < 1/2$ and $\|L_\tau\|_X < 1/2$ are satisfied for all $h \in [0, h^*]$. Let $Q^0 \in X$ be an initial value of (4.6) and set $Q_h^0 := P_h Q^0$ to be the initial data of (4.20). The following results hold true:

(i) W_{bh}^0 given by $W_{bh}^0 = L_{\tau,h}Q_h^0$ are convergent to $W_b^0 = L_\tau Q^0$ in the sense of $\|F_h W_{bh}^0 - W_b^0(0)\|_X \to 0$ as $h \to 0$.

(ii) W_{fh}^0 given by $W_{fh}^0 = (I_h - L_{\tau,h})^{-1}Q_h(0)$ are convergent to $W_f^0 = (I - L_\tau)^{-1}Q^0$ in the sense of $\|F_h W_{fh}^0 - W_f^0(0)\|_X \to 0$ as $h \to 0$.

(iii) The discrete controls $u_h(t)$ given by (4.25) corresponding to W_{fh}^0 are convergent to $u(t)$ constructed by (4.15) related to W_f^0 in $L^2([0,\tau]; \mathbb{C}^4)$.

Proof. By Theorem 3.4 and Remark 3.1, for arbitrary $Q \in X$, as long as $\|F_h P_h Q - Q\|_X \to 0$, there hold

$$\|F_h T_h(\tau)P_h Q - T(\tau)Q\|_X \to 0, \quad \|F_h T_h^*(\tau)P_h Q - T^*(\tau)Q\|_X \to 0.$$

Let $\mu = \sup_{h\in[0,h^*]} \|T_h(\tau)\|_{X_h} < \infty$. Then $\mu < 1$ because $\|T_h(\tau)\|_{X_h} < 1$ and $\|T(\tau)\|_X < 1$.

(i) Since $P_h F_h = I_h$, by the continuity of $T_h^*(\tau)$ and $T^*(\tau)$,

$$F_h W_{bh}^0 - W_b^0 = F_h T_h^*(\tau) T_h(\tau) Q_h^0 - T^*(\tau) T(\tau) Q^0 = F_h T_h^*(\tau) P_h [F_h T_h(\tau) P_h Q^0] - T^*(\tau) [F_h T_h(\tau) P_h Q^0] + T^*(\tau) [F_h T$$

By (4.26), both the sums of the first two terms and the last two terms are convergent to zero as $h \to 0$. This proves (i).

(ii) Firstly, for positive integer n, it is easy to obtain that

$$F_h L_{\tau,h}^n Q_h^0 - L_\tau^n Q^0 = F_h L_{\tau,h}^{n-1} P_h [F_h L_{\tau,h} P_h Q^0] - L_\tau^{n-1} [F_h L_{\tau,h} P_h Q^0] + L_\tau^{n-1} [F_h L_{\tau,h} P_h Q^0] - L_\tau^n Q^0.$$

By induction, $\mu < 1$ and the identity (4.28), we know that

$$\|F_h L_{\tau,h}^n Q_h^0 - L_\tau^n Q^0\|_X \to 0 \quad \text{as } h \to 0.$$

Secondly, we prove that the convergence of (4.29) is uniform for $n \in \mathbb{N}$. Indeed, by assertion (i) just proved, for any $\varepsilon > 0$, there exists $h' \in (0, h^*)$ such that

$$\|F_h L_{\tau,h} P_h Q^0 - L_\tau Q^0\|_X < \varepsilon(1-\mu) \quad \text{for all } h \in (0, h').$$

Thus, by Theorem 4.2, since $\|L_{\tau,h}\|_{X_h} < 1/2$ and $\|L_\tau\|_X < 1/2$, it has

$$\|F_h L_{\tau,h}^n P_h Q^0 - L_\tau^n Q^0\|_X \le \sum_{k=0}^{n-1} \mu^k \|F_h L_{\tau,h} P_h Q^0 - L_\tau Q^0\|_X < \varepsilon$$

for every positive integer n. On the other hand, since $\|L_{\tau,h}\|_{X_h} < 1/2$, we can select a positive integer k such that

$$\|L_{\tau,h}^n\|_{X_h} < \varepsilon \quad \text{and} \quad \|L_\tau^n\|_X < \varepsilon \quad \text{for all } n > k.$$

It then follows from (4.28)-(4.31) that for $n > k$ and $h \in (0, h')$,

$$\|F_h L_{\tau,h}^n Q_h^0 - L_\tau^n Q^0\|_X < 2\varepsilon.$$

Choose $h'' \in (0, h^*)$ so that

$$\|F_h L_{\tau,h}^n Q_h^0 - L_\tau^n Q^0\|_X < 2\varepsilon \quad \text{for } n = 0, 1, \cdots, k \text{ and } h \in (0, h'').$$

(4.32) and (4.33) imply that $\|F_h L_{\tau,h}^n Q_h^0 - L_\tau^n Q^0\|_X < 2\varepsilon$ for all $n \in \mathbb{N}$ and $h \in (0, h'')$, i.e., $F_h L_{\tau,h}^n Q_h^0$ are uniformly convergent to $L_\tau^n Q^0$ for all $n \in \mathbb{N}$.

Finally, by basic theory of analysis, we have that

$$\|F_hW_{fh}^0-W_f^0\|_X = \lim_{n\to\infty}\|F_h(I_h-L_{\tau,h})^{-1}P_hQ^0-(I-L_\tau)^{-1}Q^0\|_X = \lim_{n\to\infty}\left\|\sum_{n=0}^{\infty}[F_hL_{\tau,h}^nP_hQ^0 - L_\tau^nQ^0]\right\|_X = 0,$$

which shows that the second assertion (ii) holds.

(iii) Consider the mixed variable

$$M_h(t) = F_h[W_{fh}(t) + W_{bh}(t)] - [W_f(t) + W_b(t)],$$

in which $W_f(t)$ and $W_b(t)$ satisfy (4.16) and (4.17), and $W_{fh}(t)$ and $W_{bh}(t)$ satisfy (4.23) and (4.24), respectively. Thus $M_h(t)$ satisfies

$$\begin{cases}\dot{M}_h(t) = A[F_hW_{fh}(t) - W_f(t)] + A^*[F_hW_{bh}(t) - W_b(t)],\\ M_h(0) = F_h[W_{fh}(0) + W_{bh}(0)] - [W_f(0) + W_b(0)].\end{cases}$$

Let $E_{M_h}(t)$ be the energy of $M_h(t)$. Then

$$E_{M_h}(t) = \langle M_h(t), M_h(t)\rangle_X.$$

It follows from the same calculation as that of Lemma 2.2 that

$$\dot{E}_{M_h}(t) = -[w_{2,N+1}(t) + w_{b,2,N+1}(t)]^2 - [w_{4,N+1}(t) + w_{b,4,N+1}(t)]^2.$$

By the expressions of $u_h(t)$ and $u(t)$, we obtain

$$\int_0^\tau \|u_h(t)-u(t)\|_{\mathbb{C}^2}^2\,dt = \int_0^\tau \dot{E}_{M_h}(t)\,dt = E_{M_h}(0)-E_{M_h}(\tau) \le \|F_hW_{fh}^0-W_f^0\|_X^2+\|F_hW_{bh}^0-W_b^0\|_X^2.$$

By the assumptions and assertions, we finally arrive at $\|u_h(t)-u(t)\|_{L^2([0,\tau];\mathbb{C}^4)} \to 0$ as $h \to 0$.

The inverses of operators $I_h - L_{\tau,h}$ and $I - L_\tau$ are involved in Theorem 4.3 and are given by Neumann series. It is well known that the curse of dimensionality may happen. In this case, the numerical approximations of controls given in Theorem 4.3 cannot be realized from a computational point of view. To get off this hook, we truncate the Neumann series of the inverse of $I_h - L_{\tau,h}$ in light of a priori error and further obtain precise error estimates between the discrete system controls and continuous system ones. We elaborate this problem in the following Theorem 4.4.

Theorem 4.4. Let τ, Q^0, Q_h^0, W_{bh}^0, and W_f^0 be the same as those in Theorem 4.3. Set $W_{f,n,h}^0 = \sum_{i=0}^{n} L_{\tau,h}^i Q_h^0$. Replacing the initial value W_{fh}^0 of (4.23) by $W_{f,n,h}^0$, the corresponding solution of (4.23) is denoted by $W_{n,h}(t)$. Then for any $\varepsilon > 0$, there exist $h_n \in (0,1)$ and $N \in \mathbb{N}$ such that the approximating controls

$$u_{n,h}(t) := B_h^*[W_{n,h}(t) + W_{bh}(t)]$$

satisfy

$$\|u_{n,h}(t) - u(t)\|_{L^2([0,\tau];\mathbb{C}^2)} < \varepsilon \quad \text{for all } h \in (0, h_n).$$

Proof. It is easy to see that

$$\|u_{n,h}(t)-u(t)\|_{L^2([0,\tau];\mathbb{C}^2)}^2 \leq 2\|u_{n,h}(t)-u_h(t)\|_{L^2([0,\tau];\mathbb{C}^2)}^2 + 2\|u_h(t)-u(t)\|_{L^2([0,\tau];\mathbb{C}^2)}^2.$$

Based on assertion (iii) of Theorem 4.3, there exist $h_n \in (0, h^*)$ such that

$$\|u_h(t) - u(t)\|_{L^2([0,\tau];\mathbb{C}^2)}^2 < \frac{\varepsilon}{2} \quad \text{for all } h \in (0, h_n).$$

Furthermore, by definitions of $u_{n,h}(t)$ and $u_h(t)$, we consider variables

$$M_{n,h}(t) = [W_{n,h}(t) + W_{bh}(t)] - [W_{fh}(t) + W_{bh}(t)]$$

which satisfy

$$\begin{cases} \dot{M}_{n,h}(t) = \mathcal{A}_h M_{n,h}(t), \\ M_{n,h}(0) = W_{f,n,h}^0 - W_{fh}^0. \end{cases}$$

On the one hand, we have

$$\|M_{n,h}(t)\|_{X_h} \leq K\eta^{n+1}(1-\eta)^{-1}\|Q_h^0\|_{X_h},$$

where K is given in Theorem 2.6 and $\eta = \|L_{\tau,h}\|_{X_h}$. This means that when $n > (\ln \eta)^{-1} \ln(\varepsilon/(2K\|Q_h^0\|_{X_h}))$, there holds

$$\|M_{n,h}(t)\|_{X_h} < \varepsilon \quad \text{for any } t \in [0,\tau] \text{ and } h \in (0, h^*).$$

On the other hand, if let $E_{M_{n,h}}(t)$ be the energy of (4.40), then we have

$$E_{M_{n,h}}(t) = \langle M_{n,h}(t), M_h(t)\rangle_{X_h} \leq \|M_{n,h}(t)\|^2_{X_h}.$$

From the same calculations as that in the proof of assertion (iii) of Theorem 4.3, we obtain

$$\dot{E}_{M_{n,h}}(t) = -[w_{n,2,N+1}(t) - w_{2,N+1}(t)]^2 - [w_{n,4,N+1}(t) - w_{4,N+1}(t)]^2.$$

By the expressions of $u_{n,h}(t)$ and $u_h(t)$ and using (4.41), we finally obtain

$$\|u_{n,h}(t) - u_h(t)\|^2_{L^2([0,\tau];\mathbb{C}^2)} = \int_0^\tau \dot{E}_{M_{n,h}}(t)\,dt = E_{M_{n,h}}(0) - E_{M_{n,h}}(\tau) < \frac{\varepsilon}{2}.$$

Combining (4.37)-(4.38) and (4.42) gives (4.36).

5. Concluding remarks

In this paper, we propose a general finite-difference based semi-discrete scheme for a Timoshenko beam under boundary control modeled as a port-Hamiltonian system. A key feature of this semi-discrete scheme is that the discretized systems are still port-Hamiltonian systems and possess some common properties of the continuous system uniformly. This advantage is reflected from different perspectives. Firstly, it allows preservation of certain properties such as energy conservation and control design from infinite-dimensional model to finite-dimensional approximation. Secondly, this discretization preserves the uniform exponential stability with respect to the step size. Thirdly, the proof for the uniform exponential stability of the semi-discrete systems is parallel to the continuous counterpart, which is somehow an indirect Lyapunov method and is different from existing approaches because the Lyapunov function is not directly positive definite and must be combined with the C_0-semigroup property. More importantly, the semi-discrete systems are uniformly exactly controllable and the discrete controls are strongly convergent to the continuous counterpart. Although we focus only on the Timoshenko beam equation which is a typical coupled system, the methodology developed in this paper is potentially applicable to other coupled systems and more complicated port-Hamiltonian systems. Some other important control subjects like the convergence of the minimal energy controls that drive the system state to rest in finite time in exact controllability and other control problems discussed in [10] and [30] are worthy of being investigated in the future.

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Note: Figure translations are in progress. See original paper for figures.