Vacuum stability conditions of the general two-Higgs-doublet potential

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Date: 2023-12-04T00:00:00+00:00

Abstract

In this paper, we present a method to calculate the boundedness from below or vacuum stability of the scalar potential for a general CP-violating two-Higgs-doublet model using the concepts of co-positivity and gauge orbit spaces. Additionally, semi-positive definiteness is proven for a class of fourth-order two-dimensional complex tensors.

Full Text

Vacuum Stability Conditions of the General Two-Higgs-Doublet Potential

Yisheng Song*

Abstract

In this paper, we present novel analytical expressions for the bounded-from-below or vacuum stability conditions of the scalar potential for a general two-Higgs-doublet model using the concepts of co-positivity and gauge orbit spaces. More precisely, we establish several sufficient conditions and necessary conditions for the vacuum stability of the general 2HDM potential, respectively. We also provide an equivalent condition for the vacuum stability of the general 2HDM potential in theory, and then apply it to derive the analytical necessary conditions. Meanwhile, the semi-positive definiteness is proved for a class of 4th-order 2-dimensional complex tensors.

Keywords: Co-positivity, Complex tensors, CP violation, 2HDM

Introduction

In 1973, the first two-Higgs-doublet model (2HDM) was presented by Lee [1, 2]. Subsequently, Weinberg [3] proposed a general multi-Higgs potential model.

Since then, the stability of the scalar Higgs potential has been an important problem in the Standard Model (SM) at high energies. The bounded-from-below (BFB) or vacuum stability of the SM is very notable in the particle physics community. One of the simplest extensions of the SM Higgs sector is the 2HDM [1, 4]. It is well-known that the most general Higgs potential for a 2HDM with Higgs doublets Φ_1 and Φ_2 can be written [5-8] as:

$$V_H(\Phi_1,\Phi_2) = \mu_{11}\Phi_1^*\Phi_1+\mu_{22}\Phi_1^*\Phi_2+\mu_2^*\Phi_2-(\mu_{12}\Phi_2^*\Phi_2)^2\Phi_1)^2+\lambda_2(\Phi_1^*\Phi_1)(\Phi_2^*\Phi_2)+\lambda_4(\Phi_1^*\Phi_2)(\Phi^*+\lambda_1(\Phi^*+\lambda_3(\Phi^*+(\Phi$$

where Φ^* denotes the Hermitian conjugate of Φ. The parameters $_{11}$, $_{22}$ and λ (i = 1, 2, 3, 4) are real, while $_{12}$ and λ (i = 5, 6, 7) are complex.

The Higgs potential of such a 2HDM may be expressed [7, 9] as:

$$V_H(\Phi_1,\Phi_2) = \sum_{a,b=1} \mu_{ab}\Phi_a^*\Phi_b + \sum_{i,j,k,l=1} t_{ijkl}(\Phi_i^*\Phi_j)(\Phi_k^*\Phi_l),$$

where, by definition, t_{ijkl} = t_{klij}, t_{ijkl} = t^{jilk}, and *{ab}* = ^_{ba}. The quartic part:

$$V_4(\Phi_1,\Phi_2) = \sum_{i,j,k,l=1} t_{ijkl}(\Phi_i^*\Phi_j)(\Phi_k^*\Phi_l),$$

gives a 4th-order 2-dimensional complex tensor T = (t_{ijkl}) with components: $t_{1111} = \lambda_1$, $t_{2222} = \lambda_2$, $t_{1122} = t_{2211} = \lambda_3$, $t_{1221} = t_{2112} = t_{1212} = \lambda_5$, $t_{2121} = t_{1112} = t_{1211} = t_{1222} = t_{2212} = \lambda_6$, $t_{1121} = t_{2111} = \lambda_7$.

The stability of the 2HDM potential requires that there is no direction in field space along which the potential tends to minus infinity, i.e., it is BFB. In general, the quartic part of the scalar potential, V_4, is non-negative for arbitrarily large values of the component fields, but the quadratic part, V_2, can take negative values for at least some field values [7]. Considering only the quartic part V_4, the condition for stability (BFB) of the scalar potential in the 2HDM is equivalent to the co-positivity or semi-positive definiteness of the tensor T = (t_{ijkl}) given by the Higgs quartic couplings λ , i.e., $V_4(\Phi_1, \Phi_2) \geq 0$.

When $\lambda_6 = \lambda_7 = 0$, the vacuum stability conditions of the 2HDM potential [4, 10, 11, 13-15] are: $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 + 2\sqrt{}(\lambda_1\lambda_2) \geq 0$, $\lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{}(\lambda_1\lambda_2) \geq 0$.

In 2011, Battye et al. [16] employed Sylvester' s criterion and Lagrange multipliers for the general CP-violating 2HDM. In 2016, Kannike [17-19] presented vacuum stability conditions for the scalar potential of two Higgs doublets in the 2HDM with explicit CP conservation. Chauhan [20] derived analytical necessary and sufficient conditions for vacuum stability in the left-right symmetric model and gave sufficient conditions for successful symmetry breaking. Recently, Song [21] showed analytical sufficient and necessary conditions for the co-positivity of the tensor T = (t_{ijkl}) with real numbers λ (i = 5, 6, 7), and moreover, obtained the vacuum stability conditions for the 2HDM with explicit CP conservation. Bahl et al. [22] gave analytical sufficient conditions for vacuum stability for the 2HDM potential with CP conservation and CP violation, respectively, where the vacuum stability condition for the 2HDM potential with CP violation depends on the Lagrange multiplier . For more details about BFB or vacuum stability conditions of the 2HDM potential, see Refs. [11,12,23–26] for 2HDM with CP conservation; Refs. [8,24] for the most general 2HDM; Refs. [7,22,24] for 2HDM with CP conservation and CP violation; Ref. [27] for 2HDM handled numerically and other references not cited here.

In this paper, we provide three new analytical sufficient conditions for the bounded-from-below or vacuum stability of the scalar potential for a general 2HDM using the co-positivity of 4th-order 2-dimensional symmetric real tensors, which differs from the approach of Bahl et al. [22].

For example, consider: $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$, $\lambda_4 = 1$, $\lambda_5 = -1 - 3i$, $\lambda_6 = \lambda_7 = -2i$.

Obviously, $\mathrm{Re}\lambda_6 = \mathrm{Re}\lambda_7 = 0$, $\lambda_3 + 2\sqrt{(\lambda_1\lambda_2)} = -1 + 2 > 0$, $\lambda_3 + \lambda_4 - |\mathrm{Re}\lambda_5| + 2\sqrt{(\lambda_1\lambda_2)} = -1 + 1 - 1 + 2 > 0$, $\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7 > 0$, $-|\mathrm{Im}\lambda_5| + 2\sqrt{(\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7)} = -3 + 4 > 0$.

These parameters meet conditions (I) and (III), which means $V_4(\Phi_1, \Phi_2) \geq 0$. It is obvious that conditions (IV) and (III) are also fulfilled. Here, conditions (I), (III) and (IV) will be derived in Section 3. However, they cannot satisfy condition Eq. (5.20) of Bahl et al. [22], i.e.:

$\lambda_1\lambda_2 - (\lambda_3 + |\lambda_4| + |\lambda_5|) = 3 - (-1 + 1 + 10) < 0$, $\lambda_1\lambda_2 + \lambda_3 - (|\lambda_4| + |\lambda_5| + 4\sqrt{(\lambda_1\lambda_2)}) = 1 - 1 - (1 + 10 + 8) < 0$.

A sufficient and necessary condition for the vacuum stability of the general 2HDM potential is given in theory, which contains the vacuum stability condition of the general 2HDM potential with Z_2 symmetry as a special case. We then apply this conclusion to derive analytical necessary conditions for the vacuum stability of a general 2HDM scalar potential. Meanwhile, analytical sufficient and necessary conditions are obtained for the semi-positive definiteness of a class of 4th-order 2-dimensional complex tensors.

2 Co-positivity Criteria

The co-positivity of a matrix M = (_{ij}) has been applied to test vacuum stability in the 2HDM in Refs. [17-20]. It is well-known that a 2×2 symmetric real matrix M = (_{ij}) is co-positive, i.e., for all non-negative vectors $x = (x_1, x_2)$ ^{2}, the quadratic form x Mx = $_{11}x_1^2 + 2\ _{12}x_1x_2 +\ _{22}x_2^2 \geq 0$, if and only if [28–30]:

$_{11} \geq 0$, $_{22} \geq 0$, and $_{12}$ + √($_{11}$ $_{22}$) ≥ 0.

The co-positivity of symmetric real tensors has been used in the SM literature to obtain vacuum stability conditions in Refs. [17, 21, 31–34]. A 4th-order n-dimensional symmetric real tensor T = (t_{ijkl}) is co-positive [35–40] if for all non-negative vectors x = (x_1, x_2, ···, x) ^{n}, the quartic form:

$$Tx^4 = \sum_{i,j,k,l=1} t_{ijkl} x_i x_j x_k x_l \geq 0.$$

Let $t_{1111} = \alpha_0 > 0$ and $t_{2222} = \alpha_4 > 0$. Recently, Song and Li [32] presented analytical expressions for the co-positivity of a 4th-order 2-dimensional symmetric tensor T using an updated version [37] of Ulrich and Watson' s result [41]. Let T = (t_{ijkl}) be a 4th-order 2-dimensional symmetric real tensor with entries $t_{1111} = \alpha_0$, $t_{2222} = \alpha_4$, $t_{1112} = \alpha_1$, $t_{1122} = \alpha_2$, $t_{1222} = \alpha_3$. Then the quartic form:

$$Tx^4 = \alpha_0 x_1^4 + \alpha_1 x_1^3 x_2 + \alpha_2 x_1^2 x_2^2 + \alpha_3 x_1 x_2^3 + \alpha_4 x_2^4 \geq 0$$

for all $x_1 \geq 0$, $x_2 \geq 0$ if and only if:

(1) $\Delta \leq 0$, $\alpha_1 \geq 0$, $\alpha_3 \geq 0$, $2\sqrt{(\alpha_0\alpha_4)} + \alpha_3\sqrt{(\alpha_0/\alpha_4)} > 0$; $\alpha_0\alpha_4 + \alpha_2 \geq 0$; α_4 - $\alpha_3\sqrt{(\alpha_0\alpha_2\alpha_4)} + 2\alpha_0\alpha_4\sqrt{(\alpha_0\alpha_4)}$,

(2) $\Delta \geq 0$, $|\alpha_1\alpha_4 - \alpha_3\alpha_0| \leq 4\sqrt{(\alpha_0\alpha_2\alpha_4)} + 2\alpha_0\alpha_4\sqrt{(\alpha_0\alpha_4)}$, (i) $-2\sqrt{(\alpha_0\alpha_4)} \leq \alpha_2 \leq 6\sqrt{(\alpha_0\alpha_4)}$, (ii) $\alpha_2 > 6\sqrt{(\alpha_4)} + \alpha_3\sqrt{(\alpha_0/\alpha_4)} \geq -4\sqrt{(\alpha_0\alpha_2\alpha_4)} - 2\alpha_0\alpha_4\sqrt{(\alpha_0\alpha_4)}$,

where $\Delta = 4(12\alpha_0\alpha_4 - 3\alpha_1\alpha_3 + \alpha_{21}\alpha_4)^2 - 27\alpha_0{\alpha_2}^3 - 27{\alpha_2}^2)^3 - (72\alpha_0\alpha_2\alpha_4 + 9\alpha_1\alpha_2\alpha_3 - 2\alpha_3$.

Song and Qi [33, Theorem 3.7] gave a stronger sufficient condition for the co-positivity of a symmetric real tensor T = (t_{ijkl}). That is, T $x^4 \geq 0$ for all $x_1 \geq 0$, $x_2 \geq 0$ if:

$\beta = \alpha_1 + 4\sqrt{(\alpha_2 - 6\alpha_0\alpha_4)} + 2\sqrt{(\beta\gamma)} \geq 0$, $\gamma = \alpha_3 + 4\sqrt{(\alpha_2 - 6\alpha_0\alpha_4)} + 2\sqrt{(\beta\gamma)} \geq 0$.

Song [21] obtained an analytical sufficient and necessary condition for the co-positivity of a symmetric real tensor T(,) = (t_{ijkl}(,)) with parameters [0, 1] and [0, 2π], where: $t_{1111} = \Lambda_1$, $t_{2222} = \Lambda_2$, $t_{1122} = t_{1112} = (\Lambda_3 + \Lambda_4\ ^2 + \Lambda_5\ ^2 \cos 2\ )$, Λ_6 cos , $t_{1222} = \Lambda_7$ cos .

That is, with $\Lambda_1 > 0$, $\Lambda_2 > 0$, the quartic form:

$$T(\rho,\theta)x^4 = \Lambda_1 x_1^4 + \Lambda_2 x_2^4 + 2(\rho\Lambda_6 cos\theta)x_1^3 x_2 + (\Lambda_3 + \Lambda_4\rho^2 + \Lambda_5\rho^2 cos2\theta)x_1^2 x_2^2 + 2(\rho\Lambda_7 cos\theta)x_1 x_2^3 \geq 0$$

for all $x_1 \geq 0$, $x_2 \geq 0$ if and only if:

(b) $\Delta \geq 0$, Λ_3 + 2√(Λ_2) - Λ_7(i) - 2(a) $\Lambda_6 = \Lambda_7 = 0$, Λ_3 + 2√($\Lambda_1\Lambda_2$) ≥ 0, Λ_3 + Λ_4 - |Λ_5| + 2√($\Lambda_1\Lambda_2$) ≥ 0, Λ_3 + Λ_4 - Λ_5 + 2√($\Lambda_1\Lambda_2$) ≥ 0; √($\Lambda_1\Lambda_2$) ≥ 0, (cid:113) Λ_1| ≤ 2√($\Lambda_1\Lambda_2$)(Λ_3 + Λ_4 + Λ_5) + 2$\Lambda_1\Lambda_2$√($\Lambda_1\Lambda_2$), $\Lambda_1\Lambda_2 \leq \Lambda_3$ + Λ_4 + Λ_5 ≤ 6(ii) Λ_3 + Λ_4 + Λ_5 > 6√(Λ_2) + Λ_7√($\Lambda_1\Lambda_2$) and (cid:113) Λ_1| ≤ 2√($\Lambda_1\Lambda_2$)(Λ_3 + Λ_4 + Λ_5) - 2$\Lambda_1\Lambda_2$√($\Lambda_1\Lambda_2$), √($\Lambda_1\Lambda_2$),

where Δ = 4(12$\Lambda_1\Lambda_2$ - 12$\Lambda_6\Lambda_7$ + (Λ_3 + Λ_4 + Λ_5)^2)^3 - (72$\Lambda_1\Lambda_2$(Λ_3 + Λ_4 + 6Λ_2)^2. Λ_5) + 36$\Lambda_6\Lambda_7$(Λ_3 + Λ_4 + Λ_5) - 2(Λ_3 + Λ_4 + Λ_5)^3 - 108$\Lambda_1\Lambda_{27}$ - 108Λ_2.

3 Vacuum Stability of the General 2HDM Potential

3.1 Sufficient Conditions

In this section, we present the vacuum stability conditions of the 2HDM potential with explicit CP violation. We rewrite the quartic part of the 2HDM potential as:

$$V_4(\Phi_1,\Phi_2) = \sum_{i,j,k,l=1} t_{ijkl}(\Phi_i^*\Phi_j)(\Phi_k^*\Phi_l) = \lambda_1(\Phi_1^*\Phi_1)^2+\lambda_2(\Phi_2^*\Phi_2)^2+\lambda_3(\Phi_1^*\Phi_1)(\Phi_2^*\Phi_2)+\lambda_4(\Phi_1^*\Phi_2)(\Phi_2^*\Phi_1)+(\Phi_1^*\Phi_2)$$

Let   = |Φ | = √(Φ Φ) denote the modulus of Φ for i = 1, 2. Then Φ $_1\Phi_2$ = $_1$ $_2$ e^{i } and Φ $_2\Phi_1$ = $_1$ $_2$ e^{-i }, where i^2 = -1 and   [0, 1] is the orbit space parameter [10, 17, 20]. Let λ_5 = |λ_5|e^{iϕ_5}, λ_6 = |λ_6|e^{iϕ_6}, λ_7 = |λ_7|e^{iϕ_7}, where ϕ is the argument of the complex number λ (k = 5, 6, 7). Then λ $_5$ = |λ_5|e^{-iϕ_5}, λ $_6$ = |λ_6|e^{-iϕ_6}, λ $_7$ = |λ_7|e^{-iϕ_7}.

We therefore have:

$$V_4(\Phi_1,\Phi_2) = \lambda_1\phi_1^4+\lambda_2\phi_2^4+\lambda_3\phi_1^2\phi_2^2+\lambda_4\rho^2\phi_1^2\phi_2^2+|\lambda_5|\rho^2(e^{i(\phi_5+2\theta)}+e^{-i(\phi_5+2\theta)})\phi_1^2\phi_2^2+2|\lambda_6|\rho(e^{i(\phi_6+\theta)}+e^{-i(\phi_6+\theta)})\phi_1^3\phi_2$$

This can be rewritten as:

$$V_4(\Phi_1,\Phi_2) = \lambda_1\phi_1^4+\lambda_2\phi_2^4+(\lambda_3+\lambda_4\rho^2+|\lambda_5|\rho^2\cos(\phi_5+2\theta))\phi_1^2\phi_2^2+2|\lambda_6|\phi_1^3\phi_2\rho\cos(\phi_6+\theta)+2|\lambda_7|\phi_1\phi_2^3\rho\cos(\phi_7+\theta).$$

Expanding the cosine terms gives:

$$V_4(\Phi_1,\Phi_2) = \lambda_1\phi_1^4+\lambda_2\phi_2^4+(\lambda_3+\lambda_4\rho^2+|\lambda_5|\rho^2(\cos\phi_5\cos2\theta-\sin\phi_5\sin2\theta))\phi_1^2\phi_2^2+2|\lambda_6|\phi_1^3\phi_2\rho(\cos\phi_6\cos\theta-\sin\phi_6 \sin$$

Since Reλ = |λ |cosϕ and Imλ = |λ |sinϕ (k = 5, 6, 7), and noting that sin2 = 2sin cos , we obtain:

$$V_4(\Phi_1,\Phi_2) = \lambda_1\phi_1^4+\lambda_2\phi_2^4+(\lambda_3+\lambda_4\rho^2+\mathrm{Re}\lambda_5\rho^2\cos 2\theta)\phi_1^2\phi_2^2+2(\rho\mathrm{Re}\lambda_7\cos\theta)\phi_1\phi_2^3+2(\rho\mathrm{Re}\lambda_6\cos\theta)\phi_1^3\phi_2-2\rho\phi_1\phi_2(\mathrm{I}$$

We can separate this into two parts:

$$V_4(\Phi_1,\Phi_2) = V_4'(\phi_1,\phi_2) + V_4''(\phi_1,\phi_2),$$

where:

$$V_4'(\phi_1,\phi_2) = \lambda_1\phi_1^4+\lambda_2\phi_2^4+(\lambda_3+\lambda_4\rho^2+\mathrm{Re}\lambda_5\rho^2\cos 2\theta)\phi_1^2\phi_2^2+2(\rho\mathrm{Re}\lambda_7\cos\theta)\phi_1\phi_2^3+2(\rho\mathrm{Re}\lambda_6\cos\theta)\phi_1^3\phi_2,$$

and:

$$V_4''(\phi_1,\phi_2) = -2(\rho\sin\theta)[(\rho\cos\theta)\mathrm{Im}\lambda_5\phi_1\phi_2 + \mathrm{Im}\lambda_7\phi_2^2 + \mathrm{Im}\lambda_6\phi_1^2]\phi_1\phi_2.$$

Applying the co-positivity condition for a real tensor (9) with $\Lambda = \lambda$ (i = 1, 2, 3, 4) and $\Lambda = \mathrm{Re}\lambda$ (k = 5, 6, 7), we obtain that $\lambda_1 > 0$, $\lambda_2 > 0$, and V' _4($_1$, $_2$) ≥ 0 for all $_1$, $_2$ if and only if:

(I) $\mathrm{Re}\lambda_6 = \mathrm{Re}\lambda_7 = 0$, $\lambda_3 + 2\surd(\lambda_1\lambda_2) \geq 0$, and $\lambda_3 + \lambda_4$ - $|\mathrm{Re}\lambda_5| + 2\surd(\lambda_1\lambda_2) \geq 0$;

(II) $\mathrm{Re}\lambda_6 \neq 0$ or $\mathrm{Re}\lambda_7 \neq 0$, $\Delta \geq 0$, $\surd(\lambda_1\lambda_2) \geq 0$, $\lambda_3 + 2\surd(\lambda_1\lambda_2) \geq 0$, $\lambda_3 + \lambda_4$ - $\mathrm{Re}\lambda_5 + 2\surd(\lambda_1\lambda_2) \geq 0$, $|\mathrm{Re}\lambda_6\lambda_2$ - $\mathrm{Re}\lambda_7\lambda_1| \leq 2\surd(\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5)) + 2\lambda_1\lambda_2\surd(\lambda_1\lambda_2)$, (i) -$2\surd(\lambda_1\lambda_2) \leq \lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 \leq 6\surd(\lambda_1\lambda_2)$, and (ii) $\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 > 6\surd(\lambda_1\lambda_2)$ and $|\mathrm{Re}\lambda_6\lambda_2 + \mathrm{Re}\lambda_7\lambda_1| \leq 2\surd(\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5))$ - $2\lambda_1\lambda_2\surd(\lambda_1\lambda_2)$, where:

$$\Delta = 4(12\lambda_1\lambda_2-12\mathrm{Re}\lambda_6\cdot\mathrm{Re}\lambda_7+(\lambda_3+\lambda_4+\mathrm{Re}\lambda_5)^2)^3-(72\lambda_1\lambda_2(\lambda_3+\lambda_4+\mathrm{Re}\lambda_5)+36\mathrm{Re}\lambda_6\cdot\mathrm{Re}\lambda_7(\lambda_3+\lambda_4+\mathrm{Re}\lambda_5)-2(\lambda_3+$$

After simple calculations (with sin $\neq 0$), we find that V'' _4($_1$, $_2$) ≥ 0 for all $_1$, $_2$ if and only if:

$$\begin{cases} \mathrm{Im}\lambda_6\phi_1^2 + (\rho\cos\theta)\mathrm{Im}\lambda_5\phi_1\phi_2 + \mathrm{Im}\lambda_7\phi_2^2 \geq 0, & \sin\theta < 0, \\ \mathrm{Im}\lambda_6\phi_1^2 + (\rho\cos\theta)\mathrm{Im}\lambda_5\phi_1\phi_2 + \mathrm{Im}\lambda_7\phi_2^2 \leq 0, & \sin\theta > 0. \end{cases}$$

By the co-positivity of a real matrix (6) with $_{11} = \mathrm{Im}\lambda_6$, $_{12}$ = (cos)$\mathrm{Im}\lambda_5$, and $_{22} = \mathrm{Im}\lambda_7$, we obtain that $\mathrm{Im}\lambda_6\ _1{}^2$ + (cos)$\mathrm{Im}\lambda_5\ _1\ _2 + \mathrm{Im}\lambda_7\ _2{}^2 \geq 0$ for all $_1$, $_2$ if and only if:

$$\mathrm{Im}\lambda_6 \geq 0, \quad \mathrm{Im}\lambda_7 \geq 0, \quad (\rho\cos\theta)\mathrm{Im}\lambda_5 + 2\sqrt{\mathrm{Im}\lambda_6\cdot\mathrm{Im}\lambda_7} \geq 0.$$

Case 1: $\mathrm{Im}\lambda_5 \geq 0$. The function g(cos) = (cos)$\mathrm{Im}\lambda_5$ + 2√($\mathrm{Im}\lambda_6$ · $\mathrm{Im}\lambda_7$) reaches its minimum at cos = -1, so for all cos [-1, 1], we have:

$$(\rho\cos\theta)\mathrm{Im}\lambda_5 + 2\sqrt{\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7} \geq 0 \iff -\mathrm{Im}\lambda_5 + 2\sqrt{\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7} \geq 0.$$

Case 2: $\mathrm{Im}\lambda_5 < 0$. The function g(cos) = (cos)$\mathrm{Im}\lambda_5$ + 2√($\mathrm{Im}\lambda_6$ · $\mathrm{Im}\lambda_7$) reaches its minimum at cos = 1, so for all cos [-1, 1], we have:

$$(\rho\cos\theta)\mathrm{Im}\lambda_5 + 2\sqrt{\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7} \geq 0 \iff \mathrm{Im}\lambda_5 + 2\sqrt{\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7} \geq 0.$$

Therefore, for any real $\mathrm{Im}\lambda_5$, we have:

$$(\rho\cos\theta)\mathrm{Im}\lambda_5 + 2\sqrt{\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7} \geq 0 \iff -|\mathrm{Im}\lambda_5| + 2\sqrt{\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7} \geq 0.$$

Similarly, $\mathrm{Im}\lambda_6\ {}_1^2$ + (cos)$\mathrm{Im}\lambda_5\ {}_1\ {}_2$ + $\mathrm{Im}\lambda_7\ {}_2^2 \leq 0$ for all ${}_1$, ${}_2$ if and only if -$\mathrm{Im}\lambda_6\ {}_1^2$ - (cos)$\mathrm{Im}\lambda_5\ {}_1\ {}_2$ - $\mathrm{Im}\lambda_7\ {}_2^2 \geq 0$ for all ${}_1$, ${}_2$, which is equivalent to the co-positivity of the 2 × 2 matrix M = (_{ij}) with entries ${}_{11}$ = -$\mathrm{Im}\lambda_6$, ${}_{12}$ = -(cos)$\mathrm{Im}\lambda_5$, ${}_{22}$ = -$\mathrm{Im}\lambda_7$. This means:

$$-\mathrm{Im}\lambda_6 \geq 0, \quad -\mathrm{Im}\lambda_7 \geq 0, \quad -(\rho\cos\theta)\mathrm{Im}\lambda_5 + 2\sqrt{\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7} \geq 0,$$

or equivalently:

$$\mathrm{Im}\lambda_6 \leq 0, \quad \mathrm{Im}\lambda_7 \leq 0, \quad -(\rho\cos\theta)\mathrm{Im}\lambda_5 + 2\sqrt{\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7} \geq 0.$$

The function f(cos) = -(cos)$\mathrm{Im}\lambda_5$ + 2√($\mathrm{Im}\lambda_6$ · $\mathrm{Im}\lambda_7$) reaches its minimum at cos = 1 (when $\mathrm{Im}\lambda_5 \geq 0$) or -1 (when $\mathrm{Im}\lambda_5 < 0$). Hence, for any real $\mathrm{Im}\lambda_5$:

$$-(\rho\cos\theta)\mathrm{Im}\lambda_5 + 2\sqrt{\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7} \geq 0 \iff -|\mathrm{Im}\lambda_5| + 2\sqrt{\mathrm{Im}\lambda_6 \cdot \mathrm{Im}\lambda_7} \geq 0.$$

Thus, we conclude that V'' _4(${}_1$, ${}_2$) ≥ 0 for all ${}_1$, ${}_2$ if and only if:

(III) $\mathrm{Im}\lambda_6$ · $\mathrm{Im}\lambda_7 \geq 0$ and -|$\mathrm{Im}\lambda_5$| + 2√($\mathrm{Im}\lambda_6$ · $\mathrm{Im}\lambda_7$) ≥ 0.

In summary, we have proved that the analytic conditions (I), (II), and (III) ensure the vacuum stability of the 2HDM potential with explicit CP violation. At the same time, we obtain the semi-positive definiteness of a 4th-order 2-dimensional complex tensor T = (t_{ijkl}) defined by Eq. (5).

From Eq. (8), we may also obtain a stronger sufficient condition. That is, $V_4(\Phi_1, \Phi_2) \geq 0$ for all Φ_1, Φ_2 if for all [0, 1] and all [0, 2π]:

$$\beta(\theta) = 2\rho|\lambda_6|\cos(\phi_6+\theta)+4\sqrt{(\lambda_3+\lambda_4\rho^2+|\lambda_5|\rho^2\cos(\phi_5+2\theta))-6\sqrt{\lambda_1\lambda_2}} \geq 0,$$

$$\gamma(\theta) = 2\rho|\lambda_7|\cos(\phi_7+\theta)+4\sqrt{(\lambda_3+\lambda_4\rho^2+|\lambda_5|\rho^2\cos(\phi_5+2\theta))-6\sqrt{\lambda_1\lambda_2}} \geq 0,$$

and

$$\lambda_1\lambda_2 + 2\sqrt{\beta(\theta)\gamma(\theta)} \geq 0.$$

It is obvious that the above inequality system holds if:

$$\beta = 2\sqrt{\lambda_3 - 6\sqrt{\lambda_1\lambda_2}} - |\lambda_6| \geq 0,$$

$$\gamma = 2\sqrt{\lambda_3 - 6\sqrt{\lambda_1\lambda_2}} - |\lambda_7| \geq 0,$$

and

$$\lambda_1\lambda_2 + 4\sqrt{\beta\gamma} \geq 0,$$

$$\lambda_3 + \lambda_4 - |\lambda_5| - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} \geq 0.$$

Similarly, V' _4($_1$, $_2$) $\geq$ 0 for all $_1$, $_2$ if:

(IV) β' = 2√(λ_3 - 6√($\lambda_1\lambda_2$)) - |Reλ_6| $\geq$ 0, γ' = 2√(λ_3 - 6√($\lambda_1\lambda_2$)) - |Reλ_7| $\geq$ 0, $\lambda_1\lambda_2$ + 4√(β' γ ') $\geq$ 0, and λ_3 + λ_4 - |Reλ_5| - 6√($\lambda_1\lambda_2$) + 4√(β' γ') $\geq$ 0.

Remark 3.1. Four analytical sufficient conditions are:

(1) Reλ_6 = Reλ_7 = 0, with conditions (I) and (III);
(2) Re$\lambda_6 \neq 0$ or Re$\lambda_7 \neq 0$, with conditions (II) and (III);
(3) Conditions (IV) and (III);
(4) Conditions (IV).

These analytical sufficient conditions are obtained from the real and imaginary parts of complex numbers, not only dependent on the norm. Thus they differ from those of Bahl et al. [22]. For example, with $|\lambda_6| = |\lambda_7| = 2$, $\beta = \gamma = 0$, λ_1 = λ_2 = 1, λ_3 = 6, λ_4 = 2, λ_5 = -1 - i, λ_6 = λ_7 = -1 - i, we have $\lambda_1\lambda_2$ + 4√($\beta\gamma$) = 0, λ_3 - 6√($\lambda_1\lambda_2$) + 4√($\beta\gamma$) = 2 > 0. These parameters meet condition (IV), which means $V_4(\Phi_1, \Phi_2) \geq 0$. It is obvious that conditions (II), (IV), and (III) are also fulfilled. However, they cannot satisfy condition Eq. (5.20) of Bahl et al. [22]:

$$\lambda_1\lambda_2 - (\lambda_3 + |\lambda_4| + |\lambda_5|) = 3 - (6+2+2) < 0,$$

$$\lambda_1\lambda_2 + \lambda_3 - (|\lambda_4| + |\lambda_5| + 4\sqrt{\lambda_1\lambda_2}) = 1 + 6 - (2+2+8) < 0.$$

See also the example in the introduction.

3.2 Sufficient and Necessary Conditions

In this subsection, we rewrite $V_4(\Phi_1, \Phi_2)$ as follows (with $\lambda_1 > 0$, $\lambda_2 > 0$):

$$V_4(\Phi_1, \Phi_2) = A\rho^2 + B\rho + C = f(\rho),$$

where:

$$A = a\phi_1^2\phi_2^2, \quad B = b\phi_1\phi_2, \quad C = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + \lambda_3\phi_1^2\phi_2^2,$$
$$a = \lambda_4 - \mathrm{Re}\lambda_5 + 2(\mathrm{Re}\lambda_5\cos\theta - \mathrm{Im}\lambda_5\sin\theta)\cos\theta = \lambda_4 + |\lambda_5|\cos(\phi_5 + 2\theta),$$
$$b = 2(\mathrm{Re}\lambda_6\phi_1^2 + \mathrm{Re}\lambda_7\phi_2^2)\cos\theta - 2(\mathrm{Im}\lambda_6\phi_1^2 + \mathrm{Im}\lambda_7\phi_2^2)\sin\theta = 2|\lambda_6|\phi_1^2\cos(\phi_6+\theta) + 2|\lambda_7|\phi_2^2\cos(\phi_7+\theta).$$

The quadratic function f( ) is non-negative for [0, 1] if and only if its minimum is non-negative on this interval, which requires non-negative values at the boundary points = 0, 1 and at any unique minimum point $_0$ = -B/(2A) [0, 1] (when A > 0). That is:

$$f(\rho) \geq 0 \iff f(-B/(2A)) \geq 0, \quad f(0) \geq 0, \quad f(1) \geq 0,$$

with the conditions:

$$4AC - B^2 \geq 0, \quad -B/(2A) \in [0, 1], \quad A > 0.$$

The behavior of f( ) is illustrated in Figure 1 [Figure 1: see original paper].

Proposition 1. $V_4(\Phi_1, \Phi_2) \geq 0$ if and only if $4AC - B^2 \geq 0$, $-2A \leq B \leq 0$, and $C \geq 0$.

It is obvious that if $\lambda_6 = \lambda_7 = 0$, then B = 0, the symmetry axis is at -B/(2A) = 0, and hence $V_4(\Phi_1, \Phi_2) \geq 0$ if and only if $C \geq 0$ and $A + C \geq 0$.

For the 2HDM with Z_2 symmetry [12], the quartic part of the general 2HDM scalar potential is:

$$V_4(\Phi_1, \Phi_2) = \lambda_1(\Phi_1^*\Phi_1)^2 + \lambda_2(\Phi_2^*\Phi_2)^2 + \lambda_3(\Phi_1^*\Phi_1)(\Phi_2^*\Phi_2) + \lambda_4(\Phi_1^*\Phi_2)(\Phi_2^*\Phi_1) + (\Phi_1^*\Phi_2)^2 + (\Phi_2^*\Phi_1)^2.$$

Therefore, $V_4(\Phi_1, \Phi_2) \geq 0$ if and only if:

$$C = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + \lambda_3\phi_1^2\phi_2^2 \geq 0,$$
$$A + C = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 + |\lambda_5|\cos(\phi_5 + 2\theta))\phi_1^2\phi_2^2 \geq 0.$$

It is clear that $C \geq 0$ $\lambda_3 + 2\sqrt{}(\lambda_1\lambda_2) \geq 0$, and $A + C \geq 0$ $\lambda_3 + \lambda_4 + |\lambda_5|\cos(\phi_5$ + 2) + $2\sqrt{}(\lambda_1\lambda_2) \geq 0$ for all [0, 2π] $\lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{}(\lambda_1\lambda_2) \geq 0$.

Corollary 2. $V_4(\Phi_1, \Phi_2) \geq 0$ if and only if $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 + 2\surd(\lambda_1\lambda_2) \geq 0$, and $\lambda_3 + \lambda_4 - |\lambda_5| + 2\surd(\lambda_1\lambda_2) \geq 0$.

This condition (V) is well-known for the inert doublet model [4,10,11, 18, 27].

3.3 Necessary Conditions

In this subsection, we rewrite $V_4(\Phi_1, \Phi_2)$ as (with $\lambda_1 > 0$, $\lambda_2 > 0$):

$$V_4(\Phi_1, \Phi_2) = f(\rho) = A\rho^2 + B\rho + C,$$

$$f(1) = A+B+C = \lambda_1\phi_1^4+\lambda_2\phi_2^4+(\lambda_3+\lambda_4-\mathrm{Re}\lambda_5+2\mathrm{Re}\lambda_5\cos^2\theta)\phi_1^2\phi_2^2+2(\mathrm{Re}\lambda_6\cos\theta)\phi_1^3\phi_2+2(\mathrm{Re}\lambda_7\cos\theta)\phi_1\phi_2^3-2(\mathrm{s}$$

Obviously, f( ) $\geq$ 0 for all [0, 1] implies f(0) $\geq$ 0 and f(1) $\geq$ 0. Then $V_4(\Phi_1, \Phi_2) \geq 0$ implies f(0) = C $\geq$ 0, which is equivalent to:

$$\lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0.$$

This is a necessary condition for the vacuum stability of the general 2HDM potential. Clearly, another necessary condition is that f(1) = A + B + C $\geq$ 0. By Eq. (18), A + B + C can be regarded as a quartic form in two parameters t = sin and s = cos with $s^2 + t^2 = 1$. When s = sin = 0 and t = cos = $\pm$1, the inequality:

$$\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5)\phi_1^2\phi_2^2 \pm 2\mathrm{Re}\lambda_6\phi_1^3\phi_2 \pm 2\mathrm{Re}\lambda_7\phi_1\phi_2^3 \geq 0$$

holds if and only if (using Eq. (7)):

(1) $\Delta \leq 0$, $\mathrm{Re}\lambda_6\lambda_2 + \mathrm{Re}\lambda_7\lambda_1 > 0$;
(2) $\mathrm{Re}\lambda_6 \geq 0$, $\mathrm{Re}\lambda_7 \geq 0$, $\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 + 2\surd(\lambda_1\lambda_2) \geq 0$;
(3) $\Delta \geq 0$, $|\mathrm{Re}\lambda_6\lambda_2 - \mathrm{Re}\lambda_7\lambda_1| \leq 2\surd(\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5)) + 2\lambda_1\lambda_2\surd(\lambda_1\lambda_2)$, (i) $-2\surd(\lambda_1\lambda_2) \leq \lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 \leq 6\surd(\lambda_1\lambda_2)$, (ii) $\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 > 6\surd(\lambda_1\lambda_2)$, and $\surd(\lambda_1\lambda_2) \leq \lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 \leq 6\surd(\lambda_1\lambda_2)$.

The equivalent conditions for the negative case are:

(1) $\Delta \leq 0$, $-\mathrm{Re}\lambda_6\lambda_2 - \mathrm{Re}\lambda_7\lambda_1 > 0$; (2) $-\mathrm{Re}\lambda_6 \geq 0$, $-\mathrm{Re}\lambda_7 \geq 0$, $\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 + 2\surd(\lambda_1\lambda_2) \geq 0$; (3) $\Delta \geq 0$, $|-\mathrm{Re}\lambda_6\lambda_2 + \mathrm{Re}\lambda_7\lambda_1| \leq 2\surd(\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5)) + 2\lambda_1\lambda_2\surd(\lambda_1\lambda_2)$, (i) $-2\surd(\lambda_1\lambda_2) \leq \lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 \leq 6\surd(\lambda_1\lambda_2)$, (ii) $\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 > 6\surd(\lambda_1\lambda_2)$, and $-\mathrm{Re}\lambda_6\lambda_2 - \mathrm{Re}\lambda_7\lambda_1 \geq -2\surd(\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5)) - 2\lambda_1\lambda_2\surd(\lambda_1\lambda_2)$.

This is equivalent to:

$$\mathrm{Re}\lambda_6 = \mathrm{Re}\lambda_7 = 0, \quad \lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0,$$

or:

$$\mathrm{Re}\lambda_6 \neq 0 \text{ or } \mathrm{Re}\lambda_7 \neq 0, \quad \Delta \geq 0, \quad \sqrt{\lambda_1\lambda_2} \geq 0,$$

with either: (a) -2√($\lambda_1\lambda_2$) ≤ λ_3 + λ_4 + Reλ_5 ≤ 6√($\lambda_1\lambda_2$), |Re$\lambda_6\lambda_2$ - Re$\lambda_7\lambda_1$| ≤ 2√($\lambda_1\lambda_2$(λ_3 + λ_4 + Reλ_5)) + 2$\lambda_1\lambda_2$√($\lambda_1\lambda_2$), or (b) λ_3 + λ_4 + Reλ_5 > 6√($\lambda_1\lambda_2$), |Re$\lambda_6\lambda_2$ + Re$\lambda_7\lambda_1$| ≤ 2√($\lambda_1\lambda_2$(λ_3 + λ_4 + Reλ_5)) - 2$\lambda_1\lambda_2$√($\lambda_1\lambda_2$).

Thus, a necessary condition for $V_4(\Phi_1, \Phi_2) \geq 0$ is:

(VII) λ_3 + λ_4 + Reλ_5 + 2√($\lambda_1\lambda_2$) ≥ 0.

Similarly, if t = cos = 0 and s = sin = $\$\pm\$1$, the inequality:

$$\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 - \mathrm{Re}\lambda_5)\phi_1^2\phi_2^2 \pm 2\mathrm{Im}\lambda_6\phi_1^3\phi_2 \pm 2\mathrm{Im}\lambda_7\phi_1\phi_2^3 \geq 0$$

is equivalent to:

$$\sqrt{\lambda_1\lambda_2} \geq 0; \quad \mathrm{Im}\lambda_6 = \mathrm{Im}\lambda_7 = 0, \quad \lambda_3 + \lambda_4 - \mathrm{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0;$$

or:

$$\mathrm{Im}\lambda_6 \neq 0 \text{ or } \mathrm{Im}\lambda_7 \neq 0, \quad \Delta' \geq 0,$$

with either: (a) -2√($\lambda_1\lambda_2$) ≤ λ_3 + λ_4 - Reλ_5 ≤ 6√($\lambda_1\lambda_2$), |Im$\lambda_6\lambda_2$ - Im$\lambda_7\lambda_1$| ≤ 2√($\lambda_1\lambda_2$(λ_3 + λ_4 - Reλ_5)) + 2$\lambda_1\lambda_2$√($\lambda_1\lambda_2$), or (b) λ_3 + λ_4 - Reλ_5 > 6√($\lambda_1\lambda_2$), |Im$\lambda_6\lambda_2$ + Im$\lambda_7\lambda_1$| ≤ 2√($\lambda_1\lambda_2$(λ_3 + λ_4 - Reλ_5)) - 2$\lambda_1\lambda_2$√($\lambda_1\lambda_2$),

where:

$$\Delta' = 4(12\lambda_1\lambda_2 - 12\mathrm{Im}\lambda_6\cdot\mathrm{Im}\lambda_7 + (\lambda_3+\lambda_4-\mathrm{Re}\lambda_5)^2)^3 - (72\lambda_1\lambda_2(\lambda_3+\lambda_4-\mathrm{Re}\lambda_5) + 36\mathrm{Im}\lambda_6\cdot\mathrm{Im}\lambda_7(\lambda_3+\lambda_4-\mathrm{Re}\lambda_5) - 2(\lambda$$

Therefore, another necessary condition for $V_4(\Phi_1, \Phi_2) \geq 0$ is:

(VIII) λ_3 + λ_4 - Reλ_5 + 2√($\lambda_1\lambda_2$) ≥ 0.

Applying Corollary 3.1 of Song and Qi [33] to $V_4(\Phi_1, \Phi_2)$:

$$V_4(\Phi_1,\Phi_2) = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + [\lambda_3 + \lambda_4\rho^2 + |\lambda_5|\rho^2\cos(\phi_5+2\theta)]\phi_1^2\phi_2^2 + 2|\lambda_6|\phi_1^3\phi_2\rho\cos(\phi_6+\theta) + 2|\lambda_7|\phi_1\phi_2^3\rho\cos(\phi_7+\theta),$$

we obtain that $V_4(\Phi_1, \Phi_2) > 0$ implies that for all [0, 1] and all [0, 2π]:

$$\times 2|\lambda_6|\rho\cos(\phi_6+\theta)\times 2|\lambda_7|\rho\cos(\phi_7+\theta)\sqrt{\lambda_2} + 2\sqrt{(\lambda_3 + \lambda_4\rho^2 + |\lambda_5|\rho^2\cos(\phi_5+2\theta)) + \lambda_2\rho\cos(\phi_6+\theta) + |\lambda_7|\sqrt{\lambda_1\rho}$$

This simplifies to:

$$\lambda_3{+}\lambda_4\rho^2{+}\mathrm{Re}\lambda_5\rho^2\cos 2\theta{+}2\sqrt{\lambda_1\lambda_2}\rho^2\cos\theta{+}\mathrm{Re}\lambda_7\sqrt{\lambda_2}\rho\cos\theta{+}\lambda_3{+}\lambda_4\rho^2{+}\mathrm{Re}\lambda_5\rho^2\cos 2\theta{+}2\sqrt{\lambda_1\lambda_2}\rho^2\sin 2\theta,$$

and then, setting = 1 and = 0 or π, the above inequality must hold:

$$(\lambda_3{+}\lambda_4{+}\mathrm{Re}\lambda_5{+}2\sqrt{\lambda_1\lambda_2}{\pm}2\sqrt{\lambda_2}{+}\mathrm{Re}\lambda_7\sqrt{\lambda_2})\quad\text{or}\quad(\lambda_3{+}\lambda_4{-}\mathrm{Re}\lambda_5{+}2\sqrt{\lambda_1\lambda_2}{\mp}2\sqrt{\lambda_2}{+}\mathrm{Im}\lambda_7\sqrt{\lambda_2}),$$

or equivalently:

$$(\lambda_3{+}\lambda_4{+}\mathrm{Re}\lambda_5{+}2\sqrt{\lambda_1\lambda_2}{+}\mathrm{Re}\lambda_7\sqrt{\lambda_2})\quad\text{and}\quad(\lambda_3{+}\lambda_4{-}\mathrm{Re}\lambda_5{+}2\sqrt{\lambda_1\lambda_2}{+}\mathrm{Im}\lambda_7\sqrt{\lambda_2}),$$

with strict positivity conditions on the real and imaginary parts.

Clearly, condition (VI) (strict inequality) is obtained when = 0. In summary, conditions (VI), (VII), (VIII), and (IX) are the necessary conditions for the vacuum stability of the general 2HDM potential.

Remark 3.2. Conditions (I) and (II) obviously satisfy (VI)-(VIII) (as necessary conditions). For (IV) and (IV), with β = 2√(λ_3 + 2√($\lambda_1\lambda_2$)) - $|\lambda_6| \geq 0$ and γ = 2√(λ_3 + 2√($\lambda_1\lambda_2$)) - $|\lambda_7| \geq 0$, we have:

$$\lambda_1\lambda_2 + 4\sqrt{\beta\gamma} \geq 0,$$
$$\lambda_3 + \lambda_4 - |\lambda_5| - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} \geq \lambda_3 - 6\sqrt{\lambda_1\lambda_2} \geq 0,$$

which implies λ_3 + 2√($\lambda_1\lambda_2$) ≥ 0. Thus condition (VI) holds. Similarly:

$$\lambda_1\lambda_2 + 4\sqrt{\beta\gamma} \geq 0,$$
$$\lambda_3 + \lambda_4 - |\lambda_5| - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} \geq \lambda_3 + \lambda_4 - |\lambda_5| - 6\sqrt{\lambda_1\lambda_2} \geq 0,$$

and since -$|\lambda_5| \leq \mathrm{Re}\lambda_5 \leq |\lambda_5|$, we have:

$$\lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq \lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0,$$
$$\lambda_3 + \lambda_4 - \mathrm{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq \lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0.$$

Thus conditions (VII) and (VIII) hold. Similarly, condition (IV) also satisfies (VI)-(VIII). Condition (IX) is a necessary condition for the strict inequality $V_4(\Phi_1, \Phi_2) > 0$.

4 Conclusions

Using the co-positive conditions of a 4th-order symmetric tensor, we have established several analytical sufficient conditions and necessary conditions for the vacuum stability of the general 2HDM potential. Specifically:

Four sufficient conditions: (1) (I) and (III); (2) (II) and (III); (3) (IV) and (III); (4) (IV).

Four necessary conditions: (VI), (VII), (VIII), and (IX).

A sufficient and necessary condition is qualitatively shown for the vacuum stability of the general 2HDM potential, which is then applied to derive analytical necessary conditions. The vacuum stability condition (V) for the Z_2-symmetric 2HDM potential emerges as a special case.

[Figure 2: see original paper] shows the relationship between these analytical conditions and the vacuum stability of the general 2HDM potential (where "→" stands for "implies").

Competing interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authors' contributions: Yisheng Song is the sole author.

Availability of data and materials: This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and there are no external data associated with the manuscript].

Funding: This work was supported by the National Natural Science Foundation of P.R. China (Grant No. 12171064), by The team project of innovation leading talent in Chongqing (No. CQYC20210309536), and by the Foundation of Chongqing Normal University (20XLB009).

Acknowledgements: The authors would like to express their sincere thanks to the Editors and the anonymous referees for their constructive comments and valuable suggestions.

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Note: Figure translations are in progress. See original paper for figures.