Characterizations of Compactness of Fuzzy Set Spaces with the Endograph Metric

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Abstract

In this paper, we present the characterizations of total boundedness, relative compactness and compactness in fuzzy set spaces equipped with the endograph metric. The conclusions in this paper significantly improve the corresponding conclusions given in our previous paper H. Huang, Characterizations of endograph metric and \$ Gamma\$-convergence on fuzzy sets, Fuzzy Sets and Systems 350 (2018), 55-84. The results in this paper are applicable to fuzzy sets in a general metric space. The results in our previous paper are applicable to fuzzy sets in the m-dimensional Euclidean space \$ mathbb{R}^m\$, which is a special type of metric space. Furthermore, based on the above results, we give the characterizations of relative compactness, total boundedness and compactness in a kind of common subspaces of general fuzzy sets according to the endograph metric. As an application, we investigate some relationship between the endograph metric and the \$ Gamma\$-convergence on fuzzy sets.

Full Text

Preamble

Characterizations of Compactness of Fuzzy Set Space with Endograph Metric

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Abstract

In this paper, we present characterizations of total boundedness, relative compactness, and compactness in fuzzy set spaces equipped with the endograph metric. The conclusions significantly improve upon corresponding results from our previous work [H. Huang, Characterizations of endograph metric and Γ-convergence on fuzzy sets, Fuzzy Sets and Systems 350 (2018), 55-84]. The results herein apply to fuzzy sets in a general metric space, whereas our previous results were limited to fuzzy sets in m-dimensional Euclidean space $\mathbb{R}^m$, which is merely a special type of metric space. Furthermore, based on these general results, we provide characterizations of relative compactness, total boundedness, and compactness for common subspaces of general fuzzy sets under the endograph metric. As an application, we investigate the relationship between the endograph metric and Γ-convergence on fuzzy sets. This paper has also been submitted to arXiv.

Keywords: Compactness; Endograph metric; Γ-convergence; Hausdorff metric

1. Introduction

Fuzzy sets are fundamental tools for investigating fuzzy phenomena [1–7]. A fuzzy set can be identified with its endograph, and the endograph metric H_{end} on fuzzy sets is defined as the Hausdorff metric applied to their endographs. The endograph metric has been shown to possess significant advantages [8–11].

Compactness is a central concept in topology and analysis with important applications [6, 12]. Characterizations of compactness in various fuzzy set spaces endowed with different topologies have attracted considerable attention [13–19].

In [18], we provided characterizations of total boundedness, relative compactness, and compactness for fuzzy set spaces equipped with the endograph metric H_{end}. However, those results applied only to fuzzy sets in m-dimensional Euclidean space $\mathbb{R}^m$, which is a special type of metric space. In theoretical research and practical applications, fuzzy sets in general metric spaces are frequently employed [1, 2, 14, 15].

This paper presents characterizations of total boundedness, relative compactness, and compactness for spaces of fuzzy sets in a general metric space equipped with the endograph metric H_{end}. We demonstrate that the characterizations given in [18] are corollaries of the more general characterizations established here. Additionally, we discuss properties of the endograph metric H_{end} and use these properties, along with our general characterizations, to describe relative compactness, total boundedness, and compactness in common subspaces of general fuzzy sets under the endograph metric H_{end}.

As an application of our compactness characterizations, we examine the relationship between the H_{end} metric and Γ-convergence on fuzzy sets.

The remainder of this paper is organized as follows. Section 2 recalls basic notions and fundamental results related to fuzzy sets, the endograph metric, and Γ-convergence. Section 3 provides representation theorems for various types of fuzzy sets used throughout the paper. Section 4 characterizes relatively compact, totally bounded, and compact sets in the space of fuzzy sets in a general metric space equipped with the endograph metric. Section 5, building on Section 4’ s results, characterizes relatively compact, totally bounded, and compact sets in common subspaces of the fuzzy set space. Section 6 applies our compactness characterizations to investigate the relationship between the endograph metric and Γ-convergence. Finally, Section 7 concludes the paper.

2. Fuzzy Sets, Endograph Metric, and Γ-Convergence

This section recalls basic notions and fundamental results concerning fuzzy sets, the endograph metric, and Γ-convergence. Readers may refer to [1-3, 20, 21] for related background.

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{R}$ the set of real numbers. For $m > 1$, let $\mathbb{R}^m$ denote the set $\{(x_1, \dots, x_m) : x_i \in \mathbb{R}, i = 1, \dots, m\}$. In what follows, $\mathbb{R}$ is also written as $\mathbb{R}^1$.

Throughout this paper, we assume X is a nonempty set and d is a metric on X. For simplicity, we also use X to denote the metric space (X, d).

The metric d on $X \times [0, 1]$ is defined as follows: for $(x, \alpha), (y, \beta) \in X \times [0, 1]$,

$$d((x, \alpha), (y, \beta)) = d(x, y) + |\alpha - \beta|.$$

Throughout this paper, we assume the metric on $X \times [0, 1]$ is d. For simplicity, we also use $X \times [0, 1]$ to denote the metric space $(X \times [0, 1], d)$.

Let $m \in \mathbb{N}$. For simplicity, $\mathbb{R}^m$ also denotes the m-dimensional Euclidean space; d_m denotes the Euclidean metric on $\mathbb{R}^m$; and $\mathbb{R}^m \times [0, 1]$ denotes the metric space $(\mathbb{R}^m \times [0, 1], d_m)$.

A fuzzy set u in X can be viewed as a function $u : X \to [0, 1]$. A subset S of X can be viewed as a fuzzy set in X. When no confusion arises, the fuzzy set corresponding to S is denoted by χ_S; that is, $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \in X \setminus S$.

For simplicity, for $x \in X$, we use $\hat{x}$ to denote the fuzzy set $\chi_{\{x\}}$ in X.

When emphasizing a specific metric space X, we write the fuzzy set corresponding to S in X as $S_F(X)$, and the fuzzy set corresponding to $\{x\}$ in X as $\hat{x}_F(X)$.

Let $F(X)$ denote the set of all fuzzy sets in X. For $u \in F(X)$ and $\alpha \in [0, 1]$, let $\{u > \alpha\}$ denote the set $\{x \in X : u(x) > \alpha\}$, and let $[u]_\alpha$ denote the α-cut of u, i.e.,

$$[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0, 1], \\ \overline{\operatorname{supp} u} = \overline{\{u > 0\}}, & \alpha = 0, \end{cases}$$

where $\overline{S}$ denotes the topological closure of S in (X, d).

Let $K(X)$ and $C(X)$ denote the set of all nonempty compact subsets of X and the set of all nonempty closed subsets of X, respectively. Let $P(X)$ denote the power set of X (the set of all subsets of X).

Let $F_{USC}(X)$ denote the set of all upper semi-continuous fuzzy sets $u : X \to [0, 1]$, i.e.,

$$F_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \cup \{\emptyset\} \text{ for all } \alpha \in [0, 1]\}.$$

Define $F_{USCB}(X) := \{u \in F_{USC}(X) : [u]_0 \in K(X) \cup \{\emptyset\}\}$ and $F_{USCG}(X) := \{u \in F_{USC}(X) : [u]_\alpha \in K(X) \cup \{\emptyset\} \text{ for all } \alpha \in (0, 1]\}$.

Clearly, $F_{USCB}(X) \subseteq F_{USCG}(X) \subseteq F_{USC}(X)$. Define $F_{CON}(X) := \{u \in F(X) : \text{for all } \alpha \in (0, 1], [u]_\alpha \text{ is connected in } X\}$, $F_{USCCON}(X) := F_{USC}(X) \cap F_{CON}(X)$, and $F_{USCGCON}(X) := F_{USCG}(X) \cap F_{CON}(X)$.

Let $u \in F_{CON}(X)$. Then $[u]_0 = \cup_{\alpha>0}[u]_\alpha$ is connected in X. The proof is as follows. If $u = \chi_\emptyset$, then $[u]_0 = \emptyset$ is connected in X. If $u \neq \chi_\emptyset$, then there exists $\alpha \in (0, 1]$ such that $[u]_\alpha \neq \emptyset$. Note that $[u]_\beta \supseteq [u]_\alpha$ when $\beta \in [0, \alpha]$. Hence $\cup_{0<\beta<\alpha}[u]_\beta$ is connected, and thus $[u]_0 = \cup_{0<\beta<\alpha}[u]_\beta$ is connected.

$F_{CON}(X) = \{u \in F(X) : \text{for all } \alpha \in [0, 1], [u]_\alpha \text{ is connected in } X\}$.

Let $F^1_{USC}(X)$ denote the set of all normal and upper semi-continuous fuzzy sets $u : X \to [0, 1]$, i.e., $F^1_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0, 1]\}$.

We introduce some subclasses of $F^1_{USC}(X)$ that will be discussed in this paper. Define $F^1_{USCB}(X) := F^1_{USC}(X) \cap F_{USCB}(X)$, $F^1_{USCG}(X) := F^1_{USC}(X) \cap F_{USCG}(X)$, $F^1_{USCCON}(X) := F^1_{USC}(X) \cap F_{CON}(X)$, and $F^1_{USCGCON}(X) := F^1_{USCG}(X) \cap F_{CON}(X)$.

Clearly, $F^1_{USCB}(X) \subseteq F^1_{USCGCON}(X) \subseteq F^1_{USCG}(X) \subseteq F^1_{USCCON}(X) \subseteq F^1_{USC}(X)$.

Let (X, d) be a metric space. We use H to denote the Hausdorff distance on $C(X)$ induced by d, i.e., $H(U, V) = \max\{H^*(U, V), H^*(V, U)\}$ for arbitrary $U, V \in C(X)$, where $H^*(U, V) = \sup_{u\in U} d(u, V) = \sup_{u\in U} \inf_{v\in V} d(u, v)$.

When no confusion arises, we also use H to denote the Hausdorff distance on $C(X \times [0, 1])$ induced by d.

The Hausdorff distance on $C(X)$ can be extended to $C(X) \cup \{\emptyset\}$ as follows:

$$H(M_1, M_2) = \begin{cases} H(M_1, M_2), & \text{if } M_1, M_2 \in C(X), \\ 1, & \text{if } M_1 = \emptyset \text{ and } M_2 \in C(X), \\ 0, & \text{if } M_1 = M_2 = \emptyset. \end{cases}$$

Remark 2.1. A function $\rho : Y \times Y \to \mathbb{R}$ is called a metric on Y if it satisfies positivity, symmetry, and the triangle inequality. In this case, (Y, ρ) is called a metric space. A function $\rho : Y \times Y \to \mathbb{R} \cup \{+\infty\}$ is called an extended metric on Y if it satisfies positivity, symmetry, and the triangle inequality. In this case, (Y, ρ) is called an extended metric space.

We observe that for any metric space (X, d), the Hausdorff distance H on $K(X)$ induced by d is a metric. Therefore, the Hausdorff distance H on $K(X \times [0,1])$ induced by d on $X \times [0,1]$ is a metric.

The Hausdorff distance H on $C(X)$ induced by d on X is an extended metric but not necessarily a metric, because $H(A, B)$ could equal $+\infty$ for certain metric spaces X and sets $A, B \in C(X)$. The Hausdorff distance H on $C(X) \cup \{\emptyset\}$ is an extended metric but not a metric.

Clearly, if H on $C(X)$ induced by d is not a metric, then H on $C(X \times [0,1])$ induced by d is also not a metric. Thus, the Hausdorff distance H on $C(X \times [0,1])$ induced by d on $X \times [0,1]$ is an extended metric but possibly not a metric.

We note that H on $C(\mathbb{R}^m)$ is an extended metric but not a metric, and the same holds for H on $C(\mathbb{R}^m \times [0,1])$.

When the Hausdorff distance H is a metric, we call it the Hausdorff metric. When H is an extended metric, we call it the Hausdorff extended metric. For simplicity, in this paper we refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric.

For $u \in F(X)$, define $\operatorname{end} u := \{(x,t) \in X \times [0,1] : u(x) \geq t\}$ and $\operatorname{send} u := \{(x,t) \in X \times [0,1] : u(x) \geq t\} \cap ([u]_0 \times [0,1])$. These are called the endograph and sendograph of u, respectively.

For $u \in F(X)$, the following properties (i)-(iii) are equivalent: (i) $u \in F_{USC}(X)$; (ii) $\operatorname{end} u$ is closed in $(X \times [0,1], d)$; (iii) $\operatorname{send} u$ is closed in $(X \times [0,1], d)$.

Proof of equivalences: - (i) $\Rightarrow$ (ii): Assume (i) holds. To show (ii), let $\{(x_n, \alpha_n)\}$ be a sequence in $\operatorname{end} u$ converging to (x, α) in $X \times [0,1]$. Since u is upper semi-continuous, $u(x) \geq \limsup_{n\to\infty} u(x_n) \geq \lim_{n\to\infty} \alpha_n = \alpha$. Thus $(x, \alpha) \in \operatorname{end} u$, proving (ii).

- (ii) $\Rightarrow$ (iii): Assume (ii) holds. Since $[u]_0 \times [0,1]$ is closed in $X \times [0,1]$, $\operatorname{send} u = \operatorname{end} u \cap ([u]_0 \times [0,1])$ is closed in $X \times [0,1]$, proving (iii).
- (iii) $\Rightarrow$ (i): Assume (iii) holds. To show (i), let $\alpha \in [0,1]$ and suppose $\{x_n\}$ is a sequence in $[u]_\alpha$ converging to x in X. We must show $x \in [u]_\alpha$. Note that $\{(x_n, \alpha)\}$ converges to (x, α) in $X \times [0,1]$, and the sequence $\{(x_n, \alpha)\}$ lies in $\operatorname{send} u$. From the closedness of $\operatorname{send} u$, it follows that $(x, \alpha) \in \operatorname{send} u$, which means $x \in [u]_\alpha$. Thus (i) holds.

For $u \in F(X)$, clearly $X \times \{0\} \subseteq \operatorname{end} u$, so $\operatorname{end} u \neq \emptyset$. We observe that $\operatorname{send} u = \emptyset$ if and only if $u = \emptyset_{F(X)}$.

From the above discussion, $u \in F_{USC}(X)$ if and only if $\operatorname{end} u \in C(X \times [0,1])$.

Kloeden [8] introduced the endograph metric H_{end}. For $u, v \in F_{USC}(X)$,

$$H_{\text{end}}(u, v) := H(\operatorname{end} u, \operatorname{end} v),$$

where H is the Hausdorff metric on $C(X \times [0,1])$ induced by d on $X \times [0,1]$.

Rojas-Medar and Román-Flores [20] introduced Γ-convergence of a sequence of upper semi-continuous fuzzy sets based on Kuratowski convergence of sets in a metric space.

Let (X, d) be a metric space, C a set in X, and $\{C_n\}$ a sequence of sets in X. The sequence $\{C_n\}$ is said to Kuratowski converge to C according to (X, d), written as $C = \lim^{(K)}_{n\to\infty} C_n$, if

$$C = \liminf_{n\to\infty} C_n = \limsup_{n\to\infty} C_n,$$

where

$$\liminf_{n\to\infty} C_n = \{x \in X : x = \lim_{n\to\infty} x_n, x_n \in C_n\},$$

$$\limsup_{n\to\infty} C_n = \{x \in X : x = \lim_{j\to\infty} x_{n_j}, x_{n_j} \in C_{n_j}\} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} C_k}.$$

When no confusion arises, we omit reference to the metric space (X, d) and simply write that $\{C_n\}$ Kuratowski converges to C or $C = \lim^{(K)}_{n\to\infty} C_n$.

Remark 2.2. Theorem 5.2.10 in [22] states that in a first countable Hausdorff topological space, a sequence of sets is Kuratowski convergent if and only if it is convergent in the Fell topology. A metric space is, of course, a first countable Hausdorff topological space.

Definition 3.1.4 in [23] gives definitions of $\liminf C_n$, $\limsup C_n$, and $\lim C_n$ for a net of subsets $\{C_n, n \in D\}$ in a topological space. When $\{C_n, n = 1, 2, ...\}$ is a sequence of subsets of a metric space, the $\liminf$, $\limsup$, and $\lim$ according to Definition 3.1.4 in [23] coincide with $\liminf_{n\to\infty} C_n$, $\limsup_{n\to\infty} C_n$, and $\lim^{(K)}_{n\to\infty} C_n$ as defined above, respectively.

Let $u, u_n \in F_{USC}(X)$ for $n = 1, 2, ...$ The sequence $\{u_n\}$ is said to Γ-converge to u, denoted by $u = \lim^{(\Gamma)}_{n\to\infty} u_n$, if $\operatorname{end} u = \lim^{(K)}_{n\to\infty} \operatorname{end} u_n$ according to $(X \times [0, 1], d)$.

The following Theorem 2.3 is a known result useful in this paper.

Theorem 2.3. Suppose C, C_n are sets in $C(X)$ for $n = 1, 2, ...$ Then $H(C_n, C) \to 0$ implies that $\lim^{(K)}_{n\to\infty} C_n = C$.

Remark 2.4. Theorem 2.3 implies that for a sequence $\{u_n\}$ in $F_{USC}(X)$ and an element u in $F_{USC}(X)$, if $H_{\text{end}}(u_n, u) \to 0$ as $n \to \infty$, then $\lim^{(\Gamma)}_{n\to\infty} u_n = u$. However, the converse is false; see Example 4.1 in [18].

Theorem 2.3 can be proved similarly to Theorem 4.1 in [18]. In Theorem 2.3, we exclude the case $C = \emptyset$.

Remark 2.5. Let $\{u_n\}$ be a sequence in $F_{USC}(X)$ and $\{v_n\}$ a subsequence of $\{u_n\}$. We observe that

$$\liminf_{n\to\infty} u_n \subseteq \liminf_{n\to\infty} v_n \subseteq \limsup_{n\to\infty} v_n \subseteq \limsup_{n\to\infty} u_n.$$

Thus, if there exists $u \in F_{USC}(X)$ with $\lim_{n\to\infty}^{(\Gamma)} u_n = u$, then $\lim_{n\to\infty}^{(\Gamma)} v_n = u$.

Clearly, $\lim_{n\to\infty}^{(\Gamma)} v_n = u$ does not necessarily imply $\lim_{n\to\infty}^{(\Gamma)} u_n = u$. A simple example is given below. For $n = 1, 2, ...$, let $v_n = \widehat{1}_F(\mathbb{R})$. For $n = 1, 2, ...$, define $u_n \in F_{USC}(\mathbb{R})$ by

$$u_n = \begin{cases} \widehat{1}_F(\mathbb{R}), & n \text{ is odd,} \\ \widehat{3}_F(\mathbb{R}), & n \text{ is even.} \end{cases}$$

Then $\{v_n\}$ is a subsequence of $\{u_n\}$. We see that $\lim_{n\to\infty}^{(\Gamma)} v_n = \widehat{1}_F(\mathbb{R})$. However, $\lim_{n\to\infty}^{(\Gamma)} u_n$ does not exist because

$$\liminf_{n\to\infty} u_n = \mathbb{R} \times \{0\} \subsetneq \operatorname{end} \widehat{1}_F(\mathbb{R}) \cup \operatorname{end} \widehat{3}_F(\mathbb{R}) = \limsup_{n\to\infty} u_n.$$

In this paper, for a metric space (Y, ρ) and a subset S of Y, we still use ρ to denote the metric induced on S by ρ.

3. Representation Theorems for Various Kinds of Fuzzy Sets

This section provides representation theorems for various types of fuzzy sets. These theorems are useful throughout the paper.

The following representation theorem should be a known result. In this paper, we assume $\sup \emptyset = 0$.

Theorem 3.1. Let Y be a nonempty set. If $u \in F(Y)$, then for all $\alpha \in (0, 1]$, $[u]_\alpha = \cap_{\beta<\alpha} [u]_\beta$.

Conversely, suppose $\{v_\alpha : \alpha \in (0, 1]\}$ is a family of sets in Y with $v_\alpha = \cap_{\beta<\alpha} v_\beta$ for all $\alpha \in (0, 1]$. Define $u \in F(Y)$ by $u(x) := \sup\{\alpha : x \in v_\alpha\}$ for each $x \in Y$. Then u is the unique fuzzy set in Y satisfying $[u]_\alpha = v_\alpha$ for all $\alpha \in (0, 1]$.

Proof. Let $u \in F(Y)$ and $\alpha \in (0, 1]$. For each $x \in Y$, $x \in [u]_\alpha \iff u(x) \geq \alpha \iff$ for each $\beta < \alpha$, $u(x) \geq \beta \iff$ for each $\beta < \alpha$, $x \in [u]_\beta$. Thus $[u]_\alpha = \cap_{\beta<\alpha} [u]_\beta$.

Conversely, suppose $\{v_\alpha : \alpha \in (0, 1]\}$ is a family of sets in Y with $v_\alpha = \cap_{\beta<\alpha} v_\beta$ for all $\alpha \in (0, 1]$. Define $u \in F(Y)$ by $u(x) := \sup\{\alpha : x \in v_\alpha\}$ for each $x \in Y$. First, we show that for each $\alpha \in (0, 1]$, $[u]_\alpha = v_\alpha$. Let $\alpha \in (0, 1]$. We need to verify both $[u]_\alpha \supseteq v_\alpha$ and $[u]_\alpha \subseteq v_\alpha$.

If $x \in v_\alpha$, then clearly $u(x) \geq \alpha$, i.e., $x \in [u]_\alpha$. Thus $[u]_\alpha \supseteq v_\alpha$.

If $x \in [u]_\alpha$, then $\sup\{\beta : x \in v_\beta\} = u(x) \geq \alpha$. Hence there exists a sequence $\{\beta_n, n = 1, 2, ...\}$ such that $1 \geq \beta_n \geq \alpha - 1/n$ and $x \in v_{\beta_n}$. Let $\gamma = \sup_{n=1}^{\infty} \beta_n$. Then $1 \geq \gamma \geq \alpha$ and thus $x \in \cap_{n=1}^{\infty} v_{\beta_n} = v_\gamma \subseteq v_\alpha$. So $[u]_\alpha \subseteq v_\alpha$.

Now we show the uniqueness of u. Assume v is a fuzzy set in Y satisfying $[v]_\alpha = v_\alpha$ for all $\alpha \in (0,1]$. Then for each $x \in Y$, $v(x) = \sup\{\alpha : x \in [v]_\alpha\} = \sup\{\alpha : x \in v_\alpha\} = u(x)$. Thus $u = v$.

Remark 3.2. We could not locate the original reference for Theorem 3.1, so we provide a proof here for the self-containment of this paper. Theorem 3.1 and its proof are essentially the same as Theorem 7.10 on page 27 of chinaXiv:202110.00083v4 and its proof, since the uniqueness of u is obvious.

From Theorem 3.1, we immediately obtain representation theorems for $F_{USC}(X)$, $F^1_{USC}(X)$, $F_{USCG}(X)$, $F_{CON}(X)$, $F_{USCB}(X)$, and $F^1_{USCB}(X)$.

Proposition 3.3. Let (X,d) be a metric space. If $u \in F_{USC}(X)$ (respectively, $u \in F^1_{USC}(X)$, $u \in F_{USCG}(X)$, $u \in F_{CON}(X)$), then: (i) $[u]_\alpha \in C(X) \cup \{\emptyset\}$ (respectively, $[u]_\alpha \in C(X)$, $[u]_\alpha \in K(X) \cup \{\emptyset\}$, $[u]_\alpha$ is connected in (X,d)) for all $\alpha \in (0,1]$, and (ii) $[u]_\alpha = \cup_{\beta<\alpha}[u]_\beta$ for all $\alpha \in (0,1]$.

Conversely, suppose the family of sets $\{v_\alpha : \alpha \in (0,1]\}$ satisfies conditions (i) and (ii). Define $u \in F(X)$ by $u(x) := \sup\{\alpha : x \in v_\alpha\}$ for each $x \in X$. Then u is the unique fuzzy set in X satisfying $[u]_\alpha = v_\alpha$ for each $\alpha \in (0,1]$. Moreover, $u \in F_{USC}(X)$ (respectively, $u \in F^1_{USC}(X)$, $u \in F_{USCG}(X)$, $u \in F_{CON}(X)$).

Proof. The proof is routine. We show only the case of $F_{USC}(X)$; the other cases can be verified similarly.

If $u \in F_{USC}(X)$, then clearly (i) holds. From Theorem 3.1, (ii) holds.

Conversely, suppose the family of sets $\{v_\alpha : \alpha \in (0,1]\}$ satisfies conditions (i) and (ii). Define $u \in F(X)$ by $u(x) := \sup\{\alpha : x \in v_\alpha\}$ for each $x \in X$. Then by Theorem 3.1, u is the unique fuzzy set in X satisfying $[u]_\alpha = v_\alpha$ for each $\alpha \in (0,1]$. Since $\{[u]_\alpha, \alpha \in (0,1]\}$ satisfies condition (i), $u \in F_{USC}(X)$.

Proposition 3.4. Let (X,d) be a metric space. If $u \in F_{USCB}(X)$ (respectively, $u \in F^1_{USCB}(X)$), then: (i) $[u]_\alpha \in K(X) \cup \{\emptyset\}$ (respectively, $[u]_\alpha \in K(X)$) for all $\alpha \in [0,1]$, (ii) $[u]_\alpha = \cup_{\beta<\alpha}[u]_\beta$ for all $\alpha \in (0,1]$, and (iii) $[u]_0 = \cup_{\beta>0}[u]_\beta$.

Conversely, suppose the family of sets $\{v_\alpha : \alpha \in [0,1]\}$ satisfies conditions (i)-(iii). Define $u \in F(X)$ by $u(x) := \sup\{\alpha : x \in v_\alpha\}$ for each $x \in X$. Then u is the unique fuzzy set in X satisfying $[u]_\alpha = v_\alpha$ for each $\alpha \in [0,1]$. Moreover, $u \in F_{USCB}(X)$ (respectively, $u \in F^1_{USCB}(X)$).

Proof. The proof is routine. We show only the case of $F_{USCB}(X)$; the case of $F^1_{USCB}(X)$ can be verified similarly.

If $u \in F_{USCB}(X)$, then clearly (i) holds. By Theorem 3.1, (ii) holds. From the definition of $[u]_0$, (iii) holds.

Conversely, suppose the family of sets $\{v_\alpha : \alpha \in [0,1]\}$ satisfies conditions (i)-(iii). Define $u \in F(X)$ by $u(x) := \sup\{\alpha : x \in v_\alpha\}$ for each $x \in X$. Then by Theorem 3.1, u is the unique fuzzy set in X satisfying $[u]_\alpha = v_\alpha$ for each $\alpha \in (0,1]$. Clearly $[u]_0 = \cup_{\beta>0}[u]_\beta = \cup_{\beta>0}v_\beta = v_0$. Since $\{[u]_\alpha, \alpha \in [0,1]\}$ satisfies condition (i), $u \in F_{USCB}(X)$.

Similarly, we can obtain representation theorems for $F_{USCCON}(X)$, $F_{USCGCON}(X)$, $F^1_{USCCON}(X)$, etc.

Based on these representation theorems, we can define a fuzzy set or a certain type of fuzzy set by specifying the family of its α-cuts. In what follows, we will directly state that we are defining a fuzzy set or a specific type of fuzzy set without explicitly mentioning which representation theorem is used, as this will be clear from context.

4. Characterization of Compactness in $(F_{USCG}(X), H_{\mathbf{end}})$

This section characterizes relatively compact sets, totally bounded sets, and compact sets in $(F_{USCG}(X), H_{\text{end}})$. These results improve upon the characterizations of relatively compact, totally bounded, and compact sets in $(F_{USCG}(\mathbb{R}^m), H_{\text{end}})$ given in our previous work [18].

We recall the following definitions: - A subset Y of a topological space Z is compact if for every collection of open sets $\{O_i : i \in I\}$ with $Y \subset \cup_{i\in I} O_i$, there exists a finite subcollection $O_{i_1}, O_{i_2}, \dots, O_{i_n}$ such that $Y \subseteq O_{i_1} \cup O_{i_2} \cup \cdots \cup O_{i_n}$. In metric spaces, this is equivalent to every sequence in Y having a subsequence convergent in Y. - A subset Y of a topological space Z is relatively compact if its closure is compact. In metric spaces, this is equivalent to every sequence in Y having a subsequence convergent in X. - Let (X, d) be a metric space. A set $U \subseteq X$ is totally bounded if and only if for each $\varepsilon > 0$, it contains a finite ε-approximation (or ε-net) $S \subseteq U$ such that $d(x, S) < \varepsilon$ for each $x \in U$.

Let (X, d) be a metric space. If a set U is compact in (X, d), then U is relatively compact in (X, d), which in turn implies U is totally bounded in (X, d). If Y is a subset of X and $A \subseteq Y$, then A is totally bounded in (Y, d) if and only if A is totally bounded in (X, d).

Theorem 4.1. [19] Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. Then $\mathcal{D}$ is totally bounded in $(K(X), H)$ if and only if $\cup\{C : C \in \mathcal{D}\}$ is totally bounded in (X, d).

Theorem 4.2. [14] Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. Then $\mathcal{D}$ is relatively compact in $(K(X), H)$ if and only if $\cup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d).

Theorem 4.3. [19] Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. Then the following are equivalent: (i) $\mathcal{D}$ is compact in $(K(X), H)$; (ii) $\cup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d) and $\mathcal{D}$ is closed in $(K(X), H)$; (iii) $\cup\{C : C \in \mathcal{D}\}$ is compact in (X, d) and $\mathcal{D}$ is closed in $(K(X), H)$.

For $u \in F_{USC}(X)$, define $\hat{u} = \operatorname{end} u$. Let $\mathcal{A}$ be a subset of $F_{USC}(X)$. Define $\widehat{\mathcal{A}} = \{\hat{u} : u \in \mathcal{A}\}$. Clearly $F_{USC}(X)\hat{} \subseteq C(X \times [0,1])$.

Define $g : (F_{USC}(X), H_{\text{end}}) \to (C(X \times [0,1]), H)$ by $g(u) = \operatorname{end} u$. Then: - g is an isometric embedding of $(F_{USC}(X), H_{\text{end}})$ into $(C(X \times [0,1]), H)$; - $g(F_{USC}(X)) = F_{USC}(X)\hat{}$; - $(F_{USC}(X), H_{\text{end}})$ is isometric to $(F_{USC}(X)\hat{}, H)$.

The following representation theorem for $F_{USC}(X)\hat{}$ follows immediately from Proposition 3.3.

Proposition 4.4. Let U be a subset of $X \times [0,1]$. Then $U \in F_{USC}(X)\hat{}$ if and only if the following properties (i)-(iii) hold: (i) For each $\alpha \in (0,1]$, $\langle U \rangle_\alpha \in C(X) \cup \{\emptyset\}$; (ii) For each $\alpha \in (0,1]$, $\langle U \rangle_\alpha = \cup_{\beta<\alpha} \langle U \rangle_\beta$; (iii) $\langle U \rangle_0 = X$.

Here $\langle U \rangle_\alpha := \{x : (x,\alpha) \in U\}$.

Proposition 4.5. Let $U \in C(X \times [0,1])$. Then the following (i) and (ii) are equivalent: (i) For each α with $0 < \alpha \le 1$, $\langle U \rangle_\alpha = \cup_{\beta<\alpha} \langle U \rangle_\beta$; (ii) For each α, β with $0 \le \beta < \alpha \le 1$, $\langle U \rangle_\alpha \subseteq \langle U \rangle_\beta$.

Proof. The proof is routine. (i) $\Rightarrow$ (ii) is obvious.

Suppose (ii) holds. To show (i), let $\alpha \in (0,1]$. From (ii), $\langle U \rangle_\alpha \subseteq \cup_{\beta<\alpha} \langle U \rangle_\beta$. We only need to prove $\langle U \rangle_\alpha \supseteq \cup_{\beta<\alpha} \langle U \rangle_\beta$.

Let $x \in \cup_{\beta<\alpha} \langle U \rangle_\beta$. This means $(x,\beta) \in U$ for some $\beta \in [0,\alpha)$. Since $\lim_{\beta\to\alpha^-} d((x,\beta),(x,\alpha)) = 0$, the closedness of U implies $(x,\alpha) \in U$. Hence $x \in \langle U \rangle_\alpha$. Thus $\langle U \rangle_\alpha \supseteq \cup_{\beta<\alpha} \langle U \rangle_\beta$ by the arbitrariness of x in $\cup_{\beta<\alpha} \langle U \rangle_\beta$. Therefore (ii) $\Rightarrow$ (i).

Proposition 4.6. Let $U \in C(X \times [0,1])$. Then $U \in F_{USC}(X)\hat{}$ if and only if U has the following properties: (i) For each α, β with $0 \le \beta < \alpha \le 1$, $\langle U \rangle_\alpha \subseteq \langle U \rangle_\beta$; (ii) $\langle U \rangle_0 = X$.

Proof. Since $U \in C(X \times [0,1])$, we have $\langle U \rangle_\alpha \in C(X) \cup \{\emptyset\}$ for all $\alpha \in [0,1]$. Thus the desired result follows immediately from Propositions 4.4 and 4.5.

As a shorthand, we denote the sequence $x_1, x_2, \dots, x_n, \dots$ by $\{x_n\}$.

Proposition 4.7. $F_{USC}(X)\hat{}$ is a closed subset of $(C(X \times [0,1]), H)$.

Proof. Let $\{\hat{u}_n : n = 1, 2, \dots\}$ be a sequence in $F_{USC}(X)\hat{}$ with $\{\hat{u}_n\}$ converging to U in $(C(X \times [0,1]), H)$. To prove the desired result, we need to show $U \in F_{USC}(X)\hat{}$.

We claim that: (i) For each α, β with $0 \le \beta < \alpha \le 1$, $\langle U \rangle_\alpha \subseteq \langle U \rangle_\beta$; (ii) $\langle U \rangle_0 = X$.

To show (i), let $\alpha, \beta \in [0,1]$ with $\beta < \alpha$, and let $x \in \langle U \rangle_\alpha$, i.e., $(x,\alpha) \in U$. By Theorem 2.3, $\lim^{(K)}_{n\to\infty} \hat{u}_n = U$. Then there exists a sequence $\{(x_n,\alpha_n)\}$ with $(x_n,\alpha_n) \in \hat{u}_n$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} d((x_n,\alpha_n),(x,\alpha)) = 0$. Hence there exists N such that $\alpha_n > \beta$ for all $n \ge N$. Thus $(x_n,\beta) \in \hat{u}_n$ for all $n \ge N$. Since $\lim_{n\to\infty} d((x_n,\beta),(x,\beta)) = 0$, we have $(x,\beta) \in \liminf_{n\to\infty} \hat{u}_n = U$. This means $x \in \langle U \rangle_\beta$. So (i) holds.

Clearly $\langle U \rangle_0 \subseteq X$. From $\lim^{(K)}_{n\to\infty} \hat{u}_n = U$ and $\langle \hat{u}_n \rangle_0 = X$, we have $\langle U \rangle_0 \supseteq X$. Thus $\langle U \rangle_0 = X$. So (ii) holds.

By Proposition 4.6, (i) and (ii) imply $U \in F_{USC}(X)\hat{}$.

Remark 4.8. From Proposition 5.1, we can deduce that $F^1_{USC}(X)$ is a closed subset of $(F_{USC}(X), H_{\text{end}})$. Then by Proposition 4.7, $F^1_{USC}(X)$ is a closed subset of $(C(X \times [0,1]), H)$.

Let $(\widetilde{X}, \tilde{d})$ denote the completion of (X, d). We view (X, d) as a subspace of $(\widetilde{X}, \tilde{d})$.

When no confusion arises, we also use H to denote the Hausdorff metric on $C(\widetilde{X})$ induced by $\tilde{d}$, and on $C(\widetilde{X} \times [0,1])$ induced by $\tilde{d}$. We also use H_{end} to denote the endograph metric on $F_{USC}(\widetilde{X})$ defined using H on $C(\widetilde{X} \times [0,1])$.

$F(X)$ can be naturally embedded into $F(\widetilde{X})$. An embedding $j : F(X) \to F(\widetilde{X})$ is defined as follows. For $u \in F(X)$, define $j(u) \in F(\widetilde{X})$ by

$$j(u)(t) = \begin{cases} u(t), & t \in X, \\ 0, & t \in \widetilde{X} \setminus X. \end{cases}$$

Let $U \subseteq X$. If U is compact in (X, d), then U is compact in $(\widetilde{X}, \tilde{d})$. Thus if $u \in F_{USCG}(X)$, then $j(u) \in F_{USCG}(\widetilde{X})$ because $[j(u)]_\alpha = [u]_\alpha \subseteq K(\widetilde{X}) \cup \{\emptyset\}$ for each $\alpha \in (0,1]$.

We see that for $u, v \in F_{USCG}(X)$, $H_{\text{end}}(u, v) = H_{\text{end}}(j(u), j(v))$. So $j|_{F_{USCG}(X)}$ is an isometric embedding of $(F_{USCG}(X), H_{\text{end}})$ into $(F_{USCG}(\widetilde{X}), H_{\text{end}})$.

Since $(F_{USCG}(X), H_{\text{end}})$ can be embedded isometrically into $(F_{USCG}(\widetilde{X}), H_{\text{end}})$, in what follows we treat $(F_{USCG}(X), H_{\text{end}})$ as a metric subspace of $(F_{USCG}(\widetilde{X}), H_{\text{end}})$ by identifying $u \in F_{USCG}(X)$ with $j(u) \in F_{USCG}(\widetilde{X})$. Thus a subset $\mathcal{U}$ of $F_{USCG}(X)$ can be viewed as a subset of $F_{USCG}(\widetilde{X})$.

Suppose $\mathcal{U}$ is a subset of $F_{USC}(X)$ and $\alpha \in [0,1]$. For writing convenience, we denote: - $\mathcal{U}(\alpha) := \cup_{u \in \mathcal{U}} [u]_\alpha$, - $\mathcal{U}_\alpha := \{[u]_\alpha : u \in \mathcal{U}\}$.

We note that an empty union is $\emptyset$.

Lemma 4.9. Let $\mathcal{U}$ be a subset of $F_{USCG}(X)$. If $\mathcal{U}$ is totally bounded in $(F_{USCG}(X), H_{\text{end}})$, then $\mathcal{U}(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0,1]$.

Proof. The proof is similar to the necessity part of Theorem 7.8 in [19].

Let $\alpha \in (0,1]$. To show $\mathcal{U}(\alpha)$ is totally bounded in X, we need to show that every sequence in $\mathcal{U}(\alpha)$ has a Cauchy subsequence.

Let $\{x_n\}$ be a sequence in $\mathcal{U}(\alpha)$. Then there exists a sequence $\{u_n\}$ in $\mathcal{U}$ with $x_n \in [u_n]_\alpha$ for $n = 1, 2, \ldots$. Since $\mathcal{U}$ is totally bounded in $(F_{USCG}(X), H_{\text{end}})$, $\{u_n\}$ has a Cauchy subsequence $\{u_{n_l}\}$ in $(F_{USCG}(X), H_{\text{end}})$. Thus for any $\varepsilon \in (0, \alpha)$, there exists $K(\varepsilon) \in \mathbb{N}$ such that $H_{\text{end}}(u_{n_l}, u_{n_K}) < \varepsilon$ for all $l \geq K$. Consequently, $H^*([u_{n_l}]_\alpha, [u_{n_K}]_{\alpha-\varepsilon}) < \varepsilon$ for all $l \geq K$. From this and the arbitrariness of ε, $\cup_{l=1}^{\infty} [u_{n_l}]_\alpha$ is totally bounded in (X, d). Thus $\{x_{n_l}\}$, which is a subsequence of $\{x_n\}$, has a Cauchy subsequence, and so does $\{x_n\}$.

Remark 4.10. It is easy to see that for a totally bounded set $\mathcal{U}$ in $(F_{USCG}(X), H_{\mathrm{end}})$ and $\alpha \in (0,1]$, $\mathcal{U}(\alpha) = \emptyset$ is possible even if $\mathcal{U} \neq \emptyset$.

For $D \subseteq X \times [0,1]$ and $\alpha \in [0,1]$, define $\langle D \rangle_\alpha := \{x : (x,\alpha) \in D\}$.

Let $u \in F(X)$ and $0 \leq r \leq t \leq 1$. We use the symbol $\mathrm{end}_t^r\, u$ to denote the subset of $\mathrm{end}\, u$ given by

$$\mathrm{end}_t^r\, u := \mathrm{end}\, u \cap ([u]_r \times [r,t]).$$

For simplicity, we write $\mathrm{end}_1^r\, u$ as $\mathrm{end}_r\, u$. We see that $\mathrm{end}_0\, u = \mathrm{send}\, u$.

Theorem 4.11. Let $\mathcal{U}$ be a subset of $F_{USCG}(X)$. Then $\mathcal{U}$ is relatively compact in $(F_{USCG}(X), H_{\mathrm{end}})$ if and only if $\mathcal{U}(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,1]$.

Proof. *Necessity.* Suppose $\mathcal{U}$ is relatively compact in $(F_{USCG}(X), H_{\mathrm{end}})$. Let $\alpha \in (0,1]$. By Lemma 4.9, $\mathcal{U}(\alpha)$ is totally bounded, hence relatively compact in $(\widetilde{X}, \tilde{d})$.

To show $\mathcal{U}(\alpha)$ is relatively compact in (X,d), we proceed by contradiction. If not, there exists a sequence $\{x_n\}$ in $\mathcal{U}(\alpha)$ such that $\{x_n\}$ converges to $x \in \widetilde{X} \setminus X$ in $(\widetilde{X}, \tilde{d})$.

Assume $x_n \in [u_n]_\alpha$ with $u_n \in \mathcal{U}$ for $n = 1, 2, \ldots$. From the relative compactness of $\mathcal{U}$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges to $u \in F_{USCG}(X)$. Since $F_{USCG}(X)$ can be viewed as a subspace of $F_{USCG}(\widetilde{X})$, we have $\{u_{n_k}\}$ converges to u in $(F_{USCG}(\widetilde{X}), H_{\mathrm{end}})$. By Theorem 2.3, $\lim_{k\to\infty}^{(K)} \hat{u}_{n_k} = \hat{u}$ according to $(\widetilde{X} \times [0,1], \tilde{d})$. Notice that $(x_{n_k}, \alpha) \in \hat{u}_{n_k}$ for $k = 1, 2, \ldots$, and $\{(x_{n_k}, \alpha)\}$ converges to (x, α) in $(\widetilde{X} \times [0,1], \tilde{d})$. Thus $(x,\alpha) \in \hat{u}$, contradicting $x \in \widetilde{X} \setminus X$.

The necessity part of Theorem 7.10 in [19] can be verified similarly to the necessity part of this theorem.

Sufficiency. Suppose $\mathcal{U}(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,1]$. To show $\mathcal{U}$ is relatively compact in $(F_{USCG}(X), H_{\mathrm{end}})$, we need to show that every sequence in $\mathcal{U}$ has a convergent subsequence in $(F_{USCG}(X), H_{\mathrm{end}})$.

Let $\{u_n\}$ be a sequence in $\mathcal{U}$. If $\liminf_{n\to\infty} S_{u_n} = 0$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $H_{\mathrm{end}}(u_{n_k}, \emptyset_{F(X)}) \to 0$ as $k \to \infty$. Since $\emptyset_{F(X)} \in F_{USCG}(X)$, $\{u_{n_k}\}$ is a convergent subsequence in $(F_{USCG}(X), H_{\mathrm{end}})$.

If $\liminf_{n\to\infty} S_{u_n} > 0$, then there exists $\xi > 0$ and $N \in \mathbb{N}$ such that $[u_n]_\xi \neq \emptyset$ for all $n \geq N$.

We first claim the following property: **(a)** Let $\alpha \in (0,1]$ and S be a subset of $\mathcal{U}$ with $[u]_\alpha \neq \emptyset$ for each $u \in S$. Then $\{\mathrm{end}_\alpha\, u : u \in S\}$ is a relatively compact set in $(K(X \times [\alpha, 1]), H)$.

Indeed, for each $u \in S$, $\text{end}_\alpha u \in K(X \times [\alpha, 1])$. Since $\mathcal{U}(\alpha)$ is relatively compact in (X, d), $\mathcal{U}(\alpha) \times [\alpha, 1]$ is relatively compact in $(X \times [\alpha, 1], d)$. As $\cup_{u \in S} \text{end}_\alpha u \subseteq \mathcal{U}(\alpha) \times [\alpha, 1]$, the set $\cup_{u\in S} \text{end}_\alpha u$ is also relatively compact in $(X \times [\alpha, 1], d)$. Thus by Theorem 4.2, $\{\text{end}_\alpha u : u \in S\}$ is relatively compact in $(K(X \times [\alpha, 1]), H)$. So assertion (a) holds.

Take a sequence $\{\alpha_k, k = 1, 2, ...\}$ satisfying $0 < \alpha_{k+1} < \alpha_k \leq \min\{\xi, 1/k\}$ for $k = 1, 2, ...$ We see that $\alpha_k \to 0$ as $k \to \infty$.

By assertion (a), $\{\text{end}_{\alpha_1} u_n : n = N, N+1, ...\}$ is relatively compact in $(K(X \times [\alpha_1, 1]), H)$. So there exists a subsequence $\{u_n^{(1)}\}$ of $\{u_n : n \geq N\}$ and $v_1 \in K(X \times [\alpha_1, 1])$ such that $H(\text{end}_{\alpha_1} u_n^{(1)}, v_1) \to 0$. Clearly $\{u_n^{(1)}\}$ is also a subsequence of $\{u_n\}$.

Using assertion (a) again, $\{\text{end}_{\alpha_2} u_n^{(1)}\}$ is relatively compact in $(K(X \times [\alpha_2, 1]), H)$. So there exists a subsequence $\{u_n^{(2)}\}$ of $\{u_n^{(1)}\}$ and $v_2 \in K(X \times [\alpha_2, 1])$ such that $H(\text{end}_{\alpha_2} u_n^{(2)}, v_2) \to 0$.

Repeating this procedure, we obtain sequences $\{u_n^{(k)}\}$ and sets $v_k \in K(X \times [\alpha_k, 1])$ for $k = 1, 2, ...$, such that for each $k = 1, 2, ...$, $\{u_n^{(k+1)}\}$ is a subsequence of $\{u_n^{(k)}\}$ and $H(\text{end}_{\alpha_k} u_n^{(k)}, v_k) \to 0$.

We claim that: **(b)** Let $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$. Then: (i) $\langle v_{k_1} \rangle_{\alpha_{k_1}} \subseteq \langle v_{k_2} \rangle_{\alpha_{k_1}}$; (ii) $\langle v_{k_1} \rangle_\alpha = \langle v_{k_2} \rangle_\alpha$ when $\alpha \in (\alpha_{k_1}, 1]$; (iii) $v_k \subseteq v_{k+1}$ for $k = 1, 2, ...$

Note that $\{u_n^{(k_2)}\}$ is a subsequence of $\{u_n^{(k_1)}\}$ and $\alpha_{k_2} < \alpha_{k_1}$. Thus by Theorem 2.3, for each $\alpha \in [\alpha_{k_1}, 1]$,

$$\langle v_{k_1} \rangle_\alpha \times \{\alpha\} = \liminf_{n\to\infty} (\text{end}_{\alpha_{k_1}} u_n^{(k_1)} \cap (X \times \{\alpha\})) \subseteq \liminf_{n\to\infty} (\text{end}_{\alpha_{k_2}} u_n^{(k_2)} \cap (X \times \{\alpha\})) = \langle v_{k_2} \rangle_\alpha \times \{\alpha\}.$$

So (i) holds.

Let $\alpha \in [0, 1]$ with $\alpha > \alpha_{k_1}$. Observe that if a sequence $\{(x_m, \beta_m)\}$ converges to (x, α) as $m \to \infty$ in $(X \times [0, 1], d)$, then there exists M such that for all $m > M$, $\beta_m > \alpha_{k_1}$, i.e., $(x_m, \beta_m) \in X \times (\alpha_{k_1}, 1]$. Thus by Theorem 2.3, for each $\alpha \in (\alpha_{k_1}, 1]$,

$$\langle v_{k_1} \rangle_\alpha \times \{\alpha\} = \limsup_{n\to\infty} (\text{end}_{\alpha_{k_1}} u_n^{(k_1)} \cap (X \times \{\alpha\})) \supseteq \limsup_{n\to\infty} (\text{end}_{\alpha_{k_2}} u_n^{(k_2)} \cap (X \times \{\alpha\})) = \langle v_{k_2} \rangle_\alpha \times \{\alpha\}.$$

Hence by the previous inclusion and this one, $\langle v_{k_1} \rangle_\alpha = \langle v_{k_2} \rangle_\alpha$ for $\alpha \in (\alpha_{k_1}, 1]$. So (ii) holds. (iii) follows immediately from (i) and (ii).

Define a subset v of $X \times [0, 1]$ by

$$v = \cup_{k=1}^\infty v_k \cup (X \times \{0\}).$$

From assertion (b), we see that

$$\langle v\rangle_\alpha = \begin{cases} \langle v_k\rangle_\alpha, & \text{if for some } k\in\mathbb{N}, \alpha>\alpha_k, \\ X, & \text{if } \alpha=0, \end{cases}$$

and hence $v\cap(X\times(\alpha_k,1]) = v_k\cap(X\times(\alpha_k,1])\subseteq v_k$.

We show that $v\in C(X\times[0,1])$. Let $\{(x_l,\gamma_l)\}$ be a sequence in v converging to (x,γ) in $X\times[0,1]$. If $\gamma=0$, then clearly $(x,\gamma)\in v$. If $\gamma>0$, then there exists $k_0\in\mathbb{N}$ such that $\gamma>\alpha_{k_0}$. Hence there exists L such that $\gamma_l>\alpha_{k_0}$ when $l\geq L$. So by the previous observation, $(x_l,\gamma_l)\in v_{k_0}$ when $l\geq L$. Since $v_{k_0}\in K(X\times[\alpha_{k_0},1])$, it follows that $(x,\gamma)\in v_{k_0}\subset v$.

We claim that: **(c)** $\lim_{n\to\infty} H(\mathrm{end}\, u_n^{(n)}, v)=0$ and $v\in F_{USCG}(X)\hat{}$.

Let $n\in\mathbb{N}$ and $k\in\mathbb{N}$. Then by the definition of v,

$$H^*(\mathrm{end}\, u_n^{(n)}, v) = \max\{H^*(\mathrm{end}_{\alpha_k} u_n^{(n)}, v), H^*(\mathrm{end}_{\alpha_k} u_n^{(n)}, v)\} \leq \max\{H^*(\mathrm{end}_{\alpha_k} u_n^{(n)}, v_k), \alpha_k\},$$

and by the structure of v,

$$H^*(v, \mathrm{end}\, u_n^{(n)}) = \max\{\sup_{(x,\gamma)\in v\cap(X\times(\alpha_k,1])} d((x,\gamma), \mathrm{end}\, u_n^{(n)}), H^*(v\cap(X\times[0,\alpha_k]), \mathrm{end}\, u_n^{(n)})\} \leq \max\{H^*(v_k, \mathrm{end}_{\alpha_k} u$$

These inequalities imply $H(\mathrm{end}\, u_n^{(n)}, v)\leq \max\{H(\mathrm{end}_{\alpha_k} u_n^{(n)}, v_k), \alpha_k\}$.

Now we show $\lim_{n\to\infty} H(\mathrm{end}\, u_n^{(n)}, v)=0$. Let $\varepsilon>0$. Since $\alpha_k\to 0$ and for each α_k, $\lim_{n\to\infty} H(\mathrm{end}_{\alpha_k} u_n^{(n)}, v_k)=0$, there exists α_{k_0} and $N\in\mathbb{N}$ such that $\alpha_{k_0}<\varepsilon$ and $H(\mathrm{end}_{\alpha_{k_0}} u_n^{(n)}, v_{k_0})<\varepsilon$ for all $n\geq N$. Thus by the inequality above, $H(\mathrm{end}\, u_n^{(n)}, v)<\varepsilon$ for all $n\geq N$. So $\lim_{n\to\infty} H(\mathrm{end}\, u_n^{(n)}, v)=0$.

Since the sequence $\{\mathrm{end}\, u_n^{(n)}\}$ lies in $F_{USC}(X)\hat{}$ and converges to v in $(C(X\times[0,1]), H)$, Proposition 4.7 implies $v\in F_{USC}(X)\hat{}$.

Let $k\in\mathbb{N}$. Then $v_k\in K(X\times[\alpha_k,1])$, and hence $\langle v_k\rangle_\alpha\in K(X)\cup\{\emptyset\}$ for all $\alpha\in[0,1]$. So from the structure of v, $\langle v\rangle_\alpha\in K(X)\cup\{\emptyset\}$ for all $\alpha\in(0,1]$, and thus $v\in F_{USCG}(X)\hat{}$.

From assertion (c), $\{u_n^{(n)}\}$ is a convergent sequence in $(F_{USCG}(X), H_{\mathrm{end}})$. Note that $\{u_n^{(n)}\}$ is a subsequence of $\{u_n\}$. Thus the proof is complete.

Theorem 4.12. Let $\mathcal{U}$ be a subset of $F_{USCG}(X)$. Then $\mathcal{U}$ is totally bounded in $(F_{USCG}(X), H_{\mathrm{end}})$ if and only if $\mathcal{U}(\alpha)$ is totally bounded in (X,d) for each $\alpha\in(0,1]$.

Proof. *Necessity.* This is Lemma 4.9.

Sufficiency. Suppose $\mathcal{U}(\alpha)$ is totally bounded in (X,d) for each $\alpha\in(0,1]$. Then $\mathcal{U}(\alpha)$ is relatively compact in $(\widetilde{X},\tilde{d})$ for each $\alpha\in(0,1]$. Thus by Theorem 4.11,

$\mathcal{U}$ is relatively compact in $(F_{USCG}(\widetilde{X}), H_{\text{end}})$. Hence $\mathcal{U}$ is totally bounded in $(F_{USCG}(\widetilde{X}), H_{\text{end}})$. Therefore $\mathcal{U}$ is totally bounded in $(F_{USCG}(X), H_{\text{end}})$.

Theorem 4.13. Let $\mathcal{U}$ be a subset of $F_{USCG}(X)$. Then the following are equivalent: (i) $\mathcal{U}$ is compact in $(F_{USCG}(X), H_{\text{end}})$; (ii) $\mathcal{U}(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$ and $\mathcal{U}$ is closed in $(F_{USCG}(X), H_{\text{end}})$; (iii) $\mathcal{U}(\alpha)$ is compact in (X, d) for each $\alpha \in (0, 1]$ and $\mathcal{U}$ is closed in $(F_{USCG}(X), H_{\text{end}})$.

Proof. By Theorem 4.11, (i) $\Leftrightarrow$ (ii). Obviously (iii) $\Rightarrow$ (ii). We complete the proof by showing (ii) $\Rightarrow$ (iii). Suppose (ii) holds. To verify (iii), it suffices to show $\mathcal{U}(\alpha)$ is closed in (X, d) for each $\alpha \in (0, 1]$.

Let $\alpha \in (0, 1]$ and let $\{x_n\}$ be a sequence in $\mathcal{U}(\alpha)$ converging to x in (X, d). We need to show $x \in \mathcal{U}(\alpha)$.

Choose a sequence $\{u_n\}$ in $\mathcal{U}$ such that $x_n \in [u_n]_\alpha$ for $n = 1, 2, ...$, meaning $(x_n, \alpha) \in \text{end}\, u_n$ for each n.

From the equivalence of (i) and (ii), $\mathcal{U}$ is compact in $(F_{USCG}(X), H_{\text{end}})$. Thus there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in \mathcal{U}$ such that $H_{\text{end}}(u_{n_k}, u) \to 0$. Hence by Remark 2.4, $\lim^{(\Gamma)}_{k\to\infty} u_{n_k} = u$. Note that $(x, \alpha) = \lim_{k\to\infty}(x_{n_k}, \alpha)$. Since $(x_{n_k}, \alpha) \in \text{end}\, u_{n_k}$ and $\text{end}\, u = \liminf_{k\to\infty} \text{end}\, u_{n_k}$, we have $(x, \alpha) \in \text{end}\, u$. Thus $x \in [u]_\alpha$, and therefore $x \in \mathcal{U}(\alpha)$.

It can be seen that Theorem 7.11 in [19] can be verified in a manner similar to this theorem.

We [18] gave the following characterizations of compactness in $(F_{USCG}(\mathbb{R}^m), H_{\text{end}})$:

Theorem 4.14. (Theorem 7.1 in [18]) Let $\mathcal{U}$ be a subset of $F_{USCG}(\mathbb{R}^m)$. Then $\mathcal{U}$ is relatively compact in $(F_{USCG}(\mathbb{R}^m), H_{\text{end}})$ if and only if $\mathcal{U}(\alpha)$ is bounded in $\mathbb{R}^m$ for each $\alpha \in (0, 1]$.

Theorem 4.15. (Theorem 7.3 in [18]) Let $\mathcal{U}$ be a subset of $F_{USCG}(\mathbb{R}^m)$. Then $\mathcal{U}$ is totally bounded in $(F_{USCG}(\mathbb{R}^m), H_{\text{end}})$ if and only if for each $\alpha \in (0, 1]$, $\mathcal{U}(\alpha)$ is bounded in $\mathbb{R}^m$.

Theorem 4.16. (Theorem 7.2 in [18]) $\mathcal{U}$ is compact in $(F_{USCG}(\mathbb{R}^m), H_{\text{end}})$ if and only if $\mathcal{U}$ is closed in $(F_{USCG}(\mathbb{R}^m), H_{\text{end}})$ and $\mathcal{U}(\alpha)$ is bounded in $\mathbb{R}^m$ for each $\alpha \in (0, 1]$.

For a set S in $\mathbb{R}^m$, the following properties are equivalent: (i) S is bounded in $\mathbb{R}^m$; (ii) S is totally bounded in $\mathbb{R}^m$; (iii) S is relatively compact in $\mathbb{R}^m$.

Using this well-known fact, we see that Theorem 4.11 implies Theorem 4.14; Theorem 4.12 implies Theorem 4.15; and Theorem 4.13 implies Theorem 4.16.

Thus, the characterizations of relative compactness, total boundedness, and compactness in $(F_{USCG}(\mathbb{R}^m), H_{\text{end}})$ given in our previous work [18] are corollaries of the corresponding characterizations for $(F_{USCG}(X), H_{\text{end}})$ established in this section.

Furthermore, the characterizations for $(F_{USCG}(X), H_{\text{end}})$ given in this section illustrate the relationship between relative compactness, total boundedness, and compactness of a set in $F_{USCG}(X)$ and that of the union of its elements' α-cuts. From these discussions, we see that the characterizations for $(F_{USCG}(X), H_{\text{end}})$ significantly improve upon those for $(F_{USCG}(\mathbb{R}^m), H_{\text{end}})$ given in [18].

Remark 4.17. The following clauses (i) and (ii) are pointed out in Remark 5.1 of chinaXiv:202107.00011v2 (submitted on 2021-07-22): (i) $(F^1_{USC}(X), H_{\text{end}})$ can be treated as a subspace of $(C(X \times [0,1]), H)$ by viewing each $u \in F^1_{USC}(X)$ as its endograph; (ii) We can discuss properties of $(F^1_{USC}(X), H_{\text{end}})$ as a subspace of $(C(X \times [0,1]), H)$, including characterizations of total boundedness, relative compactness, and compactness.

In this paper, we treat $(F_{USC}(X), H_{\text{end}})$ as a subspace of $(C(X \times [0,1]), H)$ to discuss properties of $(F_{USC}(X), H_{\text{end}})$.

At the end of this section, we illustrate that Theorems 4.2, 4.1, and 4.3 can be seen as special cases of Theorems 4.11, 4.12, and 4.13, respectively.

We begin with some propositions. The following follows immediately from basic definitions.

Proposition 4.18. Let A be a subset of X. (i) The conditions (i-1) $A \in C(X)$ and (i-2) $\chi_A \in F_{USC}(X)$ are equivalent. (ii) The conditions (ii-1) $A \in K(X)$, (ii-2) $\chi_A \in F_{USCB}(X)$, and (ii-3) $\chi_A \in F_{USCG}(X)$ are equivalent.

Let $\mathcal{D} \subseteq P(X)$. We use the symbol $\mathcal{D}_F(X)$ to denote the set $\{C_F(X) : C \in \mathcal{D}\}$.

Let $A, B \in C(X)$. Then

$$H_{\text{end}}(\chi_A, \chi_B) = \min\{H(A, B), 1\}.$$

Proposition 4.19. Let $\mathcal{D}$ be a subset of $K(X)$. (i) $\mathcal{D}$ is totally bounded in $(K(X), H)$ if and only if $\mathcal{D}_F(X)$ is totally bounded in $(F_{USCG}(X), H_{\text{end}})$; (ii) $\mathcal{D}$ is compact in $(K(X), H)$ if and only if $\mathcal{D}_F(X)$ is compact in $(F_{USCG}(X), H_{\text{end}})$.

Proof. From the formula above, (i) follows immediately.

By the same formula, $\mathcal{D}$ is compact in $(K(X), H)$ if and only if $\mathcal{D}_F(X)$ is compact in $(K(X)_F(X), H_{\text{end}})$. Clearly $\mathcal{D}_F(X)$ is compact in $(K(X)_F(X), H_{\text{end}})$ if and only if $\mathcal{D}_F(X)$ is compact in $(F_{USCG}(X), H_{\text{end}})$. So (ii) holds.

Proposition 4.20. Let $\{A_n\}$ be a sequence of sets in $C(X)$. If $\{\chi_{A_n}\}$ converges to a fuzzy set u in $F_{USC}(X)$ according to the H_{end} metric, then there exists $A \in C(X)$ such that $u = \chi_A$ and $H(A_n, A) \to 0$ as $n \to \infty$.

Proof. We establish the following properties in turn: (i) For $x \in X$ and $\alpha, \beta \in (0,1]$, $(x, \alpha) \in \text{end}\, u$ if and only if $(x, \beta) \in \text{end}\, u$; (ii) $[u]_\alpha = [u]_\beta$ for all $\alpha, \beta \in [0,1]$; (iii) There exists $A \in C(X)$ such that $u = \chi_A$ and $H(A_n, A) \to 0$ as $n \to \infty$.

To show (i), we only need to prove that if $(x, \alpha) \in \operatorname{end} u$ then $(x, \beta) \in \operatorname{end} u$, since α and β can be interchanged.

Assume $(x, \alpha) \in \operatorname{end} u$. Since $H_{\text{end}}(\chi_{A_n}, u) \to 0$, by Theorem 2.3 and Remark 2.4, $\lim^{(\Gamma)}_{n\to\infty} \chi_{A_n} = u$. Then there exists a sequence $\{(x_n, \alpha_n)\}$ such that $(x_n, \alpha_n) \in \operatorname{end} \chi_{A_n}$ for $n = 1, 2, ...$, and $\lim_{n\to\infty} d((x_n, \alpha_n), (x, \alpha)) = 0$.

Since $\alpha > 0$, there exists N such that $\alpha_n > 0$ for all $n \geq N$. This yields $(x_n, \alpha_n) \in \operatorname{send} \chi_{A_n} = A_n \times [0, 1]$ for all $n \geq N$. Hence $(x_n, \beta) \in \operatorname{send} \chi_{A_n}$ for all $n \geq N$. Observe that $\lim_{n\to\infty} d((x_n, \beta), (x, \beta)) = 0$, i.e., $\{(x_n, \beta) : n \geq N\}$ converges to (x, β) in $(X \times [0, 1], d)$. Thus $(x, \beta) \in \liminf_{n\to\infty} \operatorname{end} \chi_{A_n} = \operatorname{end} u$. So (i) holds.

From (i), we have $[u]_\alpha = [u]_\beta$ for all $\alpha, \beta \in (0, 1]$. Then $[u]_0 = \cup_{\alpha>0}[u]_\alpha = [u]_1$. So (ii) holds.

Set $A = [u]_1$. By Proposition 5.1, $u \in F^1_{USC}(X)$. From this and (ii), it follows that $A \in C(X)$ and $u = \chi_A$.

Since $H_{\text{end}}(\chi_{A_n}, u) = H_{\text{end}}(\chi_{A_n}, \chi_A) = \min\{H(A_n, A), 1\} \to 0$ as $n \to \infty$, we obtain $H(A_n, A) \to 0$ as $n \to \infty$. So (iii) holds. This completes the proof.

Proposition 4.21. Let $\{x_n\}$ be a sequence in X. If $\{\hat{x}_n\}$ converges to a fuzzy set u in $F_{USC}(X)$ according to the H_{end} metric, then there exists $x \in X$ such that $u = \hat{x}$ and $d(x_n, x) \to 0$ as $n \to \infty$.

Proof. Note that $\hat{z} = \chi_{\{z\}}$ for each $z \in X$. Thus by Proposition 4.20, there exists $A \in C(X)$ such that $u = \chi_A$ and $H(\{x_n\}, A) \to 0$ as $n \to \infty$. Since $\lim^{(K)}_{n\to\infty}\{x_n\} = A$, it follows that A is a singleton. Set $A = \{x\}$. Then $u = \hat{x}$ and $d(x_n, x) = H(\{x_n\}, \{x\}) \to 0$ as $n \to \infty$. This completes the proof.

Proposition 4.21 is Proposition 8.15 in our paper arXiv:submit/4644498. It can also be shown directly using the idea in the proof of Proposition 4.20.

Using the idea in the proof of Proposition 8.15 in arXiv:submit/4644498, we can show that A in the proof of Proposition 4.21 is a singleton as follows. Assume A has at least two distinct elements. Pick $p, q \in A$ with $p \neq q$. For any $z \in X$, since $d(p, z) + d(q, z) \geq d(p, q)$, it follows that $\max\{d(p, z), d(q, z)\} \geq \frac{1}{2}d(p, q)$. Thus $H(A, \{x_n\}) = H^*(A, \{x_n\}) \geq \frac{1}{2}d(p, q)$, contradicting $H(A, \{x_n\}) \to 0$.

Proposition 4.22. Let $\mathcal{D}$ be a subset of $K(X)$ and $\mathcal{B}$ a subset of $C(X)$. (i) $C(X)_F(X)$ is closed in $(F_{USC}(X), H_{\text{end}})$; (ii) $K(X)_F(X)$ is closed in $(F_{USCG}(X), H_{\text{end}})$; (iii) $\mathcal{D}$ is closed in $(K(X), H)$ if and only if $\mathcal{D}_F(X)$ is closed in $(F_{USCG}(X), H_{\text{end}})$; (iv) $\mathcal{D}$ is relatively compact in $(K(X), H)$ if and only if $\mathcal{D}_F(X)$ is relatively compact in $(F_{USCG}(X), H_{\text{end}})$; (v) $\mathcal{B}$ is closed in $(C(X), H)$ if and only if $\mathcal{B}_F(X)$ is closed in $(F_{USC}(X), H_{\text{end}})$; (vi) $\mathcal{B}$ is relatively compact in $(C(X), H)$ if and only if $\mathcal{B}_F(X)$ is relatively compact in $(F_{USC}(X), H_{\text{end}})$.

Proof. From Proposition 4.20, we have that (i) is true.

By (i), the closure of $K(X)_F(X)$ in $(F_{USCG}(X), H_{\text{end}})$ is contained in $F_{USCG}(X) \cap C(X)_F(X)$. From Proposition 4.18(ii), $F_{USCG}(X) \cap C(X)_F(X) = K(X)_F(X)$. Thus the closure of $K(X)_F(X)$ in $(F_{USCG}(X), H_{\text{end}})$ is $K(X)_F(X)$. So (ii) is true.

Consider the following conditions: (a-1) $\mathcal{D}$ is closed in $(K(X), H)$; (a-2) $\mathcal{D}_F(X)$ is closed in $(K(X)_F(X), H_{\text{end}})$; (a-3) $\mathcal{D}_F(X)$ is closed in $(F_{USCG}(X), H_{\text{end}})$.

By the formula relating H_{end} and H, (a-1) $\Leftrightarrow$ (a-2). From (ii), (a-2) $\Leftrightarrow$ (a-3). Thus (a-1) $\Leftrightarrow$ (a-3). So (iii) is true.

Consider the conditions: (b-1) $\mathcal{D}$ is relatively compact in $(K(X), H)$; (b-2) $\mathcal{D}_F(X)$ is relatively compact in $(K(X)_F(X), H_{\text{end}})$; (b-3) $\mathcal{D}_F(X)$ is relatively compact in $(F_{USCG}(X), H_{\text{end}})$.

By the formula relating H_{end} and H, (b-1) $\Leftrightarrow$ (b-2). From (ii), (b-2) $\Leftrightarrow$ (b-3). Thus (b-1) $\Leftrightarrow$ (b-3). So (iv) is true.

Using the formula and (i), (v) and (vi) can be proved similarly to (iii) and (iv).

Each subset $\mathcal{D}$ of $(K(X), H)$ corresponds to a subset $\mathcal{D}_F(X)$ of $(F_{USCG}(X), H_{\text{end}})$. Using Theorems 4.11, 4.12, and 4.13, we obtain characterizations of relative compactness, total boundedness, and compactness for $\mathcal{D}_F(X)$ in $(F_{USCG}(X), H_{\text{end}})$ as follows.

Corollary 4.23. Let $\mathcal{D}$ be a subset of $K(X)$. Then $\mathcal{D}_F(X)$ is relatively compact in $(F_{USCG}(X), H_{\text{end}})$ if and only if $\cup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d).

Corollary 4.24. Let $\mathcal{D}$ be a subset of $K(X)$. Then $\mathcal{D}_F(X)$ is totally bounded in $(F_{USCG}(X), H_{\text{end}})$ if and only if $\cup\{C : C \in \mathcal{D}\}$ is totally bounded in (X, d).

Corollary 4.25. Let $\mathcal{D}$ be a subset of $K(X)$. Then the following are equivalent: (i) $\mathcal{D}_F(X)$ is compact in $(F_{USCG}(X), H_{\text{end}})$; (ii) $\cup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d) and $\mathcal{D}_F(X)$ is closed in $(F_{USCG}(X), H_{\text{end}})$; (iii) $\cup\{C : C \in \mathcal{D}\}$ is compact in (X, d) and $\mathcal{D}_F(X)$ is closed in $(F_{USCG}(X), H_{\text{end}})$.

From Proposition 4.19 and clauses (iii) and (iv) of Proposition 4.22, we obtain that Corollaries 4.23, 4.24, and 4.25 are equivalent forms of Theorems 4.2, 4.1, and 4.3, respectively. Thus we can view Theorems 4.2, 4.1, and 4.3 as special cases of Theorems 4.11, 4.12, and 4.13, respectively.

5. Characterizations of Compactness in $(F^r_{USCG}(X), H_{\text{end}})$

This section first investigates properties of the H_{end} metric. Then, based on the characterizations of relative compactness, total boundedness, and compactness in $(F_{USCG}(X), H_{\text{end}})$ from Section 4, we provide characterizations of relatively compact, totally bounded, and compact sets in $(F^r_{USCG}(X), H_{\text{end}})$ for $r \in [0, 1]$. The spaces $(F^r_{USCG}(X), H_{\text{end}})$, $r \in [0, 1]$, are subspaces of $(F_{USCG}(X), H_{\text{end}})$ where each element has maximum value r. The space $(F^1_{USCG}(X), H_{\text{end}})$ is one of these subspaces.

For $D \subseteq X \times [0,1]$, define $S_D := \sup\{\alpha : (x,\alpha) \in D\}$.

We claim that for $D, E \in C(X \times [0,1])$,

$$H(D,E) \geq |S_D - S_E|.$$

To see this, let $D, E \in C(X \times [0,1])$. If $|S_D - S_E| = 0$, the inequality holds trivially. If $|S_D - S_E| > 0$, assume $S_D > S_E$. Note that for each $(x,t) \in D$ with $t > S_E$, $d((x,t),E) \geq t - S_E$. Thus

$$H(D,E) \geq \sup\{t - S_E : (x,t) \in D \text{ with } t > S_E\} = S_D - S_E,$$

proving the inequality.

For $u \in F(X)$, define $S_u := \sup\{u(x) : x \in X\}$. We see that $S_u = S_{\text{end}\, u}$. Clearly $[u]_{S_u} = \emptyset$ is possible.

From the inequality above, we have for $u, v \in F_{USC}(X)$,

$$H_{\text{end}}(u,v) \geq |S_u - S_v|.$$

Proposition 5.1. Let u and u_n, $n = 1, 2, ...$, be fuzzy sets in $F_{USC}(X)$. If $H_{\text{end}}(u_n, u) \to 0$ as $n \to \infty$, then $S_{u_n} \to S_u$ as $n \to \infty$.

Proof. The result follows immediately from the inequality above.

For $u \in F(X)$, $\max\{u(x) : x \in X\}$ may not exist. If $\max\{u(x) : x \in X\}$ exists, then obviously $S_u = \max\{u(x) : x \in X\}$. If $[u]_{S_u} \neq \emptyset$, then since $S_u = \sup\{u(x) : x \in X\}$, it follows that $S_u = \max\{u(x) : x \in X\}$.

Proposition 5.2. (i) Let $u \in F_{USC}(X)$. If there exists $\alpha \in [0, S_u]$ with $[u]_\alpha \in K(X)$, then $[u]_{S_u} \neq \emptyset$ and $S_u = \max\{u(x) : x \in X\}$. (ii) Let $u \in F_{USCG}(X)$. Then $S_u = \max\{u(x) : x \in X\}$.

Proof. First we show (i). If $\alpha = S_u$, then $[u]_{S_u} \neq \emptyset$. If $\alpha < S_u$, pick a sequence $\{x_n\}$ in $[u]_\alpha$ with $u(x_n) \to S_u$. From the compactness of $[u]_\alpha$, there exists a subsequence $\{x_{n_k}\}$ converging to a point $x \in [u]_\alpha$. Thus $u(x) \geq \lim_{k\to\infty} u(x_{n_k}) = S_u$. Hence $u(x) = S_u$, and therefore $[u]_{S_u} \neq \emptyset$ and $S_u = \max\{u(x) : x \in X\}$. So (i) holds.

In the case $\alpha < S_u$, we can also prove $[u]_{S_u} \neq \emptyset$ as follows. Take an increasing sequence $\{\alpha_k\}$ in $[\alpha, 1]$ with $\alpha_k \to S_u^-$. Then $[u]_{\alpha_k} \in K(X)$ for each $k = 1, 2, ...$, and thus $[u]_{S_u} = \cap_{k=1}^\infty [u]_{\alpha_k} \in K(X)$. So $[u]_{S_u} \neq \emptyset$.

Now we show (ii). If $u \in F_{USCG}(X) \setminus \{\emptyset_{F(X)}\}$, then there exists $\alpha \in (0, S_u]$ such that $[u]_\alpha \in K(X)$. So from (i), $S_u = \max\{u(x) : x \in X\}$. If $u = \emptyset_{F(X)}$, then $S_u = 0 = \max\{u(x) : x \in X\}$. Thus for each $u \in F_{USCG}(X)$, $S_u = \max\{u(x) : x \in X\}$.

For $r \in [0,1]$, define:

$$F^r_{USC}(X) = \{u \in F_{USC}(X) : S_u = r\},$$

$$F'^r_{USC}(X) = \{u \in F_{USC}(X) : r = \max\{u(x) : x \in X\}\},$$
$$F^r_{USCG}(X) = \{u \in F_{USCG}(X) : S_u = r\},$$
$$F'^r_{USCG}(X) = \{u \in F_{USCG}(X) : r = \max\{u(x) : x \in X\}\},$$
$$F^r_{USCB}(X) = \{u \in F_{USCB}(X) : S_u = r\},$$
$$F'^r_{USCB}(X) = \{u \in F_{USCB}(X) : r = \max\{u(x) : x \in X\}\}.$$

We see that $F^r_{USCG}(X) = F'^r_{USCG}(X)$ and $F^r_{USCB}(X) = F'^r_{USCB}(X)$. Clearly $F'^0_{USC}(X) = \{\emptyset_{F(X)}\}$.

Proposition 5.3. Let $r \in [0,1]$. Then: (i) $F^r_{USCG}(X) = F'^r_{USCG}(X) = F^r_{USCB}(X) = F'^r_{USCB}(X)$; (ii) $F'^r_{USC}(X)$ is a closed subset of $(F_{USC}(X), H_{\text{end}})$; (iii) $F^r_{USCG}(X)$ is a closed subset of $(F_{USCG}(X), H_{\text{end}})$; (iv) $F^r_{USCB}(X)$ is a closed subset of $(F_{USCB}(X), H_{\text{end}})$.

Proof. From Proposition 5.2(ii) and the fact that $F_{USCB}(X) \subseteq F_{USCG}(X)$, we have $F^r_{USCG}(X) = F'^r_{USCG}(X) = F^r_{USCB}(X) = F'^r_{USCB}(X)$. So (i) holds.

By Proposition 5.1, (ii) holds.

From Proposition 5.1, $F'^r_{USCG}(X)$ is a closed subset of $(F_{USCG}(X), H_{\text{end}})$, and $F'^r_{USCB}(X)$ is a closed subset of $(F_{USCB}(X), H_{\text{end}})$. Then by (i), (iii) and (iv) hold.

Lemma 5.4. Let $r \in [0,1]$ and let $\mathcal{U}$ be a subset of $F^r_{USCG}(X)$. Then the following are equivalent: (i-1) $\mathcal{U}$ is relatively compact in $(F_{USCG}(X), H_{\text{end}})$; (i-2) $\mathcal{U}$ is relatively compact in $(F^r_{USCG}(X), H_{\text{end}})$; (ii-1) $\mathcal{U}$ is closed in $(F_{USCG}(X), H_{\text{end}})$; (ii-2) $\mathcal{U}$ is closed in $(F^r_{USCG}(X), H_{\text{end}})$.

Proof. Clause (iii) of Proposition 5.3 states that $F^r_{USCG}(X)$ is a closed subset of $(F_{USCG}(X), H_{\text{end}})$. From this we obtain (i-1) $\Leftrightarrow$ (i-2) and (ii-1) $\Leftrightarrow$ (ii-2).

In this paper, we assume $(r,r] = \emptyset$ for $r \in \mathbb{R}$.

Lemma 5.5. Let $r \in [0,1]$ and let $\mathcal{U}$ be a subset of $F^r_{USCG}(X)$. Then the following are equivalent: (i-1) $\mathcal{U}(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,1]$; (i-2) $\mathcal{U}(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,r]$; (ii-1) $\mathcal{U}(\alpha)$ is totally bounded in (X,d) for each $\alpha \in (0,1]$; (ii-2) $\mathcal{U}(\alpha)$ is totally bounded in (X,d) for each $\alpha \in (0,r]$; (iii-1) $\mathcal{U}(\alpha)$ is compact in (X,d) for each $\alpha \in (0,1]$; (iii-2) $\mathcal{U}(\alpha)$ is compact in (X,d) for each $\alpha \in (0,r]$.

Proof. Observe that if $\alpha \in (r,1]$ then $\mathcal{U}(\alpha) = \emptyset$. From this we obtain the equivalences (i-1) $\Leftrightarrow$ (i-2), (ii-1) $\Leftrightarrow$ (ii-2), and (iii-1) $\Leftrightarrow$ (iii-2).

Corollary 5.6. Let $r \in [0,1]$ and let $\mathcal{U}$ be a subset of $F^r_{USCG}(X)$. Then the following are equivalent: (i) $\mathcal{U}$ is relatively compact in $(F_{USCG}(X), H_{\text{end}})$; (ii) $\mathcal{U}$ is relatively compact in $(F^r_{USCG}(X), H_{\text{end}})$; (iii) $\mathcal{U}(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,1]$; (iv) $\mathcal{U}(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,r]$.

Proof. By Lemma 5.4, (i) $\Leftrightarrow$ (ii). From this and Theorem 4.11, we obtain (ii) $\Leftrightarrow$ (iii). By Lemma 5.5, (iii) $\Leftrightarrow$ (iv), completing the proof.

Corollary 5.7. Let $r \in [0,1]$ and let $\mathcal{U}$ be a subset of $F^r_{USCG}(X)$. Then the following are equivalent: (i) $\mathcal{U}$ is totally bounded in $(F_{USCG}(X), H_{\text{end}})$; (ii) $\mathcal{U}$ is totally bounded in $(F^r_{USCG}(X), H_{\text{end}})$; (iii) $\mathcal{U}(\alpha)$ is totally bounded in (X,d) for each $\alpha \in (0,1]$; (iv) $\mathcal{U}(\alpha)$ is totally bounded in (X,d) for each $\alpha \in (0,r]$.

Proof. Clearly (i) $\Leftrightarrow$ (ii). From this and Theorem 4.12, we obtain (ii) $\Leftrightarrow$ (iii). By Lemma 5.5, (iii) $\Leftrightarrow$ (iv), completing the proof.

Corollary 5.8. Let $r \in [0,1]$ and let $\mathcal{U}$ be a subset of $F^r_{USCG}(X)$. Then the following are equivalent: (i) $\mathcal{U}$ is compact in $(F_{USCG}(X), H_{\text{end}})$; (ii) $\mathcal{U}$ is compact in $(F^r_{USCG}(X), H_{\text{end}})$; (iii) $\mathcal{U}(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,1]$ and $\mathcal{U}$ is closed in $(F_{USCG}(X), H_{\text{end}})$; (iv) $\mathcal{U}(\alpha)$ is compact in (X,d) for each $\alpha \in (0,1]$ and $\mathcal{U}$ is closed in $(F_{USCG}(X), H_{\text{end}})$; (v) $\mathcal{U}(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,r]$ and $\mathcal{U}$ is closed in $(F^r_{USCG}(X), H_{\text{end}})$; (vi) $\mathcal{U}(\alpha)$ is compact in (X,d) for each $\alpha \in (0,r]$ and $\mathcal{U}$ is closed in $(F^r_{USCG}(X), H_{\text{end}})$.

Proof. Clearly (i) $\Leftrightarrow$ (ii). From this and Theorem 4.13, we obtain (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

By Lemma 5.4, $\mathcal{U}$ is closed in $(F_{USCG}(X), H_{\text{end}})$ if and only if $\mathcal{U}$ is closed in $(F^r_{USCG}(X), H_{\text{end}})$. By Lemma 5.5, $\mathcal{U}(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,1]$ if and only if $\mathcal{U}(\alpha)$ is relatively compact in (X,d) for each $\alpha \in (0,r]$. So (iii) $\Leftrightarrow$ (v).

Similarly, from Lemmas 5.4 and 5.5, we have (iv) $\Leftrightarrow$ (vi).

Thus (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi).

Remark 5.9. From Corollary 5.8 and Lemmas 5.4 and 5.5, the following properties are equivalent: (i) $\mathcal{U}$ is compact in $(F_{USCG}(X), H_{\text{end}})$; (ii) $\mathcal{U}$ is compact in $(F^r_{USCG}(X), H_{\text{end}})$; (iii) At least one of (i-1), (i-2), (iii-1), and (iii-2) in Lemma 5.5 holds, and at least one of (ii-1) and (ii-2) in Lemma 5.4 holds; (iv) All of (i-1), (i-2), (iii-1), and (iii-2) in Lemma 5.5 hold, and all of (ii-1) and (ii-2) in Lemma 5.4 hold.

6. An Application on the Relationship Between H_{end} Metric and Γ-Convergence

As an application of the characterizations of relative compactness, total boundedness, and compactness given in Section 4, we discuss the relationship between the H_{end} metric and Γ-convergence on fuzzy sets.

Proposition 6.1. Let S be a nonempty subset of $F_{USC}(X)$. Let $u \in S$ and $\{u_n\}$ a sequence in S. Then the following are equivalent: (i) $H_{\text{end}}(u_n, u) \to 0$; (ii) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ and $\{u_n, n = 1, 2, ...\}$ is relatively compact in $(F_{USC}(X), H_{\text{end}})$; (iii) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ and $\{u_n, n = 1, 2, ...\}$ is relatively compact in (S, H_{end}); (iv)

$\lim^{(\Gamma)}_{n\to\infty} u_n = u$ and $\{u_n, n = 1, 2, ...\} \cup \{u\}$ is compact in $(F_{USC}(X), H_{\text{end}})$; (v) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ and $\{u_n, n = 1, 2, ...\} \cup \{u\}$ is compact in (S, H_{end}).

Proof. To show (i) ⇒ (v), assume (i) holds. By Theorem 2.3 and Remark 2.4, $\lim^{(\Gamma)}_{n\to\infty} u_n = u$. Clearly $\{u_n, n = 1, 2, ...\} \cup \{u\}$ is compact in $(F_{USC}(X), H_{\text{end}})$. So (v) holds.

It can be seen that $\{u_n, n = 1, 2, ...\} \cup \{u\}$ is compact in $(F_{USC}(X), H_{\text{end}})$ if and only if it is compact in (S, H_{end}). So (v) ⇔ (iv).

If $\{u_n, n = 1, 2, ...\} \cup \{u\}$ is compact in (S, H_{end}), then $\{u_n, n = 1, 2, ...\}$ is relatively compact in (S, H_{end}) because it is a subset of $\{u_n, n = 1, 2, ...\} \cup \{u\}$. So (iv) ⇒ (iii).

Clearly, if $\{u_n, n = 1, 2, ...\}$ is relatively compact in (S, H_{end}), then it is relatively compact in $(F_{USC}(X), H_{\text{end}})$. So (iii) ⇒ (ii).

To show (ii) ⇒ (i), we proceed by contradiction. Assume (ii) holds but (i) fails; that is, $H_{\text{end}}(u_n, u) \not\to 0$. Then there exists $\varepsilon > 0$ and a subsequence $\{v_n^{(1)}\}$ of $\{u_n\}$ such that $H_{\text{end}}(v_n^{(1)}, u) \geq \varepsilon$ for all $n = 1, 2, ...$

Since $\{u_n, n = 1, 2, ...\}$ is relatively compact in $(F_{USC}(X), H_{\text{end}})$, there exists a subsequence $\{v_n^{(2)}\}$ of $\{v_n^{(1)}\}$ and $v \in F_{USC}(X)$ such that $H_{\text{end}}(v_n^{(2)}, v) \to 0$. Hence by Theorem 2.3 and Remark 2.4, $\lim^{(\Gamma)}_{n\to\infty} v_n^{(2)} = v$. Since $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, Remark 2.5 gives $u = v$. Thus $H_{\text{end}}(v_n^{(2)}, u) \to 0$, contradicting the construction of $\{v_n^{(1)}\}$.

Since we have shown (i) ⇒ (v), (v) ⇔ (iv), (iv) ⇒ (iii), (iii) ⇒ (ii), and (ii) ⇒ (i), the proof is complete.

We can also show this theorem as follows. First, prove (i) ⇔ (iii) ⇔ (iv) by verifying (i) ⇒ (iv) ⇒ (iii) ⇒ (i) (the proof of (i) ⇒ (iv) is similar to (i) ⇒ (v), and (iii) ⇒ (i) is similar to (ii) ⇒ (i)). Then setting $S = F_{USC}(X)$, we obtain (i) ⇔ (ii) from (i) ⇔ (iii), and (i) ⇔ (v) from (i) ⇔ (iv). Thus (i), (ii), (iii), (iv), and (v) are all equivalent.

Proposition 6.2. Let $u \in F_{USCG}(X)$ and $\{u_n\}$ a sequence in $F_{USCG}(X)$. Then the following are equivalent: (i) $H_{\text{end}}(u_n, u) \to 0$; (ii) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ and $\cup_{n=1}^{\infty}[u_n]_\alpha$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$; (iii) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, $\cup_{n=1}^{\infty}[u_n]_\alpha \cup [u]_\alpha$ is compact in (X, d) for each $\alpha \in (0, 1]$, and $\{u_n, n = 1, 2, ...\} \cup \{u\}$ is closed in $(F_{USCG}(X), H_{\text{end}})$; (iv) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$, $\cup_{n=1}^{\infty}[u_n]_\alpha \cup [u]_\alpha$ is compact in (X, d) for each $\alpha \in (0, 1]$, and $\{u_n, n = 1, 2, ...\} \cup \{u\}$ is compact in $(F_{USCG}(X), H_{\text{end}})$.

Proof. The result follows from Proposition 6.1, Theorem 4.11, and Theorem 4.13. The proof is routine.

Set $S = F_{USCG}(X)$ in Proposition 6.1. Then we obtain that the following conditions (a), (b), and (c) are equivalent: (a) $H_{\text{end}}(u_n, u) \to 0$; (b) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$

and $\{u_n, n = 1, 2, ...\}$ is relatively compact in $(F_{USCG}(X), H_{\text{end}})$; (c) $\lim^{(\Gamma)}_{n\to\infty} u_n = u$ and $\{u_n, n = 1, 2, ...\} \cup \{u\}$ is compact in $(F_{USCG}(X), H_{\text{end}})$.

Condition (a) is (i). By Theorem 4.11, (b) $\Leftrightarrow$ (ii). By Theorem 4.13, (c) $\Leftrightarrow$ (iv). We see that (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii). So from (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c), we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proposition 6.3. Let $\mathcal{D}$ be a nonempty subset of $C(X)$. Let $A \in \mathcal{D}$ and $\{A_n\}$ a sequence in $\mathcal{D}$. Then the following are equivalent: (i) $H(A_n, A) \to 0$; (ii) $\lim^{(K)}_{n\to\infty} A_n = A$ and $\{A_n, n = 1, 2, ...\}$ is relatively compact in $(C(X), H)$; (iii) $\lim^{(K)}_{n\to\infty} A_n = A$ and $\{A_n, n = 1, 2, ...\}$ is relatively compact in $(\mathcal{D}, H)$; (iv) $\lim^{(K)}_{n\to\infty} A_n = A$ and $\{A_n, n = 1, 2, ...\} \cup \{A\}$ is compact in $(C(X), H)$; (v) $\lim^{(K)}_{n\to\infty} A_n = A$ and $\{A_n, n = 1, 2, ...\} \cup \{A\}$ is compact in $(\mathcal{D}, H)$.

Proof. The proof is similar to that of Proposition 6.1.

Proposition 6.4. Let $A \in K(X)$ and $\{A_n\}$ a sequence in $K(X)$. Then the following are equivalent: (i) $H(A_n, A) \to 0$; (ii) $\lim^{(K)}_{n\to\infty} A_n = A$ and $\cup_{n=1}^{\infty} A_n$ is relatively compact in (X, d); (iii) $\lim^{(K)}_{n\to\infty} A_n = A$, $\cup_{n=1}^{\infty} A_n \cup A$ is compact in (X, d), and $\{A_n, n = 1, 2, ...\}$ is a closed set in $(K(X), H)$; (iv) $\lim^{(K)}_{n\to\infty} A_n = A$, $\cup_{n=1}^{\infty} A_n \cup A$ is compact in (X, d), and $\{A_n, n = 1, 2, ...\} \cup \{A\}$ is compact in $(K(X), H)$.

Proof. The result follows from Proposition 6.3 and Theorems 4.2 and 4.3. The proof is routine and similar to that of Proposition 6.2.

Set $\mathcal{D} = K(X)$ in Proposition 6.3. Then we obtain that the following conditions (a), (b), and (c) are equivalent: (a) $H(A_n, A) \to 0$; (b) $\lim^{(K)}_{n\to\infty} A_n = A$ and $\{A_n, n = 1, 2, ...\}$ is relatively compact in $(K(X), H)$; (c) $\lim^{(K)}_{n\to\infty} A_n = A$ and $\{A_n, n = 1, 2, ...\} \cup \{A\}$ is compact in $(K(X), H)$.

Condition (a) is (i). By Theorem 4.2, (b) $\Leftrightarrow$ (ii). By Theorem 4.3, (c) $\Leftrightarrow$ (iv). We see that (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii). So from (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c), we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Remark 6.5. Let $u \in F_{USCG}(X)$ and $\{u_n\}$ a sequence in $F_{USC}(X)$. Let $\alpha \in (0, 1]$. Since $[u]_\alpha$ is compact in X, the conditions: (a) $\cup_{n=1}^{\infty}[u_n]_\alpha$ is relatively compact in (X, d), and (b) $\cup_{n=1}^{\infty}[u_n]_\alpha \cup [u]_\alpha$ is relatively compact in (X, d), are equivalent. So "$\cup_{n=1}^{\infty}[u_n]_\alpha$ is relatively compact in (X, d)" can be replaced by "$\cup_{n=1}^{\infty}[u_n]_\alpha \cup [u]_\alpha$ is relatively compact in (X, d)" in clause (ii) of Proposition 6.2. Similar replacements can be made in Propositions 6.1, 6.3, and 6.4.

Propositions 6.1, 6.2, 6.3, and 6.4 can be shown in different ways. Below we provide alternative proofs.

Proposition 6.3 implies Proposition 6.1. Let S be a nonempty subset of $F_{USC}(X)$. Let $u \in S$ and $\{u_n\}$ a sequence in S.

Set $A = \text{end}\, u$, and for $n = 1, 2, ...$, set $A_n = \text{end}\, u_n$ in Proposition 6.3. Set

$\mathcal{D} = \{\operatorname{end} u : u \in F_{USC}(X)\}$ in Proposition 6.3. Then from (i) ⇔ (iii) in Proposition 6.3, we obtain (i) ⇔ (ii) in Proposition 6.1.

Set $\mathcal{D} = \{\operatorname{end} u : u \in S\}$ in Proposition 6.3. Then from (i) ⇔ (iii) in Proposition 6.3, we obtain (i) ⇔ (iii) in Proposition 6.1.

Similarly, we can show that (i) ⇔ (iv) and (i) ⇔ (v) in Proposition 6.1 follow from the corresponding equivalences in Proposition 6.3. Thus Proposition 6.3 implies Proposition 6.1.

Proposition 6.3, Theorem 4.11, and Theorem 4.13 imply Proposition 6.2. Let $u \in F_{USCG}(X)$ and $\{u_n\}$ a sequence in $F_{USCG}(X)$.

Set $A = \operatorname{end} u$, and for $n = 1, 2, ...$, set $A_n = \operatorname{end} u_n$ in Proposition 6.3. Set $\mathcal{D} = \{\operatorname{end} u : u \in F_{USCG}(X)\}$ in Proposition 6.3. Then from (i) ⇔ (iii) in Proposition 6.3 and Theorem 4.11, we obtain (i) ⇔ (ii) in Proposition 6.2. Similarly, from (i) ⇔ (iv) in Proposition 6.3 and Theorem 4.13, we obtain (i) ⇔ (iv) in Proposition 6.2. Since (iv) ⇒ (iii) ⇒ (ii) in Proposition 6.2, it follows that (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) in Proposition 6.2. Thus Proposition 6.3, Theorem 4.11, and Theorem 4.13 imply Proposition 6.2.

Proposition 6.1 implies Propositions 6.3 and 6.4. Proposition 6.2 implies Proposition 6.4.

Proposition 6.6. (i) Let $A \in C(X)$ and $\{A_n\}$ a sequence in $C(X)$. Then $H_{\text{end}}(\chi_{A_n}, \chi_A) \to 0$ if and only if $H(A_n, A) \to 0$. (ii) Let $A \in P(X)$ and $\{A_n\}$ a sequence in $P(X)$. Then $\lim_{n\to\infty}^{(\Gamma)} \chi_{A_n} = \chi_A$ if and only if $\lim_{n\to\infty}^{(K)} A_n = A$. (iii) Let $\mathcal{D}$ be a subset of $C(X)$ and $\mathcal{B}$ a subset of $\mathcal{D}$. Then $\mathcal{B}$ is totally bounded (respectively, relatively compact, compact, closed) in $(\mathcal{D}, H)$ if and only if $\mathcal{B}_F(X)$ is totally bounded (respectively, relatively compact, compact, closed) in $(\mathcal{D}_F(X), H_{\text{end}})$.

Proof. (i) and (iii) follow immediately from the formula $H_{\text{end}}(\chi_A, \chi_B) = \min\{H(A, B), 1\}$. (ii) follows from the definitions of Kuratowski convergence and Γ-convergence.

Let $\mathcal{D}$ be a nonempty subset of $C(X)$. Let $A \in C(X)$ and $\{A_n\}$ a sequence in $\mathcal{D}$.

Set $S = C(X)_F(X)$ in Proposition 6.1. Then from (i) ⇔ (iii) in Proposition 6.1, we have: (c-1) $H_{\text{end}}(\chi_{A_n}, \chi_A) \to 0$ if and only if $\lim_{n\to\infty}^{(\Gamma)} \chi_{A_n} = \chi_A$ and $\{\chi_{A_n}, n = 1, 2, ...\}$ is relatively compact in $(C(X)_F(X), H_{\text{end}})$.

By Proposition 6.6, (c-1) means that (i) ⇔ (ii) in Proposition 6.3.

Set $S = \mathcal{D}_F(X)$ in Proposition 6.1. Then from (i) ⇔ (iii) in Proposition 6.1, we have: (c-2) $H_{\text{end}}(\chi_{A_n}, \chi_A) \to 0$ if and only if $\lim_{n\to\infty}^{(\Gamma)} \chi_{A_n} = \chi_A$ and $\{\chi_{A_n}, n = 1, 2, ...\}$ is relatively compact in $(\mathcal{D}_F(X), H_{\text{end}})$.

By Proposition 6.6, (c-2) means that (i) ⇔ (iii) in Proposition 6.3.

Similarly, using Proposition 6.6, we can show that (i) $\Leftrightarrow$ (iv) in Proposition 6.1 implies (i) $\Leftrightarrow$ (iv) and (i) $\Leftrightarrow$ (v) in Proposition 6.3.

By clauses (i) and (ii) of Proposition 6.6 and clause (iii) of Proposition 4.22, we see that Proposition 6.2 implies Proposition 6.4.

Using level characterizations of H_{end} and Γ-convergence on fuzzy sets, it is easy to show that Proposition 6.4 also implies Proposition 6.2.

7. Conclusion

This paper presents characterizations of total boundedness, relative compactness, and compactness in $(F_{USCG}(X), H_{\text{end}})$, where X is a general metric space. Based on these results, we also provide characterizations of total boundedness, relative compactness, and compactness in $(F^r_{USCG}(X), H_{\text{end}})$ for $r \in [0,1]$. The spaces $(F^r_{USCG}(X), H_{\text{end}})$, $r \in [0,1]$, are metric subspaces of $(F_{USCG}(X), H_{\text{end}})$.

The conclusions in this paper significantly improve upon corresponding results in our previous work [18], where we gave characterizations of total boundedness, relative compactness, and compactness in $(F_{USCG}(\mathbb{R}^m), H_{\text{end}})$. The space $\mathbb{R}^m$ is a special type of metric space.

We discuss the relationship between the H_{end} metric and Γ-convergence as an application of the compactness characterizations established in this paper.

The results in this paper have potential applications in research on fuzzy sets involving the endograph metric and Γ-convergence.

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Note: Figure translations are in progress. See original paper for figures.