A Reflective Thought Experiment on Ordinal Utility Theory

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Abstract

This paper designs a thought experiment using the control variable method and reductio ad absurdum to demonstrate that at least one problem renders the internal logic of ordinal utility theory inconsistent—that is, when only ordinal utility information of the article is obtained, many fundamental axioms of ordinal utility theory cannot hold, which can explain why ordinal utility theory cannot be established. More importantly, this experimental method demonstrates the form and properties of actual ordinal ranking, thereby illustrating the viewpoint of this paper from both positive and negative perspectives: that ordinal utility theory cannot be established and that utility is essentially cardinal in nature.

Full Text

Preamble

A Thought Experiment Reconsidering Ordinal Utility Theory

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This paper designs a thought experiment using the control variable method and reduction to absurdity to demonstrate that at least one fundamental problem renders the internal logic of ordinal utility theory inconsistent. Specifically, when only ordinal utility information of alternatives is available, many basic axioms of ordinal utility theory cannot hold, which explains why ordinal utility theory itself cannot be established. More importantly, this experimental method reveals the form and properties of genuine ordinal ranking, thereby illustrating the paper' s argument from both positive and negative perspectives: ordinal utility theory is untenable, and utility is essentially cardinal in nature.

Keywords: Consumer behavior; Ordinal utility; Cardinal utility; Thought experiment
JEL Classification: D11

Utility is a psychological phenomenon representing the satisfaction a consumer derives from a good or service, occupying a foundational position in economic theory. Throughout the long development of economic theory, two distinct theories of utility measurement have emerged: "cardinal utility theory" and "ordinal utility theory." Cardinal utility was a concept widely used in 19th and early 20th-century Western economics, based on the idea that utility is measurable and calculable. The unit of measurement was called the "util," allowing utility magnitudes to be expressed with cardinal numbers (1, 2, 3). Ordinal utility theory, by contrast, maintains that utility as a psychological phenomenon cannot be measured or aggregated; it can only represent degrees and rankings of satisfaction. Thus, "utility" can only be expressed with ordinal numbers (1st, 2nd, 3rd).

Early scholars studying utility theory employed cardinal utility theory, while ordinal utility concepts were first proposed in the late 19th century (Fisher 1892). Due to its so-called revealed preference characteristic (ordinal preferences being directly observable), ordinal utility theory was considered more scientific than cardinal utility theory and continued to develop and refine. By the 1930s, it had replaced cardinal utility theory as the mainstream analytical paradigm in modern utility theory (Hicks and Allen 1934). However, this status has proven unstable. Since its inception, ordinal utility theory has faced various challenges and questions that profoundly highlight its internal logical contradictions and conflicts with intuitive experience.

First, the emergence of expected utility theory represented the first major challenge to ordinal utility theory (Morgenstern and von Neumann 1953). Von Neumann and Morgenstern established expected utility theory based on axiomatic assumptions using logical and mathematical tools, creating a framework for analyzing rational human choice under uncertainty. Expected utility refers to the weighted average utility of various outcomes a consumer might obtain under uncertain conditions. The expected utility function is cardinal because it only permits positive affine transformations. Following this development, utility theory faced an awkward situation: utility could be either ordinal or cardinal depending on conditions. This dilemma was "resolved through subtle and complex interpretations: the index (expected utility value) does not reveal outcomes (behaviors) but merely serves as a mathematical tool indirectly expressing one' s attitude toward risk" (Allais and Hagen 2012). Yet where do outcomes come from without attitudes? Economists have accepted the coexistence of two logically contradictory utility theories.

The second challenge to ordinal utility theory is the inconsistency between preferences and welfare. Ordinal utility theory assumes that preferences and welfare are consistent and that people' s choices maximize their utility. However, extensive research and intuitive experience tell us that inconsistency between

the two is extremely common. Maximizing psychological satisfaction may be the purpose or intention of behavior but is far from the result of all behavior. Generally, correct choices are based on consumers' accurate cognition of corresponding goods' utility. When this cognition is insufficient or erroneous, inconsistency between welfare and preferences arises (Ng 2005). Studying behavior alone cannot explain whether consumers' varied behaviors achieve utility maximization or grasp the meaning of such behavior.

Third, causal inversion represents another logical contradiction in ordinal utility theory because the theory focuses on what consumers choose while ignoring why they choose it (Sen 1979). Moreover, it arbitrarily defines the result of their choice behavior as individuals obtaining maximum psychological satisfaction. Thus, ordinal utility theory treats the cause of choice behavior as the result of choice behavior—namely, utility maximization. The problem is that verifying this result using ordinal utility theory can only be limited to introspection because satisfaction cannot be observed through explicit behavior. In this way, when choice is used to completely define utility, economists can call any result of a person' s action "utility maximization" (Thurow 1983). Within this logical framework, it is neither verifiable nor falsifiable (Ye 2003). Therefore, the method used by ordinal utility theory cannot satisfy logical positivism and can only be considered to satisfy behaviorism. How, then, can this theory explain the causes of consumer behavior with evidence? This cause can only be explained by their past actions—that is, what everyone does is what they do (Sweezy 1934). If someone asks why the sun rises in the east and sets in the west, the answer according to ordinal utility theory' s logic would necessarily be that it has always risen in the east and set in the west. Thus, ordinal utility theory has an inherent logical problem: cause is treated as result, and result is treated as cause.

Next, an internal conclusion of ordinal utility theory denies the law of diminishing marginal utility (Jehle and Reny 2011). This assertion is equivalent to denying the application of intuition in economics. Consequently, such a natural, everyday psychological law with strong economic significance, acknowledged by almost everyone, is canceled and denied. Many economists believe this deviates from the essence and core of the marginal revolution. Therefore, they accuse abandoning the concept of diminishing marginal utility of being an amputation (Bernardelli 1938) or throwing out the baby of marginal utility with the bathwater of measurable utility (Rothbard 1956).

Furthermore, the method of constructing real-valued functions to express preferences in ordinal utility theory has also been criticized. Although existence theorems for utility functions have been formally proven (Debreu 1954; Debreu 1964), many problems in the application of utility functions do not conform to mathematical logic. For example, William Barnett II (2003) insists that "neoclassical utility functions are an invalid means of analyzing consumer behavior for three reasons: first and most importantly, because such functions and their attendant orderings are cardinal in nature, not ordinal; second, because relative

to the set of bundles relevant to actual humans, such functions are not continuous and therefore not differentiable; third, because these functions do not correctly, consistently, and properly include dimensions/units." However, without utility functions, the application scope of ordinal utility theory would be severely limited. Notably, the design of the thought experiment in Barnett' s paper appendix is very close to this paper' s experimental design. This paper will discuss their relevance in the "Further Analysis" section.

Most importantly, continuous progress in neuroscience has provided material conditions for finding the neural basis corresponding to subjective satisfaction, giving rise to neuroeconomics. Neuroeconomics is a new interdisciplinary field that uses neuroscience technology to study the neural mechanisms of economic decision-making. Its experimental methods can directly address the theme of utility measurement. In just over 20 years, neuroeconomics has developed rapidly with significant results (Camerer et al. 2005). Various empirical methods of neural observation have been widely applied, and computational models of various utilities in the human brain are continuously being constructed (Bayer and Glimcher 2005; Montague et al. 2004). Based on predictions, this technology may enable direct measurement of utility because any data measured by experimental means in natural sciences must be cardinal in nature. Therefore, utility must also be cardinal rather than ordinal.

However, because corresponding experimental technology has not yet reached a very profound and sophisticated level, it seems premature to use obtained data as the foundation for establishing cardinal utility theory (Zizzo 2002). Given the defects of ordinal utility theory and the continuous development of utility measurement capabilities, the revival of cardinal utility theory is the general trend.

In recent years, scholars such as Kahneman et al. (1997), Köbberling (2006), Mandler (2006), and more neuroeconomists like Glimcher et al. (2005), Padoa-Schioppa and Assad (2006), Phillips et al. (2007) have explicitly advocated that cardinal utility should be used to explain relevant economic phenomena.

This paper' s argument is not based on neuroeconomics methods or the aforementioned questions surrounding ordinal utility theory. Instead, it is based on a more fundamental question about ordinal utility theory: Can a person rank goods according to preferences without pre-existing cardinal utility? Or, if a person does not know the quantity of satisfaction, what is the basis for comparing preferences for goods? For example, Chinese economist Ye Hang (2003) argues that it is absurd and counterintuitive for something to be rankable (by any standard) yet not describable by quantity—a situation that does not exist in any other scientific measurement field.

This paper aims to design these intuitive questions as a thought experiment to show that ordinal utility theory' s attempt to explain utility levels by skipping utility measurement is an inappropriate method that is not only counterintuitive but also illogical. Ordinalists mix the cardinal attributes of utility into the

basic assumptions of ordinal utility theory, creating so-called pure ordinal utility theory. This approach leads to logical inconsistency: establishing ordinal utility theory must be based on the existence of utility' s cardinal nature. This paper only discusses whether ordinal utility theory holds under certainty conditions because expected utility theory already demonstrates that under uncertainty, utility can only exist in cardinal form. This paper' s significance lies in its strict negation of the rationality of common ordinal preferences' existence, taking a key step toward clarifying utility' s cardinal nature, aiming to return the research focus of utility theory to cardinal utility theory and renew the face of modern economics. Moreover, it can inspire future economists to conduct further analysis and exploration of problems that ordinal utility theory cannot solve based on this foundation.

The paper proceeds as follows: Section 2 designs a thought experiment to prove that utility is cardinal rather than ordinal and demonstrates what constitutes genuine ordinal preferences. Section 3 compares cardinal preferences with real ordinal preferences, showing their fundamental differences. It further analyzes the experimental process to explain the root of ordinal utility theory' s internal logical inconsistency, then illustrates the connection between this paper' s research method and other scholars' methods, and further explains why this paper' s method can resolve this controversy. Finally, Section 4 concludes that under conditions where only ordinal utility information is obtained, most basic axioms of ordinal utility theory do not hold. Therefore, ordinal utility theory is untenable, and utility is essentially cardinal.

2. Methodology

This section first introduces some basic concepts that may be frequently used in other studies. These concepts make this paper' s experimental and analytical processes clear and concise. Next, based on the ideas of the control variable method and proof by contradiction, a new experiment is proposed. Its purpose is to demonstrate that under conditions of obtaining pure ordinal utility information, the completeness and transitivity axioms of ordinal utility theory cannot hold. It should be noted that these experiments can be conducted in actual or virtual environments. Because they are very concise and logical, the results will not differ between the two cases. Although the experimental steps are simple, people may find them not easy to understand. To represent a more comprehensible explanation of such experiments, this section first uses a simple height-ranking thought experiment as an introductory experiment (Experiment 1) before the main experiment (Experiment 2) to illustrate the basic logic of subsequent experiments. Finally, to make the argument more complete and rigorous, a counterexample (Experiment 3) is proposed to demonstrate what constitutes genuine ordinal preferences and their characteristics and fundamental properties. To express the logic and steps of the experiments rigorously and vividly, Experiment 2 is presented in a more concrete form. Other experiments are given in simplified form.

2.1 Ordinal and Cardinal Information

Ordinal information refers to information that only allows people to obtain the ranking of certain attributes. Cardinal information refers to information that enables people to obtain the quantity of certain attributes. Where does such ordinal or cardinal information come from? It may come from language, observation of reality, or other means. Therefore, cardinal and ordinal information themselves are not necessarily cardinal or ordinal but are impressions or feelings. Ordinal or cardinal information may be vague or precise, certain or probable. Additionally, order information can be strict or non-strict. Here it should be noted that most ordinal information exists based on cardinal attributes. For example, exam rankings are based on exam scores. But this is not always the case. Official rank, for instance, is an ordinal ranking that may not be based on cardinal attributes. This will be reflected in Experiment 3 later.

Ordinal or cardinal information may have many properties, but those most relevant to this study are as follows: obtaining cardinal information of an attribute simultaneously obtains its ordinal information, but the obtained ordinal information does not include the corresponding cardinal information. That is, cardinal information contains more information. For example, knowing the specific heights of all students in a class naturally reveals their height ranking. However, knowing their height ranking does not mean simultaneously knowing their specific heights. After clarifying the characteristics of cardinal and ordinal information, the concepts of ordinal utility theory and cardinal utility theory can be expressed more clearly. Ordinal utility theory refers to a consumer behavior theory where consumers can arbitrarily rank preferences based on ordinal utility information of commodity bundles in a specific commodity set. Cardinal utility theory refers to a consumer behavior theory where consumers can only arbitrarily rank preferences when they obtain cardinal utility information of commodity bundles in a specific commodity set.

Therefore, the focus of debate between cardinal utility theorists and ordinal utility theorists is whether consumers base their ranking of goods on ordinal or cardinal utility information of commodity bundles when ordering goods according to preferences. That is, under the premise of only obtaining ordinal utility information, whether consumer behavior can satisfy the three most fundamental axioms of preference relations, including completeness, reflexivity, and transitivity axioms. If this premise can hold, then ordinal utility theory can be established. However, under conditions where cardinal utility information is obtained, consumer behavior certainly satisfies these axioms. The fact that consumer behavior satisfies these axioms under such conditions does not mean ordinal utility theory holds.

Based on the above analysis, the key method to resolve the debate between cardinal utility theory and ordinal utility theory lies in creating a situation where consumers only know the ordinal utility information of commodity bundles to see whether the basic axioms of consumer preferences can hold. That situation

is when consumers only know the preference ranking of commodity bundles but are unclear about the degree of preference for the bundles. In reality, such situations are rare but do exist. What is needed now is to design an experiment to demonstrate this.

The following is the specific experimental and analytical process.

Experiment 1: Suppose a subject simultaneously meets three people. No one tells them the exact heights or height rankings of these three people, and the subjects themselves do not know. The expression of the experiment can be simplified by specially arranging for the three people to have different heights, which can be told to subjects in advance. Next, because the three people have different heights, there must be a tallest, second tallest, and shortest. Subjects are required to name these individuals A, B, and C according to their inevitable height ranking. At this point, subjects have obtained the height ranking information of the three people and corresponding codes: A > B > C. Whether this information is pure ordinal can be determined through a test. For example, asking subjects about these three people' s height order, they will answer A > B > C. Asking subjects which group, (A, B) or (C), is taller overall, subjects must answer that (A, B) is taller than (C). Asking subjects which group, (A) or (B, C), is taller overall, subjects certainly cannot answer because ordinals cannot be used for arithmetic operations. This method can confirm that subjects have obtained pure ordinal information about the three people' s heights at this time. However, if subjects can see the actual heights of these three people—that is, obtain cardinal information about their heights—they can obviously answer these three questions reasonably. This experiment concludes that pure ordinal information about certain attributes can be obtained through some means. Generally, it is impossible to rank any combination of pure ordinal information.

At this point, readers may feel confused. Subjects do not know who A, B, and C correspond to. In fact, this does not matter because after subjects see the actual heights, A, B, and C must correspond to the first, second, and third tallest people, respectively. As long as no one is switched during the process, the actual identities of the three people can be directly matched to the codes after seeing the specific heights. Therefore, the corresponding question is not important. The specific principle behind this thinking method can be explained as follows: Because the human mind has the ability to imagine, it can anticipate the inevitable results of specific experimental steps in advance, perform virtual mental operations, and thus obtain the pure ordinal information of any measurable cardinal attribute. People can explain the measurement characteristics of specific attributes by requiring subjects to use these pure ordinal information for arbitrary combinations and ordering.

Experiment 2: This experiment represents a class of thought experiments seeking to falsify ordinal utility theory and is the simplest form. The experiment requires one experimenter and at least one subject. Subjects must satisfy three assumptions of general consumer behavior, as well as monotonicity and commodity continuity assumptions, and the experimental goods must be good

goods for subjects.

Prepare three opaque boxes, all containing the same good, such as apples. These apples should have no obvious difference in quality or size. The number of apples in each box can be the same or different. For example, the experimenter can place six, four, and three apples in the first, second, and third boxes, respectively. The key is to ensure subjects do not know the number of apples in each box.

Take out an apple, let the subject look at or taste it, then say to the subject: "The boxes all contain such apples, but the number is unknown. Now, you should name the commodity bundles a, b, c (from large to small; non-strict) according to their function. This designation does not consider visual observation at this time but is based on the inevitable result after opening the boxes and seeing the specific goods." Obviously, the naming of a, b, c is unrelated to the order of the boxes and only relates to the subject' s preference order for the commodity bundles in the boxes.

At this time, the experimenter is very clear that the subject' s preference information for the experimental product is a (6 apples) $\succsim$ b (4 apples) $\succsim$ c (3 apples). However, the only information the subject knows is a $\succsim$ b $\succsim$ c. At this point, the boxes remain closed. Ask the subject the following three questions:

Question 1: Do you prefer a or b?
Question 2: b and c are combined or regarded as a commodity bundle d (subjects are informed of this combination procedure, the same below). Do you prefer a or d?
Question 3: a and b are combined or regarded as commodity bundle e. Which do you prefer, e or c?

Let the subject answer these three questions.

Open the boxes and let subjects see the specific commodity bundles inside. Have them match the code names with the observed actual commodity bundles. The ordering result is certain: 6 apples $\succsim$ 4 apples $\succsim$ 3 apples. The naming result is a: 6 apples, b: 4 apples, c: 3 apples.

Ask the subject these three questions again:
Question 4: Do you prefer a or b?
Question 5: Do you prefer a or d?
Question 6: Do you prefer e or c?

Let the subject answer these three questions.

Then analyze the results and reasons for the subject's answers to these questions.

For Question 1, a rational subject can make a reasonable choice and will certainly choose bundle A (a) to maximize utility. Because the condition that the subject prefers A (a) over b is known, there may be indifference between A (a) and b, but choosing A (a) absolutely has no loss.

For Question 2, a rational subject aiming to maximize utility cannot make a

choice. Because a $\succsim$ b $\succsim$ c is the only preference information they know at this time, they do not know whether they prefer A(a) or d (b, c). Only after opening the boxes and matching the codes with specific commodity bundles can subjects say whether they prefer a or d. Even if they made a choice before opening the boxes, it would be a blind choice unrelated to utility maximization. Therefore, if subjects rashly choose, they may regret their choice.

Moreover, in this experiment, due to the application of the monotonicity assumption, larger numbers mean greater utility. Subjects cannot obtain the ranking of the combined bundles themselves by knowing the quantity ranking of the bundles themselves, and thus cannot reasonably answer Question 2. It can be seen that the monotonicity assumption implies to some extent that utility is essentially cardinal. The fact that subjects cannot reasonably answer Question 2 shows that knowing only the preference order of each component of a commodity bundle is not necessarily a means to understand the overall ranking of the bundle. According to mathematical principles, ordinals cannot be added or subtracted.

For Question 3, according to the monotonicity assumption, subjects can make reasonable choices, but only in some simple cases. For example, if there are more commodity bundles, Question 3 may not necessarily be answerable rationally.

Questions 4, 5, and 6 serve as control groups, demonstrating that when all utility information of commodity bundles is known to subjects, any combination of commodity bundles can be ordered, satisfying the completeness assumption.

Then, based on experimental results and mathematical principles, the logical contradiction of ordinal utility theory is derived. According to commodity continuity, since commodity bundles are ordinal overall, parts of commodity bundles are also ordinal in nature. Then, according to mathematical principles and experimental results (the answer to Question 2), it is clear that for two commodity bundles, knowing only the ranking of utility of their parts likely makes their overall utility incomparable. This contradicts the empirical fact that subjects knowing the ranking of each part of a bundle can always obtain the ranking of the entire bundle. It also contradicts the completeness assumption of consumer behavior. From this, it can be concluded that the utility of goods cannot be purely ordinal.

Experiment 2 has specific requirements for the quantity and type of experimental goods. All commodity bundles must simultaneously contain one or more goods, and the quantities of various goods need to increase monotonically in the same direction. Therefore, a more general form can be considered: there are no restrictions on the types or quantities of goods in the commodity bundles themselves. This situation is actually no different from Experiment 2; only the experimenter' s language prompts to subjects are somewhat different. Here we briefly introduce this experiment. For example, still prepare three opaque boxes. The experimenter can place any type and quantity of items in each box and inform subjects of this arrangement. The rest of this experiment still follows the

steps of Experiment 2. Experiments with this change can still explain the view that ordinal utility theory does not hold. Since no matter what items are used for experiments like Experiment 2, the key question of Experiment 2 can be reduced to a prototype problem: they only know their preference order for three commodity bundles is a $\succsim$ b $\succsim$ c, and cannot conclude whether they prefer A(a) or (b, c) (ii). This violates the completeness assumption and empirical facts, so ordinal utility theory does not hold.

Since Experiment 2 can lead to infinite experiments, the number of commodity bundles they contain is not equal. The common conditions established by these experiments are that there must first be three or more commodity bundles. If commodity bundles are divisible, this condition is also satisfied. The number of elements in commodity bundles is essentially cardinal, and subjects' preference order for these commodity bundles is related to the quantity of the commodity elements themselves. Moreover, subjects are allowed to combine any two commodity bundles into one bundle, and commodity bundles have an indifference relation before and after combination. If these experimental preconditions are met, under the pure ordinal utility information conditions of Experiment 2 or its derivative experiments, subjects' preferences do not satisfy the completeness axiom and naturally do not satisfy the transitivity axiom. Only the reflexivity axiom is satisfied.

Additionally, experiments similar to Experiment 2 can show that various dimensions of indifference curves cannot be obtained using given preference rankings of each commodity element. This process is simple and omitted.

2.3 Real Ordinal Preferences

The experimental method adopted in this paper can also reveal which preferences are truly ordinal preferences. According to the previous analysis, as long as consumers can rank their preferences arbitrarily under conditions where only ordinal utility information is obtained, genuine ordinal preferences must exist. This paper introduces an official ranking, which is a purely ordinal ranking. Suppose a department uses this ranking method as a demand rule for specific purposes. Then a pure ordinal preference can be formed, called "official rank preference," because the rules of the two are the same. Using official ranking as an example, this paper' s viewpoint is verified.

Official rankings are common. For example, a general is higher than a major general, and a lieutenant is higher than a second lieutenant. This single-person situation is easy to understand. Here is a set of official ranking rules involving multiple people. It is named Official Rank Matrix Sorting Rule 1. The specific content is that all official ranks can be sorted from high to low. Regardless of the official rank categories in two official rank matrices, always first compare the highest official rank, and the matrix with the higher highest rank is placed first. If the highest ranks are the same, compare the next highest rank, and then sort sequentially. However, the same official rank should be regarded as an official

rank element in the rank matrix. If all official ranks are the same, the two have an indifferent relationship.

After defining the sorting rule in this way, this study conducted Experiment 3, which is similar to Experiments 1 and 2. In front of subjects are four bags containing many cards with different official ranks that can be sorted from high to low. Moreover, subjects know these cards are in the bags and are familiar with Official Rank Matrix Sorting Rule 1, but do not know which cards are in which bag. This experiment assumes subjects will use Official Rank Matrix Sorting Rule 1 to sort them and ensures that the official rank matrices in the four bags are not indifferent to subjects, which should be explained to subjects. Additionally, subjects are told, "You can sort and name them from front to back according to your sorting method." Suppose a $\succ$ b $\succ$ c $\succ$ d is the subject' s sorting and naming result.

Ask the subject the following three questions:
Question 1: Which ranks higher, a or b?
Question 2: a and d are combined or regarded as official rank matrix e. Similarly, b and c are combined or regarded as official rank matrix f (subjects are informed of this combination procedure, the same below). Which ranks higher, e or f?
Question 3: a and b are combined or regarded as official rank matrix g. Additionally, c and d are combined or regarded as official rank matrix h. Which ranks higher, g or h?

Let the subject answer these three questions.

For the first question, subjects will undoubtedly choose A (a), the same answer as Question 1 in Experiment 2, for basically the same reason. For the second and third questions, from the subject' s sorting results and sorting rules, it can be known that e or g must rank first. Because A (a) ranks first, there is an element with the highest official rank among them. Even after merging with the last official rank matrix, the highest official rank in this bundle is still higher than the rank in f and naturally higher than the rank in h. It can now be predicted that after opening these bags, no matter which official rank cards are inside, if subjects are asked Questions 1, 2, and 3 again, their answers will not change.

Therefore, this experiment shows that under conditions of obtaining pure ordinal utility information, official rank matrices can be sorted, which is a pure ordinal sorting. It also illustrates that official rank preference is pure ordinal preference. The following statement expresses the logic of this experiment. Assuming the sorting relationship of four official rank matrices is a $\succ$ b $\succ$ c $\succ$ d, under Official Rank Matrix Sorting Rule 1, it can be obtained that e (a, d) $\succ$ f (b, c) and g (a, b) $\succ$ h (c, d). Moreover, (a, b) $\succ$ (c, d) does not change with sorting rules.

What happens if the official rank rule is changed? First, name it Official Rank Matrix Sorting Rule 2. Specifically, when sorting two official rank matrices, first compare the rank of the first element, and the matrix with the higher rank is placed first. If the same, compare the rank of the second element, continue this

process until the last element is reached. If all official ranks are the same, it is an indifferent relationship.

It can be imagined that after switching to such a rule, people will still get the same results as Experiment 3, which are still examples of pure ordinal sorting. Additionally, this sorting rule feels familiar. Is this the sorting rule of lexicographic preferences? An interesting thing happens! This is not a lexicographic preference rule. The lexicographic preference sorting rule can be described as: assuming $X = \mathbb{R}^2_+$, if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$, then $x \succsim y$. The content of Official Rank Matrix Sorting Rule 2 is: first assume all elements of the official rank matrix are pure ordinals, if $x_1 \succ y_1$ or $x_1 \sim y_1$ and $x_2 \succsim y_2$, then $x \succsim y$. This content shows that elements in the official rank matrix are ordinal, while elements in lexicographic preferences are cardinal. Lexicographic preferences can be verified as cardinal preferences through a thought experiment similar to Experiment 1. Incidentally, the sorting rule for words in English dictionaries is the same as Official Rank Matrix Sorting Rule 2 because English letters can be sorted as a $\succ$ b $\succ$ c, etc. These elements are essentially ordinal and can only be sorted, not added or subtracted. Therefore, lexicographic order is an inaccurate name; perhaps "numerical order" is more accurate because when comparing the size of two numbers, the lexicographic preference sorting rule is basically applied —that is, first compare the size of the highest digit, then compare the size of the second digit, and so on. However, comparing numbers leads to magnitude, while comparing lexicographic preferences leads to ranking. This also shows that lexicographic preference is a fundamental preference.

Observation and inductive analysis of real life show that such pure ordinal preferences are generally group preferences. For example, enterprises simply reward employees based on their position levels. Do individual consumers have pure ordinal preferences? If so, under what circumstances would they have such preferences? These questions can be further explored. However, at least most common individual preferences are cardinal. In this way, this paper illustrates from both positive and negative aspects that ordinal utility theory is invalid and shows the form of genuine ordinal preferences. The use of proof by contradiction and the control variable method can be seen in the experimental process. Since ordinalists believe utility quantities are ordinal, to refute ordinal utility theory, a situation must be created where consumers only know the ordinal utility information of commodity bundles, keeping other conditions unchanged, and then compare it with the situation where consumers know all utility information of commodity bundles to see whether the conclusions of ordinal utility theory hold. Then, conclusions are drawn. The reasonable use of these methods in experimental design can ensure that the conclusions drawn in this paper are scientifically valid.

3. Further Analysis

The experimental process and preliminary analysis of the above two experiments show that this paper' s experiments can distinguish real ordinal preferences from

cardinal preferences. The next task is to analyze the characteristics of these two preferences to explain why they are fundamentally different. Simple observation reveals that elements of commodity bundles with cardinal preferences are all cardinals that can be used for arithmetic operations. Moreover, the degree of preference for commodity bundles is strictly related to the quantity of the commodity elements themselves. Therefore, the degree of preference for commodity bundles increases or decreases with the increase or decrease of commodity elements. Thus, the two must form a mapping between cardinal members. Then, commodity bundles with such cardinal elements must correspond to cardinal preferences. The result of ordering two cardinal preference commodity bundles must be related to the quantity of cardinal elements of the two components themselves. For example, ordering two commodity bundles (a: three oranges, d: two apples) and (b: three apples, c: two oranges). The ranking of both bundles is related to the quantity of corresponding commodity elements. If the quantity of one element is changed, the order of the two bundles may change. However, if only the ordinal preference information of elements of two commodity bundles is known, such as (a: 1st, d: 4th), (b: 2nd, c: 3rd), then ranking is impossible due to insufficient information. This situation can also be found in other measurement fields, such as the height ranking in Experiment 1. Obviously, the quantity of all commodity elements analyzed by the indifference curve analysis method used in economics textbooks is the cardinal that can be added and subtracted. Therefore, the ordinal preferences analyzed by indifference curve analysis are actually cardinal preferences.

Next, discuss the case of real ordinal preferences. In real ordinal preferences, elements of commodity bundles must be ordinal, and multi-element ordinal matrices generally cannot be directly sorted. The premise of arbitrarily sorting ordinal matrices is first to set an external rule specifying a certain form of priority. For example, before establishing rules, directly writing ranks (General, Corporal), (Major General, Sergeant). At this time, the two are not comparable. However, if a sorting rule similar to Official Rank Matrix Sorting Rule 1 or 2 is set in advance, ordinal matrices can be arbitrarily sorted. Therefore, there is such a logical contradiction in ordinal utility theory: multiple commodity bundles can be arbitrarily combined into one commodity bundle. However, multiple ordinals cannot be arbitrarily combined into one ordinal. Commodity bundles can be arbitrarily sorted by preference, but ordinal matrices cannot be arbitrarily sorted. Furthermore, known behavior can obtain an ordinal; conversely, given an ordinal, behavior may not necessarily be obtained. The analysis also shows that real ordinal preferences cannot be cardinal preferences, and cardinal preferences cannot be real ordinal preferences.

This mathematical conclusion can be used in the analysis of Experiment 3—that is, generally, two matrices containing several pure ordinal elements cannot be directly sorted unless all elements of one matrix are ranked in front. This conclusion is not unfamiliar to us because it already exists in economic theory in the form of Arrow' s Impossibility Theorem (Arrow 1950). This paper does not explain the homology of mathematical principles applied by the two.

Here it only needs to be noted that the situation of "all elements ranked in front" in Experiment 3 corresponds to the situation in social choice problems that is completely consistent with each individual' s preference order, while Official Rank Matrix Sorting Rules 1 and 2 correspond to rank dictatorship rules. Specifically, the highest-ranked person first makes a preference order, and their preference order serves as the social preference order. If they are indifferent, the second-ranked person decides, until the lowest-ranked person makes the decision. These are the two situations where group preference matrix sorting can be obtained; other situations without external rules cannot be sorted. Therefore, if commodity utility is purely ordinal, without external rules, people can only order some indivisible single-element commodity bundles, while common multi-element commodity bundles generally cannot be ordered.

However, Arrow' s Impossibility Theorem is based on ordinal utility theory. Why are the mathematical principles used in this theorem applied to the falsification of ordinal utility theory? This paper believes that utility quantity is essentially cardinal, which determines consumer choice behavior. But at the current level of measurement ability, only preference order in explicit behavior can be observed. Based on this preference ordering, Arrow created the impossibility theorem, which is reasonable. However, if utility quantity is simply defined as ordinal from the perspective of explicit behavior, it can be seen that if ordinals are directly substituted into commodity bundle matrices, the resulting ordinal matrices cannot be sorted. More importantly, the sorting rules for ordinal preference matrices and cardinal preference matrices are inconsistent. Sorting ordinal preference matrices requires an external rule to determine a specific form of priority. The sorting of the entire ordinal preference matrix is replaced by the corresponding sorting of one or several elements, while cardinal preference matrices have no such external rules but are sorted directly by intuition. This shows that the nature of these two preferences is fundamentally different.

This paper's experiments and corresponding analysis partially summarize the debate between ordinal utility theory and cardinal utility theory. Here we explain the relationship between this paper' s experimental method and the methods used by other economists, as well as its relative superiority. Generally, four methods are used to analyze this controversy. The first is the cognitivist approach, which studies and expands cardinal utility theory through introspection, such as (Allais 1994a; Allais 1994b). This method is considered inconsistent with logical positivism and is not widely recognized in the economics community today. The second uses neuroeconomics experimental methods. However, due to current technological limitations, experimental results lack decisive persuasive power for establishing cardinal utility theory. The third is the mathematical method, analyzing the characteristics and axioms of consumer preferences to solve this problem through mathematics and reasoning, such as (Barzilai 2013; Köbberling 2006). However, as can be seen from the analysis process in this paper, as long as it is admitted that ordinal preference relations contain the completeness axiom, completely solving this problem through mathematical derivation is challenging. In fact, cardinal information of commodity utility has

already been mixed into preference relations. At this time, preference relations must simultaneously contain ordinal and cardinal attributes. If cardinal information cannot be removed and the cardinal nature of the completeness axiom cannot be fundamentally explained, the key conclusion falsifying ordinal utility theory cannot be reached. This basic axiom of consumer behavior with both cardinal and ordinal properties makes mathematical experts in economics suspect whether more advanced mathematical knowledge is needed to solve this problem. The fourth method analyzes people' s internal utility cognition through behaviorist approaches and is the most concise and persuasive method at the current technical level. This method is mainly achieved by analyzing specific behaviors or conducting behavioral experiments, such as (Barnett 2003; Handa 1977). Among them, Barnett' s research contains an experiment in its appendix that is closest to this paper' s experimental design. The research mainly illustrates that ordinal utility theory unreasonably uses utility functions. Several thought experiments in the paper will achieve this purpose. In fact, this is a certain conclusion. But ordinalists believe that utility functions are merely a convenient and heuristic tool for explaining consumer behavior without real economic significance. Therefore, even if problems occur in the operation process, they cannot affect the overall situation. However, the thought experiment in the article appendix proposes a breakthrough viewpoint.

First, this method can separate the ordinal and cardinal information of commodity bundles for consideration. Second, it shows that preference ranking cannot be done arbitrarily without knowing all cardinal utility information. The only weakness of this experiment is that it does not design an actual step to obtain pure ordinal utility information of real commodity bundles. This point is crucial for thoroughly solving this problem. Although this paper' s experiments are simple, they reasonably include these three key points, thus reaching the expected conclusion.

This paper mainly uses thought experiments to illustrate problems in ordinal utility theory and endorses the view that utility quantity is essentially cardinal. However, this study excludes endorsement of some existing views or conclusions about cardinal utility theory, such as using "util" as a unit of utility. Util is a purely imaginary utility unit without scientific basis. The setting of units must be based on pre-existing quantities of specific attributes. For example, a gram is defined as the mass of one cubic centimeter of pure water. If one uses util as a unit, when a person eats an apple, it can be said that they obtain 100 utils, 10 utils, or any other result. This situation never occurs when reasonably set measurement units are used in any other scientific field. When no measurement unit is reasonably set and can be standardized, it is easy to obtain ordinal information from cardinal data; however, cardinal data cannot be obtained. For example, even if several trees stand before us, without a ruler with specific units, we cannot know their specific values, but the corresponding ranking is easily obtained. The emergence of ordinal utility theory is related to the lack of effective utility units and the availability of ordinal ranking information. In summary, cardinal utility theory is imperfect and needs further revision and

improvement.

Based on the description and analysis of the thought experiments, this paper concludes: In most cases, consumers cannot rank their preferences based on pure ordinal utility information. Therefore, under conditions where pure ordinal utility information is obtained, the completeness and transitivity axioms of consumer preferences are invalid. This also shows that ordinal utility theory is invalid, and consumers rely on cardinal utility information of commodity bundles when ranking preferences. However, expressing cardinal preferences is a complex problem. Economists cannot recognize the lack of a revealed method. Solving this problem requires further research on methods for setting utility units. However, this paper' s experimental and analytical process shows that experiments can verify whether consumer preferences are cardinal or ordinal, and ordinalists' characterization of utility quantity as ordinal is a fallacy of misplaced abstraction.

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Note: Figure translations are in progress. See original paper for figures.