Invariant Manifold Growth Formula and Its Application in Magnetic Confinement Fusion

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Abstract

For the invariant manifolds of hyperbolic cycles in three-dimensional vector fields, this paper presents their growth formula in cylindrical coordinates. The initial growth direction of these manifolds depends on the Jacobian matrix of the Poincaré map, and the paper provides an evolution formula for this matrix on the cycle. This evolution formula is also applicable to any finite-dimensional autonomous continuous-time dynamical systems. This paper constructs non-Möbius-type and Möbius-type saddle cycles, as well as artificial X-cycles, to demonstrate the application of the formula. For a realistic numerical example, a magnetic field time slice from the operation of the EAST tokamak is taken as a demonstration.

Full Text

Preamble

Invariant Manifold Growth Formula in Cylindrical Coordinates and Its Application for Magnetically Confined Fusion

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Abstract

For three-dimensional vector fields, the governing formula for invariant manifolds grown from a hyperbolic cycle is derived in cylindrical coordinates. The initial growth directions depend on the Jacobian matrices of the Poincaré map on that cycle, for which an evolution formula is deduced to reveal the relationship among Jacobians of different Poincaré sections. This evolution formula also applies to cycles in arbitrary finite n-dimensional autonomous continuous-time dynamical systems. Non-Möbiusian/Möbiusian saddle cycles and a dummy X-cycle are constructed analytically as demonstrations. A real-world numerical example analyzing a magnetic field timeslice from the EAST tokamak is presented.

Keywords: magnetic topology, tokamak, invariant manifold

Introduction

In the tokamak community, magnetic topology is most often assumed to consist of nested flux surfaces in the Grad-Shafranov equation, EFIT, VMEC, etc. Based on this assumption, researchers have established a dedicated theory of magnetic coordinates in which all magnetic field lines on a flux surface are straight. Nevertheless, three-dimensional (3D) effects are ubiquitous in real-world fusion experiments since every facility in fusion machines, except the central solenoid and poloidal field coils, is inherently 3D. For example, toroidal field coils exhibit a ripple effect, microwave and radio-frequency wave heating impose their distinctive localized distribution patterns of induced current, and neutral beam injection (NBI) has an obvious non-axisymmetric current effect. Furthermore, most plasma instability modes, such as tearing modes and sawtooth modes, imply significant 3D topology changes of the magnetic field. The 3D effect is unavoidable in magnetically confined fusion research, thereby requiring deeper comprehension of global field structure.

An enormous amount of fusion research has attempted to stimulate a chaotic field layer at the plasma boundary by either resonant magnetic perturbation (RMP) coils or other means to mitigate destructive type-I edge localized modes (ELMs). The theoretical basis was established in 1983 by Cary and Littlejohn to estimate how wide island chains become when an axisymmetric magnetic field is perturbed by a non-axisymmetric one, after which hundreds of researchers implemented RMP coils to mitigate and suppress ELMs. The method, now called magnetic spectrum analysis, heavily relies on Fourier transforms of the radial component of the perturbation field, which is a linear operation in functional spaces. Therefore, the utility of magnetic spectrum analysis is limited inside the plasma and becomes less accurate as the perturbation strengthens, because Fourier transforms are merely linear operations and cannot explain nonlinear behavior.

With the aid of modern dynamical systems theory, the structure of a 3D vector field can be comprehended and analyzed in terms of invariant manifolds. Various numerical methods have been developed to grow them. Kuznetsov and Meijer systematically investigated the bifurcation behavior of 1D and 2D maps when their parameters change, presenting diverse analytic and numerical methods to study maps. However, the methods employed by mathematicians are too general to capture the essence of 3D vector fields, leaving the analysis as complicated as before.

Since the magnetic field dominates plasma transport in magnetically confined fusion machines, evidence of transversely intersecting invariant manifolds should be easy to observe, which is a signature of chaos. Invariant manifolds are essential for determining chaotic field regions that induce mixing effects inside the plasma. The plasma edge is not suitable for characterization by a single closed surface when 3D effects are strong, for which it is proposed to use the notion of invariant manifolds of outermost saddle cycle(s). In fact, this has been observed in some existing simulation and experimental results. On EAST, helical current filaments induced by lower hybrid wave heating impacted the plasma edge topology and caused evident splitting of strike points in experiments.

If RMP coils are imposed to suppress ELMs in tokamaks, the heat flux pattern at the divertor exhibits a complex toroidally asymmetric distribution, which poses challenges for ITER and DEMO divertor designs. Simulation results demonstrate that divertor plasma regions with connection to the bulk plasma are dragged further outward when asymmetry is intensified. The field line connection length and magnetic footprint (how deeply field lines penetrate into the bulk plasma) distribution at the divertor are usually ribbon-like. While RMP does mitigate or even suppress ELMs, which would otherwise cause intolerable transient particle and heat flux, significant heat fluxes may arise far from the strike point originally designed for axisymmetric cases, possibly damaging fragile divertor components. Moreover, it remains unknown whether the total heat flux leaked from the bulk plasma is reinforced by the perturbation.

Previous research has attempted to draw the two transversely intersecting manifolds of hyperbolic cycles for 3D toroidal vector fields. Ottino from the fluid mechanics community, and Roeder, Rapoport, and Evans from the fusion community have drawn relevant figures. Abdullaev has deduced an approximate (to first order in ϵ because the Poincaré integral is used) analytic implicit expression of the invariant manifolds of the outermost X-cycle when a single-null configuration is perturbed. This paper carries forward that research and directly derives the intrinsic analytic formula of invariant manifolds of hyperbolic cycles without requiring approximation.

Section 2 explains the definitions and notations used in this paper. The theoretical summary is presented in Section 3, while detailed proofs are provided in Appendix A. Section 4 offers intuitive examples to help understand the theory: non-Möbiusian/Möbiusian saddle cycles and a dummy X-cycle are constructed analytically, followed by a real-world numerical example analyzing a magnetic field timeslice from EAST. Section 5.1 and Appendix B provide comparisons with other works, and Section 5.2 presents the conclusion.

2. Definitions and Notations Explained

Vector fields are presumed to be at least once continuously differentiable, i.e., of class C^1. Although a vector field is denoted by $\mathbf{B}$, it is worth emphasizing that it does not need to be divergence-free in this paper.

The flow $X(x_0, t)$ induced by the field $\mathbf{B}$ and the corresponding flow $X_{\mathrm{pol}}(x_0, \phi_s, \phi_e)$ in cylindrical coordinates play a central role in the theoretical derivation, especially the latter. (A flow is often denoted by ϕ_t or $\phi(t, x_0)$ in other literature, but ϕ has been used as the azimuthal angle in this paper. X and X_{pol} denote the field line tracing (FLT) flows instead.) Notations used are listed in Table 1 . The symbol D serves as a differentiation operator with respect to the initial condition x_0, as distinguished from those with respect to time-like variables t, ϕ_s, or ϕ_e.

Table 1: Notations

Cartesian	Cylindrical
FLT ODE system: $\dot{x} = \mathbf{B}(x)$	$dx^R/d\phi = RB^R/B^\phi, dx^Z/d\phi = RB^Z/B^\phi$
Initial condition: (x_{0x}, x_{0y}, x_{0z})	$x_0 = (x_0^R, x_0^Z)$
Flow: $X(x_0, t)$	$X_{\mathrm{pol}}(x_0, \phi_s, \phi_e)$, subscripts s for start, e for end
$\partial_t X(x_0, t) = \mathbf{B}(X(x_0, t))$	$R\mathbf{B}_{\mathrm{pol}}/B^\phi = \partial_{\phi_e} X_{\mathrm{pol}}(x_0, \phi_s, \phi_e)$
Differentiation w.r.t. x_0: $DX(x_0, t) := \partial X(x_0, t)/\partial(x_{0x}, x_{0y}, x_{0z})$	$DX_{\mathrm{pol}}(x_0, \phi_s, \phi_e) := \partial X_{\mathrm{pol}}(x_0, \phi_s, \phi_e)/\partial(x_0^R, x_0^Z)$

By the chain rule, $\frac{d}{dt} DX(x_0, t) = \partial \mathbf{B}(X(x_0, t)) DX(x_0, t)$. Similarly, $\frac{d}{d\phi_e} DX_{\mathrm{pol}}(x_0, \phi_s, \phi_e) = A(\phi_e) DX_{\mathrm{pol}}(x_0, \phi_s, \phi_e)$, where $A(\phi_e) := \frac{\partial}{\partial(R,Z)}(R\mathbf{B}_{\mathrm{pol}}/B^\phi)|_{X_{\mathrm{pol}}(x_0,\phi_s,\phi_e),\phi_e}$.

Naturally, the Poincaré map $P(x_0, \phi)$, with $x_0 = (x_0^R, x_0^Z)$, is defined by the standard R-Z semi-infinite planes $\Sigma(\phi)$ at ϕ angles, recording the strike points crossing the planes in one direction. If a trajectory flies around $\Sigma(\phi)$ through the opposite semi-infinite plane $\Sigma(\phi + \pi)$, then by definition the Poincaré map does not record the strike point on $\Sigma(\phi + \pi)$. Since the flow $X_{\mathrm{pol}}(x_0, \phi_s, \phi_e)$ and the map $P(x_0, \phi)$ are used frequently in this paper, different terminologies distinguish continuous-time dynamics from discrete-time dynamics: equilibrium point, trajectory, and cycle refer to continuous-time dynamics, while fixed point, orbit, and periodic orbit refer to discrete-time dynamics.

This paper follows the definition of stable and unstable manifolds given by Jacob Palis and Welington de Melo (p. 73 for a hyperbolic fixed point and p. 98 for a hyperbolic cycle). In both cases, the manifolds are defined by ω- and α-limit sets, allowing unified manifold definitions in continuous and discrete dynamics by considering manifolds of a hyperbolic invariant set S (defined at p. 160 for the case of a map), since both hyperbolic fixed points and hyperbolic cycles are hyperbolic invariant sets. Note that a hyperbolic periodic orbit is also a hyperbolic invariant set, which can be handled by the unified definition.

Let (M, f) be a dynamical system, where f is a map P or a vector field $\mathbf{B}$ defined on the manifold M. The stable and unstable manifolds of a hyperbolic invariant set S for f are defined by ω- and α-limit sets respectively:

$$W^s(f,S) := \{p \in M : \omega(f,p) = S\},$$
$$W^u(f,S) := \{p \in M : \alpha(f,p) = S\}.$$

The term "hyperbolic" ensures that nearby trajectories/orbits on stable (resp. unstable) manifolds approach (resp. depart from) S at an exponential rate. Without this property, the sets defined above are only qualified to be named stable (resp. unstable) sets, because the stable manifold theorem requires S to be hyperbolic. Colloquially, the stable (resp. unstable) manifold is the set of points that flow into (resp. out of) S at an exponential rate.

Kuznetsov and Meijer simply write $f^k(x) \to S$ as $k \to \infty$ without explaining what they mean by "converging to a set S," which is suspected to be an informal denotation of ω- and α-limit sets.

For notational convenience, arguments can be omitted when suitable: $DX(t)$ and $DX_{\mathrm{pol}}(\phi_s, \phi_e)$ are short for $DX(x_0, \phi_t)$ and $DX_{\mathrm{pol}}(x_0, \phi_s, \phi_e)$ by omitting x_0; $DP^m(\phi)$ for $DP^m(x_0, \phi)$; and $W^{u/s}(S)$ for $W^{u/s}(f, S)$.

3. From Field Line Tracing to Invariant Manifolds

DX and DX_{pol} imply the change of differential volume and area during FLT, respectively. Suppose a 2D map is written as $(x, y) \mapsto (u, v)$. The differential area expands, shrinks, or remains constant after being mapped, as revealed by the exterior product of differential 1-forms: $du \wedge dv = (\partial_x u dx + \partial_y u dy) \wedge (\partial_x v dx + \partial_y v dy) = (\partial_x u \partial_y v - \partial_y u \partial_x v) dx \wedge dy$.

As $X_{\mathrm{pol}}(\phi_s, \phi_e)$ is a typical 2D map from section $\Sigma(\phi_s)$ to $\Sigma(\phi_e)$, the determinant of $DX_{\mathrm{pol}}(\phi_s, \phi_e)$, denoted by $|DX_{\mathrm{pol}}(\phi_s, \phi_e)|$, is indeed the same as $\partial_x u \partial_y v - \partial_y u \partial_x v$. One might be curious about the geometric meaning of $|DX_{\mathrm{pol}}(\phi_s, \phi_e)|$ and conjecture that it must be related to the divergence of the field, since it is well-known that for $|DX(x_0, t)|$:

$$|DX(x_0,t)| = e^{\int_0^t \mathrm{tr}(\partial \mathbf{B}(X(x_0,\tau)))d\tau} = e^{\int_0^t \nabla \cdot \mathbf{B}(X(x_0,\tau))d\tau}$$

which indicates that for a divergence-free field, $|DX(x_0, t)|$ is always constant. A similar formula for $|DX_{\mathrm{pol}}(\phi_s, \phi_e)|$ is deduced (proof in Appendix A.1) to reveal the relationship between $|DX_{\mathrm{pol}}(\phi_s, \phi_e)|$ and the divergence along the corresponding trajectory $X_{\mathrm{pol}}(\phi_s, \phi)$, $\phi_s \leq \phi \leq \phi_e$, as shown below:

$$|DX_{\mathrm{pol}}(\phi_s,\phi_e)| = \exp\left(\int_{\phi_s}^{\phi_e} [R(\nabla \cdot \mathbf{B})/B^\phi] d\phi\right) \cdot (B^\phi|_{\phi_s} / B^\phi|_{\phi_e})$$

which applies to 3D vector fields of class C^1, regardless of whether they are divergence-free.

More importantly, $DP^m(x_0, \phi)$ with x_0 on an X-cycle γ of m toroidal turn(s) determines the two X-leg directions of the X-point. If B^ϕ is positive everywhere, $DP^m(x_0, \phi) = DX_{\text{pol}}(x_0, \phi, \phi + 2m\pi)$. It is the two eigenvectors of $DP^m(x_0, \phi)$ that dictate the directions in which the two invariant manifolds of that hyperbolic cycle initially grow. However, $DP^m(x_0, \phi)$ itself is more difficult to solve for than its determinant, which is constant and independent of ϕ for the cycle. This is because the right-hand side of the determinant equation becomes constant when $\phi_e = \phi_s + 2m\pi$. Unlike the determinant, the matrix $DP^m(x_0, \phi)$ varies with the azimuthal angle ϕ. The evolution rule of $DP^m(x_0, \phi)$ with respect to ϕ is revealed in Section 3.1.

To calculate $DX_{\text{pol}}(\phi_s, \phi_e)$ by integrating the ODE requires that B^ϕ on the trajectory does not change sign. Otherwise, $R\mathbf{B}_{\text{pol}}/B^\phi$ would be undefined due to zero B^ϕ. These trajectories are not useless and could be well-defined in Cartesian coordinates. Suppose a trajectory goes from ϕ_s to ϕ_e, during which B^ϕ may change sign several times. The zero B^ϕ singularities cause inconvenience for solving $DX_{\text{pol}}(\phi_s, \phi_e)$. To circumvent this issue, one can first solve for the corresponding DX and then transform its coordinates back to the cylindrical system. The following DX to DX_{pol} formula (proof in Appendix A.2) shows how to do this:

$DX_{\text{pol}} =$ (matrix expression involving trigonometric functions and field components evaluated at start and en

where the subscripts "start" and "end" mean the corresponding matrices are evaluated at the starting and ending points of the trajectory, respectively.

3.1 The Evolution of DP^m Along a Cycle

For a cycle of m toroidal turn(s), the relationship among the $DP^m(\phi)$ matrices at neighboring sections is needed; without it, calculating their eigenvectors would require unnecessarily enormous computational resources. The most primitive approach would be repeatedly integrating Eq. (3) from ϕ_s to $\phi_s + 2m\pi$ for various ϕ_s. To avoid this horribly inefficient approach, the following DP^m evolution formula is deduced (proof in Appendix A.3) to reveal how $DP^m(\phi)$ varies along the cycle:

$\frac{d}{d\phi} DP^m(\phi) = [A(\phi), DP^m(\phi)]$

where the square bracket denotes the commutator, i.e., $[A, B] = AB - BA$. In the dummy X-cycle demonstration Fig. 2 Figure 2: see original paper, Section 4.2, arrows are drawn to indicate eigenvector directions.

The DP^m evolution formula can be applied to cycles in autonomous n-dimensional flows ($n \geq 2$ and finite). For dimensions other than $n = 3$, the notation $|DX(x_0, T)|$ is preferred over $|DP^m(\phi)|$, where T is the period of the cycle, because the former does not rely on the choice of Poincaré section. The

n-dimensional version of the evolution equation in Cartesian coordinates is also provided.

Furthermore, it is desirable to deduce how an eigenvector of $DP^m(\phi)$ evolves along an X-cycle, to eliminate arbitrariness in the computed eigenvector direction, which depends on the specific eigen-decomposition numerical algorithm. If the eigenvector rotates significantly during evolution, the numerical method might give a reversed direction without consistency (i.e., the computed eigenvector may suddenly jump to the opposite side). The following DP^m eigenvector evolution formula (proof in Appendix A.4) extracts the underlying rule governing eigenvector rotation along the cycle. Let the eigenvectors of $DP^m(\phi)$ be denoted by v_i, $i \in \{1,2\}$. The derivative of the parameterization satisfies:

$$(d/d\phi)\Theta(\phi) = V^{-1}[A, DP^m]V\Lambda^{-1}$$

where $V := [v_1, v_2]$ and $\Lambda := \mathrm{diag}(\lambda_1, \lambda_2)$. In numeric implementation, this formula, though accurate, encounters issues when handling X-cycles in island chains because the two eigenvectors are so close that some matrices become nearly singular (i.e., have large condition numbers). A more robust approach is to directly use the DP^m evolution formula to evolve $DP^m(\phi)$ and later adjust eigenvector directions.

Traditionally, fixed points of 2D maps are classified as hyperbolic, elliptic, or parabolic based on their Jacobian eigenvalues. For cycles of 3D flows, the authors adopt this naming convention. It is worth emphasizing that the eigenvalues of $DP^m(\phi)$ remain constant during evolution. This λ-invariance ensures the safety of such classification, as one need not worry about $DP^m(\phi)$ eigenvalues differing at different ϕ.

If both eigenvalues of DP^m lie off the unit circle S in $\mathbb{C}$, the cycle is hyperbolic. If only one eigenvalue lies on the unit circle, the cycle is partially hyperbolic (but not hyperbolic). If both eigenvalues lie on the unit circle but neither equals 1 nor -1, the cycle is elliptic. If the two eigenvalues are identically 1 or -1, the cycle is parabolic.

Furthermore, a saddle cycle is defined as one with $|\lambda| > 1$. Saddle cycles with both eigenvalues negative (resp. positive) are called Möbiusian (resp. non-Möbiusian). Note that the Möbiusian cycle defined here differs from the classical Möbius strip. Cycles with both λ inside (resp. outside) the unit circle S are defined as sinking (resp. sourcing) cycles.

Magnetic fields are typical divergence-free fields, in which X-cycles, O-cycles, and cycles on rational flux surfaces can now be formally defined as hyperbolic (and saddle), elliptic, and parabolic, respectively.

3.2 Invariant Manifold Growth Formula in Cylindrical Coordinates

Consider that an invariant manifold of a hyperbolic cycle γ may grow endlessly. One natural parameter for the manifold is the arc length s of curves intersected by the 2D manifold $W^{u/s}(\mathbf{B}, \gamma)$ and R-Z cross-sections. The other coordinate is chosen as the azimuthal angle ϕ.

It is defined that $s = 0$ on the cycle and s increases toward positive infinity as the manifold grows away from the cycle. The diagram (Fig. 7 [Figure 7: see original paper]) in Appendix A.5 may help readers understand the geometry, illustrating relationships among the differentials used. The diagram is placed in the Appendix because readers following the proof there would need it more.

For precision, this paper defines a stable/unstable (manifold) branch as a connected component of the manifold minus its invariant set, i.e., a connected component of $W^{u/s}(f, S)\quad S$. For example, a saddle fixed point of a 2D map has 2 eigenvectors and 4 invariant branches. An invariant branch of γ is parameterized as $X^{u/s}(s, \phi) = [R(s, \phi), Z(s, \phi)]^T$, where superscripts u and s indicate whether the branch is unstable or stable.

To deduce the governing equation for $X^{u/s}(s, \phi)$, one simply analyzes the differential relationship appearing in FLT, which leads to the following invariant manifold growth formula (proof in Appendix A.5, requiring only multivariable calculus knowledge):

$$\partial_s X^{u/s}(s, \phi) = \pm \frac{R\mathbf{B}_{\text{pol}}(X^{u/s}, \phi) - \partial_\phi X^{u/s}(s, \phi)}{\| R\mathbf{B}_{\text{pol}}(X^{u/s}, \phi) - \partial_\phi X^{u/s}(s, \phi) \|}$$

with the initial condition $\partial_s X^{u/s}(s, \phi)|_{s=0}$ set to be the normalized eigenvector of $DP^m(\phi)$. Naturally, $\partial_s X^{u/s}(s, \phi)$ is $2m\pi$-periodic in ϕ for a non-Möbiusian saddle cycle. The denominator is essentially $ds/d\phi$, so the sign $\pm$ takes $+$ if the field line moves away from the cycle as ϕ increases; otherwise it takes $-$.

For Möbiusian saddle cycles, invariant manifolds can be grown similarly with subtle differences. A non-Möbiusian saddle cycle has two invariant branches for each (un)stable manifold. For a Möbiusian saddle cycle, the two branches of a (un)stable manifold are considered as a whole (since they are connected). Double the period of $X^{u/s}(s, \phi)$ in ϕ from $2m\pi$ to $4m\pi$, where $X^{u/s}(s, \phi)$ is opposite $X^{u/s}(s, \phi + 2m\pi)$ across the cycle γ. Then the growth formula works again.

The growth formula would not grow a rational flux surface from a parabolic cycle on that surface, since any field line does not cover the entire surface. To grow this surface, the DP^m evolution formula is sufficient. One simply moves the cycle in the directions of the DP^m eigenvectors step by step. The role of the DP^m evolution formula is to accelerate computation of $DP^m(\phi)$ at all sections. For an irrational flux surface, one can choose a non-invariant "cycle" that does not obey the FLT ODE system and employ the growth formula. For example, in an axisymmetric field, pick a "cycle" with constant R, Z coordinates. The growth formula then degenerates into a simpler form where $\partial_s X^{u/s}$ is parallel to $\mathbf{B}_{\text{pol}}$ and of unit length.

If knowledge is limited to one section $\Sigma(\phi)$ (so ϕ is omitted in this paragraph), the 2D invariant manifold growth formula degrades to a delay ODE describing

1D invariant manifolds for a 2D map. Let $\{x_i\}_{i=1}^m$ be a hyperbolic periodic orbit under P. Parameterize a 1D invariant branch of $W^{u/s}(P^m, x_i)$ by its arc length s as $X^{u/s}(s) : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, whose inverse is denoted by $s(X)$. Then one can acquire equations to grow the manifold by simple analysis along the branch:

$$dX^u(s)/ds = DP^m(X^u(s))/\|DP^m(X^u(s))\|$$
$$dX^s(s)/ds = DP^{-m}(X^s(s))/\|DP^{-m}(X^s(s))\|$$

where the ellipsis dots in the denominators serve to normalize and denote the same quantities as the numerators. Both equations also hold for an invariant circle C. The only modification needed is that the circle should be parameterized as a function $X(s) : \mathbb{R} \to C \subset \mathbb{R}^2$ with period equal to the circumference, whose inverse $s(X) : C \to \mathbb{R}$ is now a multivalued function.

4. Demonstration of Cycles and Invariant Manifolds

Having developed systematic theory to characterize invariant manifolds in 3D autonomous flows, the formulas have been implemented to present readers with vivid illustrations. Two analytic examples and one real-world example are exhibited in this section. As the simplest model, the first analytic example is a saddle cycle shown in Fig. 1 [Figure 1: see original paper], which can be either non-Möbiusian or Möbiusian. The next, more complicated dummy X-cycle model is shown in Fig. 2 [Figure 2: see original paper], acquired by twisting the cycle of the first model (i.e., letting the R, Z coordinates of the cycle depend on ϕ instead of being constants). In these two analytic examples, we demonstrate the technique of constructing a field by prescribing expected trajectories. Finally, a time slice of the magnetic field on EAST is taken as a real-world example, shown in Fig. 3 [Figure 3: see original paper] and Fig. 5 [Figure 5: see original paper], with RMP as the non-axisymmetric factor.

4.1 Non-Möbiusian/Möbiusian Saddle Cycles

In this section, a saddle cycle model is constructed as an analytic example. Let $B^\phi := R_0 B_0^\phi / R$ to simulate a tokamak, where B_0^ϕ denotes the toroidal field magnitude at the axis R_0. Suppose B^ϕ is positive. The trajectories on the invariant manifolds of a Möbiusian cycle are expected to follow:

$$X_{\text{pol}}(\phi) = [R_0 + |\lambda_u|^{\phi/2m\pi} \cos\theta_u(\phi), Z_0 + |\lambda_u|^{\phi/2m\pi} \sin\theta_u(\phi)]^T \text{ (unstable)}$$
$$X_{\text{pol}}(\phi) = [R_0 + |\lambda_s|^{\phi/2m\pi} \cos\theta_s(\phi), Z_0 + |\lambda_s|^{\phi/2m\pi} \sin\theta_s(\phi)]^T \text{ (stable)}$$

where λ_u, λ_s (independent of ϕ) denote the two eigenvalues of DP^m, and $\theta_u(\phi), \theta_s(\phi)$ denote the corresponding eigenvectors. If both θ_u and θ_s satisfy $\theta(\phi + 2\pi) = \theta(\phi) + (2k + 1)\pi$, $k \in \mathbb{Z}$, the cycle is Möbiusian; if they satisfy $\theta(\phi + 2\pi) = \theta(\phi) + 2k\pi$, $k \in \mathbb{Z}$, the cycle is non-Möbiusian.

Figure 1: (a, b) show the same Möbiusian saddle cycle, with the field constructed by the governing equations using parameters: $(R_0, Z_0) = (1.0, 0.0)$, $B_0^\phi = 2.5$, $\theta_u(\phi) = \phi/2 + \pi/2$, $\theta_s(\phi) = \phi/2$, $\lambda_{u/s} = -e^{\pm 1/3}$. The sprouts of

two invariant branches are plotted (red for W^u, blue for W^s). (a) and (b) draw the manifolds on $\phi \in [0, 2\pi]$ and $[0, 4\pi]$ for a half and full poloidal turn, respectively. A trajectory on the unstable branch is drawn for three toroidal turns ($\phi \in [0, 6\pi]$).

Let $\Delta X_{\text{pol}}(\phi) := X_{\text{pol}}(\phi) - [R_0, Z_0]^T$. For unstable trajectories, expanding $\Delta X_{\text{pol}}(\phi)$ yields expressions that match the prescribed eigenstructure. The FLT equation in cylindrical coordinates is $\Delta X_{\text{pol}}(\phi) = R\mathbf{B}_{\text{pol}}/B^\phi(X_{\text{pol}}(\phi), \phi)$. Expanding $R\mathbf{B}_{\text{pol}}/B^\phi$ around the cycle and matching terms allows construction of the desired field. Eigen-decomposition is employed to satisfy the first-order equations at θ_u and θ_s, requiring θ_0 to be equal for both.

The construction is completed by determining the linearized B^R, B^Z fields (note B^R, B^Z at the axis are preset to zero). An example of a Möbiusian saddle cycle is shown in Fig. 1. The non-Möbiusian case is not shown as it is easy to imagine. Some readers may wonder about higher-order terms $\partial^k \mathbf{B}_{\text{pol}}/\partial(R, Z)^k$, $k \geq 2$, which are determined by setting $\partial^k(R\mathbf{B}_{\text{pol}}/B^\phi)/\partial(R, Z)^k = 0$.

4.2 Dummy X-Cycle

In this subsection, R_0, Z_0 in the previous model are replaced with $R_c(\phi), Z_c(\phi)$, two ϕ-dependent functions. An example expression is given:

$$R_c(\phi) = R_{\text{ell}} \cos(\iota\phi + \theta) + R_{\text{ax}}$$
$$Z_c(\phi) = Z_{\text{ell}} \sin(\iota\phi + \theta) + Z_{\text{ax}}$$

where subscript c denotes cycle, ell denotes elliptic, and ax denotes axis. Similar to the first model, set $B^\phi(R, Z, \phi) = B^\phi(R) = R_{\text{ax}} B^\phi_{\text{ax}}/R$ to simulate a tokamak.

The zeroth-order field expansion terms are denoted B^R_c, B^Z_c, B^ϕ_c, evaluated on the cycle. The first-order expansion $\partial\mathbf{B}_{\text{pol}}/\partial(R, Z)$ is calculated using the same eigen-decomposition approach. The construction is completed by expanding the poloidal field around the cycle to first order.

Figure 2: (a, b, c) show the same dummy X-cycle, with the field constructed using parameters: $(R_{\text{ax}}, Z_{\text{ax}}) = (1.0, 0.0)$, $B^\phi_{\text{ax}} = 2.5$, $\iota = n/m = 1/3$, $(R_{\text{ell}}, Z_{\text{ell}}) = (0.3, 0.5)$, $\theta = \pi/9$, $\lambda_{u/s} = e^{\pm 1/5}$. (a) shows a top view; (b, c) show other views. (a, b) draw arrows for the two eigenvectors of $DP^3(\phi)$ and their opposites (blue for $\lambda < 1$, red for $\lambda > 1$), acquired via the DP^m evolution formula and consistent with the designed $\theta_{u/s}$. The sprouts of four invariant branches are plotted (orange for W^u, blue for W^s). In (a, b), manifolds are not shown on $\phi \in [6\pi - 2\pi/3, 6\pi]$ to avoid obscuring eigenvector arrows. A transparent torus with the corresponding elliptic section is drawn for reference.

4.3 A Real-World Example

The equilibrium field of EAST shot #103950 at 3500 ms from EFIT is taken as background, superimposed with a non-axisymmetric field induced by RMP coils running in $n = 1$ mode. Plasma response is not considered for simplicity. B^ϕ for this shot is negative everywhere, and $\mathbf{B}_{\text{pol}}$ at R-Z cross-sections is clockwise.

To locate periodic points of the Poincaré map, the simplest discrete Newton method $x_{j+1} = x_j - [DG(x_j)]^{-1} G(x_j)$ is employed, where $G(x) := F(x) - I$ and I is the identity map. To locate m-periodic orbits of $P(\phi)$ at section ϕ, our map F is chosen as the m-th iterate $P^m(\phi)$.

After locating the cycle, one needs $DP^m(\phi)$ at every section, which is obtained using the DP^m evolution formula. This formula is a traditional matrix ODE system. Python's `scipy.integrate.solve_ivp` and Julia's `DifferentialEquations.jl` provide suitable numerical algorithms. Next, eigen-decomposition of $DP^m(\phi)$ yields eigenvectors (e.g., via `scipy.linalg.eig`). The two eigenvectors of $DP^m(\phi)$ and their opposites give the initial growth directions of invariant manifolds.

Recall the invariant manifold growth formula requires only $R\mathbf{B}_{\text{pol}}/B^\phi$ and $\partial X^{u/s}/\partial\phi$ on the right-hand side. In numeric implementation, $R\mathbf{B}_{\text{pol}}/B^\phi$ is linearly interpolated on a regular grid of shape (n_R, n_Z, n_ϕ). For $\partial X^{u/s}/\partial\phi$, different numerical approaches exist. Two methods are exhibited below.

Figure 3: (a) 3D visualization of invariant manifolds of the lower X-cycle of EAST shot #103950 at 3500 ms (EFIT + vacuum RMP). The first wall is drawn as a transparent grey surface, with the top removed to avoid obscuring manifolds. (b) Enlarged view of (a) at $\phi = 0$ near the lower X-point.

4.3.1 Naive Field Line Tracing The simpler scheme distributes a line of Poincaré seed points along an eigenvector of a periodic point. Computing the arc length s of a branch of $W^{u/s}(P^m, x_0)$ requires ordering the FLT Poincaré orbits used to construct the manifold, achieved with assistance from DP^m eigenvalues.

Figure 4 [Figure 4: see original paper]: Illustration of the numeric algorithm to grow invariant manifolds.

Suppose x_0 is a saddle fixed point of the 2D Poincaré map P at an R-Z section. Denote the unstable eigenvalue and eigenvector of $DP(x_0)$ by λ_u and v_u. For an m-periodic saddle point, substitute $DP^m(x_0)$ for $DP(x_0)$. Seed points $(x_1, \dots, x_N)$ are distributed along v_u with equal spacing. If $\lambda_u > 1$, $P(x_1)$ will likely fall behind x_N, making $s(P(x_1)) < s(x_N)$. This complicates computing s because the order is uncertain.

Define a sequence $X := (x_1, \dots, x_N) \frown (P(x_1), \dots, P(x_N)) \frown (P^2(x_1), \dots, P^2(x_N)) \frown$ ... It is expected that $s(P(x_1)) > s(x_N)$, $s(P^2(x_1)) > s(P(x_N))$, etc. In fact, as long as $s(P(x_1)) > s(x_N)$, the conditions for $k \geq 1$ are naturally satisfied. Since $P(x) \approx x_0 + \lambda_u(x - x_0)$ near x_0 for x in direction v_u, placing x_N closer to x_0 than $P(x_1)$ ensures $s(P^k(x_N)) < s(P^{k+1}(x_1))$. This untangles orbits easily and avoids tentative manifold growth methods that need to decrease step size when local manifold curvature is large. It ensures $s(X) = (s(x_1), \dots, s(x_N), s(P(x_1)), \dots, s(P(x_N)), \dots)$ is strictly increasing, guaranteeing safe computation of s by simply accumulating segment lengths. This technique applies to hyperbolic fixed points of n-dimensional maps.

The invariant manifolds in Fig. 3 and Fig. 5 are grown using this naive FLT technique, with computed arc lengths s expressed by varying color for clarity. One immediately observes that confinement in this equilibrium relies mostly on invariant manifolds of the lower X-cycle γ_{low}, though an X-cycle also exists at the top—known as the disconnected double-null configuration.

Blue arrows in Fig. 5(c) are drawn using the invariant manifold growth formula, which takes $\partial X^{u/s}/\partial\phi$ and gives $\partial X^{u/s}/\partial s$. Evidently, $\partial X^{u/s}/\partial s$ is the manifold growth direction. A first-order central scheme calculates $\partial X^{u/s}/\partial s(s,\phi)$ using $X^{u/s}(s,\phi)$ at neighboring R-Z sections at $\phi \pm \epsilon$.

4.3.2 Discretizing the Invariant Manifold Growth PDE to an ODE System The other numerical approach transforms the invariant manifold growth formula—a PDE including ∂_s and ∂_ϕ—into a system of ODEs with s as the evolution parameter. Transect the invariant manifold by N R-Z cross-sections at $\{\phi_i\}_{i=1}^N$ to discretize it, where $\{\phi_i\}$ is an arithmetic sequence from 0 to $2m\pi$ with common difference $\Delta\phi$. The R and Z coordinates at each section become univariate functions of s, yielding a $2N$-dimensional ODE system:

$$dX^{u/s}(s,\phi_i)/ds = \pm \frac{R\mathbf{B}_{\text{pol}}/B^\phi - \Delta X^{u/s}(s,\phi_i)/\Delta\phi}{\|R\mathbf{B}_{\text{pol}}/B^\phi - \Delta X^{u/s}(s,\phi_i)/\Delta\phi\|}$$

where $\Delta X^{u/s}(s,\phi_i)/\Delta\phi$ denotes a numerical approximation of $\partial X^{u/s}(s,\phi)/\partial\phi$ using first- or second-order central differences.

This discretization works well for invariant manifolds of outermost saddle cycles but fails for $q = m/n = 3/1$ island chains, where stable and unstable branches become indistinguishable. The primary reason is suspected to be that the two eigenvectors are too close, and grid granularity is insufficient to resolve the difference needed for correct branch calculation. Although this scheme suffers such numerical instability, it is worth introducing because it directly utilizes the invariant manifold growth formula and can be useful when DP^m eigenvectors are not extremely close.

4.3.3 Notations Explained with Figures A periodic orbit for a map is best viewed as a whole to reflect intrinsic homoclinic/heteroclinic structure. One can denote $W^u(P,\{x_1,x_2,x_3\})$ for all unstable branches of the X-cycle of the $q = m/n = 3/1$ island chain in Fig. 5. This X-cycle has three strike points $\{x_1,x_2,x_3\}$ through the $\phi = 0$ section, each with two stable and two unstable branches—totaling 6 branches of $W^s(P,\{x_1,x_2,x_3\})$ and 6 branches of $W^u(P,\{x_1,x_2,x_3\})$. One can specify a branch more finely by replacing P with P^m and $\{x_1,x_2,x_3\}$ with x_i, i.e., $W^{u/s}(P^m,x_i)$, representing the two unstable/stable branches belonging to specific point x_i. Obviously:

$$W^{u/s}(P,\{x_1,x_2,x_3\}) = \bigcup_{i=1}^3 W^{u/s}(P^m,x_i)$$

The 2D manifold $W^{u/s}(\mathbf{B}, \gamma)$ consists of all corresponding 1D manifolds $W^{u/s}(P^m(\phi), x(\phi))$ for Poincaré maps $P^m(\phi)$ at all sections ϕ:

$$W^{u/s}(\mathbf{B}, \gamma) = \{x(R, Z, \phi) \in M : (R, Z) \in W^{u/s}(P^m(\phi), x(\phi)), \phi \in [0, 2m\pi)\}$$

where $x(\phi)$ denotes the (R, Z) coordinates of cycle γ at angle ϕ, and $P(\phi)$: $\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ denotes the Poincaré map at ϕ. For the X-cycle in Fig. 5, $x(\phi)$ is a 6π-periodic function representing its (R, Z) coordinates, with $P(x_1) = x_2, P(x_2) = x_3, P(x_3) = x_1$. Then $x(0) = x_1, x(-2\pi) = x_2, x(-4\pi) = x_3$ (B^ϕ is negative in this shot). Viewing the 3D Fig. 3 and 2D Fig. 5 together helps understand the relationship between 2D manifolds $W^{u/s}(\mathbf{B}, \gamma)$ and 1D manifolds $W^{u/s}(P^m(\phi), x(\phi))$.

In Fig. 5(b), some regions enclosed by invariant manifolds of γ_{low} are filled with colored scatter points. Points are marked with distinctive colors and then mapped by P^{-1}. The color pattern reflects how these regions are connected by field lines. Let x_{low} be the strike point of γ_{low} crossing the $\phi = 0$ section. All transversal intersection points of $W^u(P, x_{\text{low}})$ and $W^s(P, x_{\text{low}})$ (red and blue curves) are homoclinic to x_{low}.

Although fluxes in colored regions (e.g., regions 1 and 2 in Fig. 5(b)) are identical, this flux value likely differs from uncolored regions (e.g., regions 3 and 4). One should always be careful of this fact.

Figure 5: (a) Poincaré plot at $\phi = 0$ of EAST shot #103950 at 3500 ms (EFIT + vacuum RMP). Some invariant manifolds are grown and plotted. (b) Enlarged view near the lower X-point. Dense scatter points with color show region connectivity via field lines. (c) Further enlarged view near x_1. Blue arrows are drawn according to the growth formula.

5.1 Comparisons with Existing Works

This subsection compares various approaches for studying invariant manifolds. Most existing research has been satisfied with Floquet's normal form. Floquet theory is so successful that further exploration often ceases. Floquet's theorem governs how $DX_{\text{pol}}(\phi_s, \phi_e)$ varies with ϕ_e and guides system solution. However, it remains unclear how $DX_{\text{pol}}(\phi_s, \phi_s + 2m\pi)$ changes with ϕ_s, which is essential for reducing computational resources needed to solve initial growth directions of stable and unstable manifolds.

The work most similar to our DP^m evolution formula is by Tsutsumi, who considered the linear system $d\phi/dt = A(\phi, t)x$, where $A(\phi, t)$ is T-periodic in ϕ. Tsutsumi's theorem states that a monodromy matrix independent of t exists iff a matrix function $\Gamma(\phi, t)$ exists that is T-periodic in ϕ and satisfies $A(\phi, t) - \Gamma(\phi, t) + [A(\phi, t), \Gamma(\phi, t)] = 0$. Tsutsumi's equation is more complicated than our DP^m evolution formula because $A(\phi, t)$ depends on both ϕ and t. If $\partial A/\partial t$ vanishes, Γ is governed by the same equation as our DP^m evolution formula. Tsutsumi did not explain what Γ is or how to construct it, focusing instead on conditions for $A(\phi, t)$ such that the monodromy matrix is t-independent.

Works similar to our invariant manifold growth formula are collated in Table 2 . The classical invariance equation for n-dimensional maps might be too general for fusion scientists. T. E. Evans, J. P. England, and J. M. Ottino et al. focused on numerical or experimental methods rather than analysis.

Abdullaev has contributed to establishing manifold formulas, but his work: (a) relies on small perturbation assumptions to transform FLT to perturbed Hamiltonian form; (b) uses Poincaré integrals for first-order approximations, while our Poincaré map is exact; (c) yields implicit rather than explicit formulas; (d) uses non-standard (x, y) coordinates not necessarily perpendicular. A detailed comparison is in Appendix B.

Table 2: Manifold Expression Comparison

Reference	Method	Key Features
This work	Exact analytic formula	Cylindrical coordinates, no perturbation assumption
Abdullaev [30]	Perturbation theory	Small ϵ, implicit expression, Hamiltonian form
Evans et al. [16]	Numerical	Experimental approach, no analytic formula
England et al. [13-15]	Numerical	BVP continuation methods
Haro et al. [10]	Parameterization	General n-dim maps, complex for fusion applications

5.2 Conclusion

Figure 6 [Figure 6: see original paper]: Thought process map of this paper, where grey boxes indicate non-new content.

Although tokamak magnetic fields are mostly considered axisymmetric, almost all auxiliary heating schemes except ohmic heating are strongly localized and non-axisymmetric. Magnetic field topology dominates confined plasma behavior and the scrape-off layer. Motivated by curiosity about intrinsic characteristics of general 3D vector fields (not necessarily divergence-free), this paper establishes an analytic theory on invariant manifolds of cycles, regarding short-period cycles as the skeleton of fields.

Invariant manifolds of saddle and parabolic cycles grow from $DP^m(\phi)$ eigenvectors. The DP^m evolution formula eliminates repetitive $2m\pi$-integration of the ODE for $DP^m(\phi)$ at every angle ϕ. The primitive FLT ODE system is extended to the invariant manifold growth formula in cylindrical coordinates by analyzing relevant differentials.

It is proposed to characterize plasma edge magnetic topology using invariant manifolds of outermost saddle cycle(s) when fields are strongly non-axisymmetric. A tokamak operating in single-null (resp. double-null) mode has one (resp. two) outermost saddle cycle(s). Transversely intersecting manifolds are suspected to cause spiral ribbon-like heat deposition patterns on divertors observed in experiments, requiring further verification. In the scrape-off layer, dispersing particle and heat fluxes before they reach the divertor is an interesting problem worthy of more investigation. With drift effects included in future work, heat load patterns from diagnostics like infrared thermography should become more consistent with divertor regions covered by outermost saddle cycle manifolds. For both tokamak and stellarator communities, boundary plasma transport is essential for controlling heat loads below material limits.

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Appendix A. Proofs

A.1 The geometric meaning of $|DX_{pol}(\phi_s, \phi_e)|$

The proof shows that $|DX_{\mathrm{pol}}(\phi_s, \phi_e)| = \exp\left(\int_{\phi_s}^{\phi_e} [R(\nabla \cdot \mathbf{B})/B^{\phi}] d\phi\right) \cdot (B^{\phi}|_{\phi_s} / B^{\phi}|_{\phi_e})$ by carefully analyzing the divergence in cylindrical coordinates and using properties of the flow.

A.2 DX to DP formula

The coordinate transformation between Cartesian and cylindrical representations of the derivative is derived by analyzing differential relationships at trajectory start and end points, yielding the matrix formula connecting DX and DX_{pol}.

A.3 DP^m evolution formula

The commutator form $\frac{d}{d\phi} DP^m(\phi) = [A(\phi), DP^m(\phi)]$ is derived by differentiating the flow map, using matrix inversion formulas and periodicity properties of $A(\phi)$ on the cycle.

A.4 DP^m eigenvector evolution formula

Eigenvector rotation along the cycle is governed by a differential equation derived from differentiating the eigenvalue problem and substituting the DP^m evolution formula, showing how eigenvectors evolve while eigenvalues remain constant.

A.5 Invariant manifold growth formula

The PDE $\partial_s X^{u/s} = \pm(R\mathbf{B}_{\mathrm{pol}}/B^{\phi} - \partial_{\phi} X^{u/s})/\|...\|$ is derived by analyzing differential geometry of the manifold, relating $ds, d\phi, dR$, and dZ through multivariable calculus.

Appendix B. Comparison with Abdullaev's Work

Abdullaev's approach transforms to canonical variables (z, p_z) using normalized poloidal flux ψ_p as Hamiltonian, which requires closed flux surfaces diffeomorphic to T^2—limiting applicability to near-integrable systems. Our theory treats $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_{\mathrm{pert}}$ as a whole without perturbation distinction. Abdullaev's Poincaré map uses full poloidal turns with sections Σ_s consisting of ξ- and η-axes, yielding implicit manifold expressions valid only for infinitesimal ϵ. Our explicit formula in standard cylindrical coordinates avoids these limitations.

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Note: Figure translations are in progress. See original paper for figures.