

## Some open problems on cycles (postprint)

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**Date:** 2024-03-27T00:00:00+00:00

### Abstract

Let  $f(n)$  be the maximum number of edges in a graph on  $n$  vertices in which no two cycles have the same length. Erdős raised the problem of determining  $f(n)$ . Erdős conjectured that there exists a positive constant  $c$  such that  $ex(n, C_{2k}) \geq cn^{1+1/k}$ . Hajós conjectured that every simple even graph on  $n$  vertices can be decomposed into at most  $n/2$  cycles. We present the problems, conjectures related to these problems and we summarize the known results. We do not think Hajós' conjecture is true.

### Full Text

#### Preamble

#### Some Open Problems on Cycles

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### Abstract

Let  $f(n)$  be the maximum number of edges in a graph on  $n$  vertices in which no two cycles have the same length. Erdős raised the problem of determining  $f(n)$ . Erdős conjectured that there exists a positive constant  $c$  such that  $ex(n; C_{2k}) \geq cn^{1+1/k}$ . Hajós conjectured that every simple even graph on  $n$  vertices can be decomposed into at most  $n/2$  cycles. We present the problems and conjectures related to these questions and summarize the known results. We do not believe Hajós' conjecture is true.

**Key words:** Hajós conjecture; even graph; Turán number; cycle; maximum number of edges

**AMS Subject Classifications:** 05C35, 05C38

*Project supported by the National Science Foundation of China (No. 11101358), Innovation Program for Young Scientists of Fujian (2011J01026), NSF of Fujian (2011J05014), Fujian Provincial Training Foundation for “Bai-Quan-Wan Talents Engineering”, IRTSTFJ, Project of Fujian Education Department (JA11165) and Project of Zhangzhou Teachers College.*

## Erdős Problem 1

Let  $f(n)$  be the maximum number of edges in a graph on  $n$  vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining  $f(n)$  (see Bondy and Murty [?], p.247, Problem 11). Shi [?] proved a lower bound.

**Theorem 1.1 (Shi [?]).**  $f(n) \geq n + \lfloor \frac{\sqrt{8n-23}+1}{2} \rfloor$  for  $n \geq 3$ .

Chen, Lehel, Jacobson, and Shreve [?], Jia [?], Lai [?, ?, ?, ?, ?, ?, ?], Shi [?, ?, ?, ?, ?, ?], and Y. Shi, Y. Tang, H. Tang, L. Gong and L. Xu [?] obtained some results. Boros, Caro, Füredi and Yuster [?] proved an upper bound.

**Theorem 1.2 (Boros, Caro, Füredi and Yuster [?]).** For  $n$  sufficiently large,  $f(n) < n + 1.98\sqrt{n}$ .

Lai [?] improved the lower bound.

**Theorem 1.3 (Lai [?]).**  $f(n) \geq n + \sqrt{n}(1 - o(1))$ .

Lai [?] proposed the following conjecture:

**Conjecture 1.4 (Lai [?]).**  $f(n) = n + O(\sqrt{n})$ .

It seems difficult to prove this conjecture. It would be nice to prove one of the following weaker conjectures:

**Conjecture 1.5 (Lai [?]).**  $\liminf_{n \rightarrow \infty} \frac{f(n)-n}{\sqrt{n}} = \infty$ .

**Conjecture 1.6 (Lai [?]).**  $\liminf_{n \rightarrow \infty} \frac{f(n)-n}{\sqrt{n}} > 0$ .

Let  $f_2(n)$  be the maximum number of edges in a 2-connected graph on  $n$  vertices in which no two cycles have the same length. Shi [?] proved that:

**Theorem 1.7 (Shi [?]).** For every integer  $n \geq 3$ ,  $f_2(n) \leq n + \lfloor \frac{\sqrt{8n-15}-3}{2} \rfloor$ .

Chen, Lehel, Jacobson, and Shreve [?] proved that:

**Theorem 1.8 (Chen, Lehel, Jacobson, and Shreve [?]).**  $f_2(n) \geq n + \frac{n}{2} - o(n)$ .

Boros, Caro, Füredi and Yuster [?] improved this lower bound significantly:

**Theorem 1.9 (Boros, Caro, Füredi and Yuster [?]).**  $f_2(n) \geq n + O(n^{9/20})$ .

**Corollary 1.10 (Boros, Caro, Füredi and Yuster [?]).**  $2 \geq \limsup_{n \rightarrow \infty} \frac{f_2(n) - n}{\sqrt{n}} \geq \liminf_{n \rightarrow \infty} \frac{f_2(n) - n}{\sqrt{n}} \geq \frac{1}{2}$ .

Boros, Caro, Füredi and Yuster [?] made the following conjecture:

**Conjecture 1.11 (Boros, Caro, Füredi and Yuster [?]).**  $f_2(n) = n + O(\sqrt{n})$ .

It is easy to see that Conjecture 1.11 implies the (difficult) upper bound in the Erdős-Turán Theorem [?, ?] (see Boros, Caro, Füredi and Yuster [?]).

Markström [?] raised the following problem:

**Problem 1.12 (Markström [?]).** Determine the maximum number of edges in a Hamiltonian graph on  $n$  vertices with no repeated cycle lengths.

Let  $g(n)$  denote the least number of edges of a graph which contains a cycle of length  $k$  for every  $1 \leq k \leq n$ . Jia [?] proved the following results:

**Theorem 1.13 (Jia [?]).** When  $n$  is sufficiently large,  $n + \log_2 n - 1 \leq g(n) \leq n + \log_2 n + 1$ .

**Theorem 1.14 (Jia [?]).** For a sufficiently large positive integer  $n$ ,  $g(n) \leq n + \log_2 n + \frac{3}{2} \log_2 \log_2 n + O(1)$ .

**Corollary 1.15 (Jia [?]).** For  $n$  sufficiently large,  $g(n) = n + \log_2 n + O(\log_2 \log_2 n)$ .

Jia [?] made the following conjecture:

**Conjecture 1.16 (Jia [?]).**  $g(n) = n + \log_2 n + O(1)$  as  $n \rightarrow \infty$ .

The sequence  $(c_1, c_2, \dots, c_n)$  is the cycle length distribution of a graph  $G$  of order  $n$  where  $c_i$  is the number of cycles of length  $i$  in  $G$ . Let  $f(a_1, a_2, \dots, a_n)$  denote the maximum possible number of edges in a graph which satisfies  $c_i \leq a_i$  where  $a_i$  is a nonnegative integer. Shi posed the problem of determining  $f(a_1, a_2, \dots, a_n)$  which extended the problem due to Erdős; it is clear that  $f(n) = f(1, 1, \dots, 1)$  (see Xu and Shi [?]).

The lower bound  $f(0; 0; 2; \dots; 2)$  is given by Xu and Shi [?].

**Theorem 1.17 (Xu and Shi [?]).** For  $n \geq 3$ ,  $f(0, 0, 2, \dots, 2) \geq n - 1 + \lfloor \frac{\sqrt{11n-20}}{2} \rfloor$ , and the equality holds when  $3 \leq n \leq 10$ .

Given a graph  $H$ , what is the maximum number of edges of a graph with  $n$  vertices not containing  $H$  as a subgraph? This number is denoted  $\text{ex}(n; H)$ , and is known as the Turán number.

We denote by  $m_i(n)$  the number of cycles of length  $i$  in the complete graph  $K_n$  on  $n$  vertices. Obviously,  $\text{ex}(n; C_k) = f(0, 0, m_3(n), \dots, m_{k-1}(n), 0, m_{k+1}(n), \dots, m_n(n)) =$

$f\left(0, 0, \frac{2}{n(n-1)}, \dots, \frac{n(n-1)}{2}, 0, \frac{2}{n(n-1)}, \dots, \frac{2}{n(n-1)}\right)$ . Therefore, finding  $\text{ex}(n; C_k)$  is a special case of determining  $f(a_1, a_2, \dots, a_n)$ .

There are no good sufficient and necessary conditions for when a graph on  $n$  vertices has no two cycles of the same length. There are also no good sufficient and necessary conditions for when a 2-connected graph on  $n$  vertices has no two cycles of the same length.

## Erdős Conjecture 2

P. Erdős conjectured that there exists a positive constant  $c$  such that  $\text{ex}(n; C_{2k}) \geq cn^{1+1/k}$  (see Erdős [?]). Erdős [?] posed the problem of determining  $\text{ex}(n; C_4)$ .

Erdős [?] (published without proof) and Bondy and Simonovits [?] obtained that:

**Theorem 2.1 (Erdős [?] and Bondy and Simonovits [?]).**  $\text{ex}(n; C_{2k}) \leq c_k n^{1+1/k}$ .

Wenger [?] proved the following:

**Theorem 2.2 (Wenger [?]).**  $\text{ex}(n; C_4) \geq \left(\frac{1}{2}\right)^{3/2} n^{3/2}$ ;  $\text{ex}(n; C_6) \geq \left(\frac{1}{2}\right)^{6/5} n^{4/3}$ ;  $\text{ex}(n; C_{10}) \geq \left(\frac{1}{2}\right)^{4/3} n^{6/5}$ .

Füredi [?] proved that:

**Theorem 2.3 (Füredi [?]).** If  $q$  is a power of 2, then  $\text{ex}(q^2+q+1; C_4) = \frac{q(q+1)^2}{2}$ .

Füredi [?] also showed the following:

**Theorem 2.4 (Füredi [?]).** Let  $G$  be a quadrilateral-free graph with  $e$  edges on  $q^2 + q + 1$  vertices, and suppose that  $q \geq 15$ . Then  $e \leq \frac{q(q+1)^2}{2}$ .

**Corollary 2.5 (Füredi [?]).** If  $q$  is a prime power greater than 13,  $n = q^2 + q + 1$ , then  $\text{ex}(n; C_4) = \frac{q(q+1)^2}{2}$ .

Füredi, Naor and Verstraete [?] proved that:

**Theorem 2.6 (Füredi, Naor and Verstraete [?]).**  $\text{ex}(n; C_6) > 0.5338n^{4/3}$  for infinitely many  $n$  and  $\text{ex}(n; C_6) < 0.6272n^{4/3}$  if  $n$  is sufficiently large.

This refutes the Erdős-Simonovits conjecture from 1982 for hexagons (see [?]). A survey article on this Erdős conjecture can be found in Chung [?].

There are no good sufficient and necessary conditions for when a graph on  $n$  vertices contains  $k$  cycles. For  $k = n$ , this is the Hamiltonian problem; a survey article on the Hamiltonian problem can be found in Gould [?, ?].

## Hajós Conjecture

An Eulerian graph is a graph (not necessarily connected) in which each vertex has even degree. Let  $G$  be an Eulerian graph. A circuit decomposition of  $G$  is a set of edge-disjoint circuits  $C_1, C_2, \dots, C_t$  such that  $E(G) = C_1 \cup C_2 \cup \dots \cup C_t$ .

It is well known that every Eulerian graph has a circuit decomposition. A natural question is to find the smallest number  $t$  such that  $G$  has a circuit decomposition of  $t$  circuits. Such smallest number  $t$  is called the circuit decomposition number of  $G$ , denoted by  $cd(G)$ . For each edge  $xy \in E(G)$ , let  $m(xy)$  be the number of edges between  $x$  and  $y$ . The multiple number of  $G$  is defined by  $m(G) = \sum_{uv \in E(G)} (m(uv) - 1)$ . A reduction of a graph  $G$  is a graph obtained from  $G$  by recursively applying the following operations:

1. Delete a 2-vertex with two distinct neighbors and add a new edge joining its two neighbors.
2. If  $u$  is a 4-vertex with 4 distinct neighbors  $\{x, y, z, w\}$  such that  $xy \in E(G)$  and  $zw \notin E(G)$ , then delete  $u$  and join  $x$  and  $y$  with a new parallel edge, and add a new edge between  $z$  and  $w$ .

A reduction is proper if it is not the original graph (see [?]). The following conjecture is due to Hajós (see [?]).

**Hajós Conjecture:**  $cd(G) \leq |V(G)|$  for every simple Eulerian graph  $G$ .

Lovász [?] proved the following:

**Theorem 3.1 (Lovász [?]).** A graph of  $n$  vertices can be covered by  $\leq \lfloor n/2 \rfloor$  disjoint paths and circuits.

Jiang [?] and Seyffarth [?] considered planar Eulerian graphs.

**Theorem 3.2 (Jiang [?] and Seyffarth [?]).**  $cd(G) \leq |V(G)| - 1$  for every simple planar Eulerian graph  $G$ .

Granville and Moisiadis [?] and Favaron and Kouider [?] extended these results to multigraphs.

**Theorem 3.3 (Granville and Moisiadis [?] and Favaron and Kouider [?]).** If  $G$  is an even multigraph of order  $n$ , of size  $m$ , with  $\Delta(G) \leq 4$ , then  $cd(G) \leq n + M - 1$  where  $M = m - m'$  and  $m'$  is the size of the simple graph induced by  $G$ .

Fan and Xu [?] proved that:

**Theorem 3.4 (Fan and Xu [?]).** If  $G$  is an Eulerian graph with  $cd(G) > |V(G)| + m(G) - 1$  then  $G$  has a reduction  $H$  such that  $cd(H) > |V(H)| + m(H) - 1$  and the number of vertices of degree less than six in  $H$  plus  $m(H)$  is at most one.

**Corollary 3.5 (Fan and Xu [?]).** Hajós' conjecture is valid for projective graphs.

**Corollary 3.6 (Fan and Xu [?]).** Hajós' conjecture is valid for  $K_6$ -minor free graphs.

Xu [?] also proved the following two results:

**Theorem 3.7 (Xu [?]).** If  $G$  is an Eulerian graph such that  $cd(G) > |V(G)| + m(G) - 1$  and  $cd(H) \leq |V(H)| + m(H) - 1$  for each proper reduction of  $G$ , then  $G$  is 3-connected. Moreover, if  $S = \{x, y, z\}$  is a 3-cut of  $G$ , letting  $G_1$  and  $G_2$  be the two induced subgraphs of  $G$  such that  $V(G_1) \cap V(G_2) = S$  and  $E(G_1) \cup E(G_2) = E(G)$ , then either  $S$  is not an independent set, or  $G_1$  and  $G_2$  are both Eulerian graphs.

**Corollary 3.8 (Xu [?]).** To prove Hajós' conjecture, it suffices to prove  $cd(G) \leq |V(G)| + m(G) - 1$  for every 3-connected Eulerian graph  $G$ .

Fan [?] proved that:

**Theorem 3.9 (Fan [?]).** Every Eulerian graph on  $n$  vertices can be covered by at most  $\lfloor \frac{n-1}{2} \rfloor$  circuits such that each edge is covered an odd number of times.

This settles a conjecture made by Chung in 1980 (see [?]).

Xu and Wang [?] gave the following result:

**Theorem 3.10 (Xu and Wang [?]).** The edge set of each even toroidal graph can be decomposed into at most  $(n+3)/2$  circuits in  $O(mn)$  time, where a toroidal graph is a graph embeddable on the torus.

**Theorem 3.11 (Xu and Wang [?]).** The edge set of each toroidal graph can be decomposed into at most  $3(n-1)/2$  circuits and edges in  $O(mn)$  time.

We do not believe Hajós' conjecture is true. A related problem was conjectured by Gallai (see [?]):

**Conjecture 3.12 (Gallai's conjecture).** Every simple connected graph on  $n$  vertices can be decomposed into at most  $(n+1)/2$  paths.

Lovász [?] proved that:

**Theorem 3.13 (Lovász [?]).** If a graph has  $u$  odd vertices and  $g$  even vertices ( $g \geq 1$ ), then it can be covered by  $u/2 + g - 1$  disjoint paths.

**Theorem 3.14 (Lovász [?]).** Let a locally finite graph have only odd vertices. Then it can be covered by disjoint finite paths so that every vertex is the endpoint of just one covering path.

The path number of a graph  $G$ , denoted  $p(G)$ , is the minimum number of edge-disjoint paths covering the edges of  $G$ . Donald [?] proved the following:

**Theorem 3.15 (Donald [?]).** If a graph has  $u$  vertices of odd degree and  $g$  nonisolated vertices of even degree, then  $p(G) \leq u/2 + \lfloor g/2 \rfloor \leq \lfloor 3n/4 \rfloor$ .

Pyber [?] proved that:

**Theorem 3.16 (Pyber [?]).** A graph  $G$  of  $n$  vertices can be covered by  $n - 1$  circuits and edges.

**Theorem 3.17 (Pyber [?]).** Let  $G$  be a graph of  $n$  vertices and  $\{C_1, \dots, C_k\}$  be a set of circuits and edges such that  $\bigcup_{i=1}^k E(C_i) = E(G)$  and  $k$  is minimal. Then we can choose  $k$  different edges,  $e_i \in E(C_i)$ , such that these edges form a forest in  $G$ .

**Theorem 3.18 (Pyber [?]).** Let  $G$  be a graph of  $n$  vertices not containing  $C_4$ . Then  $G$  can be covered by  $\lfloor (n - 1)/2 \rfloor$  circuits and  $n - 1$  edges.

Pyber [?] proved that:

**Theorem 3.19 (Pyber [?]).** Every connected graph  $G$  on  $n$  vertices can be covered by  $n/2 + O(n^{3/4})$  paths.

**Theorem 3.20 (Pyber [?]).** Every connected graph on  $n$  vertices with  $e$  edges can be covered by  $n/2 + 4(e/n)$  paths.

Reed [?] proved that:

**Theorem 3.21 (Reed [?]).** Any connected cubic graph  $G$  of order  $n$  can be covered by  $\lceil n/9 \rceil$  vertex-disjoint paths.

Dean, Kouider [?] and Yan [?] proved that:

**Theorem 3.22 (Dean, Kouider [?] and Yan [?]).** If a graph (possibly disconnected) has  $u$  vertices of odd degree and  $g$  nonisolated vertices of even degree, then  $p(G) \leq u/2 + \lfloor g/2 \rfloor$ .

Fan [?] proved that:

**Theorem 3.23 (Fan [?]).** Every connected graph on  $n$  vertices can be covered by at most  $\lceil n/2 \rceil$  paths.

This settles a conjecture made by Chung in 1980 (see [?]).

**Theorem 3.24 (Fan [?]).** Every 2-connected graph on  $n$  vertices can be covered by at most  $\lfloor \frac{2n-1}{3} \rfloor$  circuits.

This settles a conjecture made by Bondy in 1990 (see [?]).

**Corollary 3.25 (Fan [?]).** Let  $G$  be a 2-edge-connected graph on  $n$  vertices. Then  $G$  can be covered by at most  $\lfloor \frac{3(n-1)}{4} \rfloor$  circuits.

Fan [?] defined a graph operation called an  $\mathcal{F}$ -operation and proved that:

**Definition 3.26 (Fan [?]).** Let  $H$  be a graph. A pair  $(S, y)$ , consisting of an independent set  $S$  and a vertex  $y \in S$ , is called an  $\mathcal{F}$ -pair if the following holds: for every vertex  $v \in S \setminus \{y\}$ , if  $d_H(v) \geq 2$ , then (a)  $d_H(u) \leq 3$  for all  $u \in N_H(v)$  and (b)  $d_H(u) = 3$  for at most two vertices  $u \in N_H(v)$ . (That is, all the neighbors of  $v$  have degree at most 3, at most two of which have degree exactly 3.) An  $\mathcal{F}$ -operation on  $H$  is either (i) add an isolated vertex or (ii) pick

an  $\mathcal{F}$ -pair  $(S, y)$  and add a vertex  $x$  joined to each vertex of  $S$ , in which case the ordered triple  $(x, S, y)$  is called the  $\mathcal{F}$ -triple of the  $\mathcal{F}$ -operation.

**Definition 3.27 (Fan [?]).** An  $\mathcal{F}$ -graph is a graph that can be obtained from the empty set via a sequence of  $\mathcal{F}$ -operations.

**Theorem 3.28 (Fan [?]).** Let  $G$  be a graph on  $n$  vertices (not necessarily connected). The  $E$ -subgraph of  $G$  is the subgraph induced by the vertices of even degree in  $G$ . If the  $E$ -subgraph of  $G$  is an  $\mathcal{F}$ -graph, then  $G$  can be decomposed into  $\lfloor n/2 \rfloor$  paths.

**Corollary 3.29 (Fan [?]).** Let  $G$  be a graph on  $n$  vertices (not necessarily connected). If each block of the  $E$ -subgraph of  $G$  is a triangle-free graph with maximum degree at most 3, then  $G$  can be decomposed into  $\lfloor n/2 \rfloor$  paths.

Harding and McGuinness [?] proved that:

**Theorem 3.30 (Harding and McGuinness [?]).** For every simple graph  $G$  with girth  $g \geq 4$ , with  $u$  vertices of odd degree and  $w$  nonisolated vertices of even degree, there is a path-decomposition having at most  $u/2 + \lfloor \frac{g+1}{2g} w \rfloor$  paths.

## Acknowledgment

The authors would like to thank Professor Y. Caro, G. Fan, R. Gould, B. Xu, and R. Yuster for their advice and for sending some papers to us. The authors would like to thank the referee for his many valuable comments and suggestions.

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