A Modified Full-Newton-Step Feasible Interior-Point Algorithm for a Class of Linear Weighted Complementarity Problems Postprint

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Abstract

As a generalization of complementarity problems, the weighted complementarity problem is an important optimization problem that can model a large class of practical equilibrium problems in economics and finance. Due to the existence of nonzero weight vectors, weighted complementarity problems are much more complex than complementarity problems, and thus algorithms for weighted complementarity problems are currently scarce. We extend the interior-point algorithm for linear optimization to weighted complementarity problems. Based on an equivalent transformation of the central path, we propose a modified full-Newton step feasible interior-point algorithm for solving a class of linear weighted complementarity problems in the nonnegative orthant. At each iteration, the algorithm does not require a line search. Under appropriate assumptions, we prove the feasibility of the algorithm and obtain its iteration complexity. Numerical experimental results demonstrate the effectiveness of the algorithm.

Full Text

A Modified Full-Newton Step Feasible Interior-Point Algorithm for a Class of Linear Weighted Complementarity Problems

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Abstract

As a generalization of complementarity problems, the weighted complementarity problem represents an important class of optimization problems that can model a wide range of practical equilibrium problems in economics and finance. Due to the presence of nonzero weight vectors, weighted complementarity problems are significantly more complex than standard complementarity problems, and consequently, algorithms for weighted complementarity problems remain relatively scarce. This paper extends interior-point algorithms from linear optimization to weighted complementarity problems. Based on an equivalent transformation of the central path, we propose a modified full-Newton step feasible interior-point algorithm for solving a class of linear weighted complementarity problems over the nonnegative orthant. The algorithm requires no line search at each iteration. Under appropriate assumptions, we prove the feasibility of the algorithm and establish its iteration complexity. Numerical experimental results demonstrate the effectiveness of the proposed algorithm.

Keywords: weighted complementarity problem; modified full-Newton step; feasible interior-point algorithm; iteration complexity

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1. Introduction

The weighted complementarity problem (weighted complementarity problem) is a novel optimization problem. Currently, theoretical and algorithmic research on weighted complementarity problems remains limited. Potra [?] first introduced the concept of weighted complementarity, which involves finding a pair of vectors belonging to the intersection of a manifold and a cone such that their product in some algebra equals a given weight vector. Potra [?] further demonstrated that quadratic programming and weighted center problems can be transformed into monotone linear weighted complementarity problems, and designed a predictor-corrector interior-point algorithm for solving monotone linear weighted complementarity problems. Smoothing Newton methods [?, ?] and interior-point algorithms [?] are effective approaches for solving linear optimization, complementarity problems [?], and weighted complementarity problems [?]. Interior-point algorithms have attracted significant attention due to their polynomial-time complexity [?].

Kojima et al. [?] presented primal-dual interior-point algorithms for linear complementarity problems. Roos [?] first proposed a full-Newton step feasible interior-point algorithm for linear programming. Zhang et al. [?] developed an infeasible interior-point algorithm for linear programming based on the modified Newton direction. Darvay [?] introduced a new full-Newton step feasible interior-point algorithm for linear programming. Asadi et al. [?] proposed a full-Newton step feasible interior-point algorithm for monotone linear weighted complementarity problems.

Building upon the modified full-Newton step approach proposed in [?], this paper presents a feasible interior-point algorithm for solving linear weighted complementarity problems over the nonnegative orthant. Our algorithm employs modified full-Newton steps, which simplifies the algorithmic analysis while ensuring that all iterates remain within a narrow neighborhood of the central path. We analyze the feasibility and convergence properties of the algorithm and establish its polynomial-time complexity.

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2. Problem Formulation and Central Path

Consider the linear weighted complementarity problem (LWCP) on $\mathbb{R}^n$: find a vector pair $(x, s) \in \mathbb{R}^n$ such that

$$\begin{aligned} &s = Mx + q, \\ &x \circ s = \omega, \\ &x \geq 0,\ \ s \geq 0, \end{aligned}$$

where $M \in \mathbb{R}^{n\times n}$ is a matrix, $q \in \mathbb{R}^n$, $\omega \in \mathbb{R}^n_+$ is the weight vector, and $\circ$ denotes the componentwise product. The strictly feasible region is defined as

$$\mathcal{F}^0 := \{(x, s) \in \mathbb{R}^n \times \mathbb{R}^n \mid s = Mx + q,\ \ x > 0,\ \ s > 0\}.$$

Let A be a full row rank matrix. The central path for LWCP(1) is defined by replacing the complementarity condition $xs - \omega = 0$ with the parameterized equation $xs - \omega = \mu e$, where $\mu > 0$ is a parameter and e is the vector of ones. For any $\mu > 0$, the system

$$\begin{aligned} &s = Mx + q, \\ &xs = \omega + \mu e, \\ &x > 0,\ \ s > 0, \end{aligned}$$

has a unique solution $(x(\mu), s(\mu))$, which we call the μ-center of LWCP(1). The set of all such μ-centers forms the central path, and the limit point of the central path as $\mu \to 0$ yields a solution to LWCP(1).

Define the proximity measure

$$\delta(x, s; \mu) := \left\| \frac{xs}{\mu} - \frac{\omega}{\mu} - e \right\|_\infty .$$

Lemma 1 [?]. If u and v are orthogonal, then $\|uv\|_\infty \leq \frac{1}{4}\|u + v\|^2_\infty$.

Since the search directions dx and ds are orthogonal, we can bound the second-order term $dxds$.

Lemma 2 [?]. If the initial point satisfies the interior-point condition, then the algorithm is well-defined.

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3. Modified Full-Newton Step Feasible Interior-Point Algorithm

Algorithm 1 (Modified Full-Newton Step Feasible Interior-Point Algorithm)

Input: Choose an appropriate parameter $\theta \in (0,1)$, accuracy parameter $\varepsilon > 0$, and initial point (x^0, s^0) with $\mu^0 > 0$ such that $\delta(x^0, s^0; \mu^0) < 1$.

Step 0: Set $k := 0$.

Step 1: If $\mu^k \le \varepsilon$, stop; otherwise, proceed to Step 2.

Step 2: Compute the Newton search direction (dx^k, ds^k) by solving the system

$$\begin{aligned} &M dx^k - ds^k = 0, \\ &s^k dx^k + x^k ds^k = \mu^k e - x^k s^k + \omega. \end{aligned}$$

Step 3: Update the iterates

$$x^{k+1} := x^k + dx^k, \quad s^{k+1} := s^k + ds^k.$$

Step 4: Update the parameter $\mu^{k+1} := (1-\theta)\mu^k$.

Step 5: Set $k := k+1$ and return to Step 1.

Output: An ε-approximate solution (x^k, s^k) of LWCP(1).

In the modified full-Newton step, we compute the search direction by solving the linear system. The algorithm does not require line search at each iteration.

4. Feasibility and Convergence Analysis

Lemma 3. Let $\delta := \delta(x, s; \mu) < 1$ and let (dx, ds) be the solution of the Newton system. Then the new iterates $(x^+, s^+) := (x + dx, s + ds)$ are strictly feasible.

Proof. The feasibility follows from the condition $\delta < 1$ and the orthogonality of dx and ds.

Lemma 4. For any $\alpha \in [0,1]$, we have $\|(x + \alpha dx)(s + \alpha ds) - \omega - \mu e\|_\infty \le (1-\alpha)\delta\mu + \alpha^2 \|dxds\|_\infty$.

Lemma 5. Let $v := \sqrt{xs}$ and $v := \sqrt{\omega/\mu}$. Then the proximity measure after a full-Newton step satisfies

$$\delta(x^+, s^+; \mu^+) \le \frac{\theta\sqrt{n} + \delta^2}{1 - \theta + \sqrt{1-\delta}}.$$

Lemma 6. If $\delta \le \tau$ for some $\tau \in (0,1)$, then the iterates remain in the narrow neighborhood of the central path.

Lemma 7. The sequence $\{\mu^k\}$ generated by the algorithm converges to zero, and the algorithm terminates after a finite number of iterations.

5. Complexity Analysis

Theorem 1. Let $\theta \in (0,1)$ and $\tau \in (0,1)$ be parameters satisfying $\tau \geq \frac{\theta\sqrt{n}+\tau^2}{1-\theta+\sqrt{1-\tau}}$. Then the algorithm terminates after at most

$$\left\lceil \frac{1}{\theta} \log \frac{\mu^0}{\varepsilon} \right\rceil$$

iterations, yielding an ε-approximate solution of LWCP(1).

Proof. From Lemma 5 and the update rule $\mu^{k+1} = (1-\theta)\mu^k$, we have

$$\mu^k \leq (1-\theta)^k \mu^0.$$

The termination condition $\mu^k \leq \varepsilon$ is satisfied when $k \geq \frac{1}{\theta} \log \frac{\mu^0}{\varepsilon}$.

The iteration complexity is therefore $O\left(\frac{1}{\theta} \log \frac{\mu^0}{\varepsilon}\right)$, which is polynomial in the problem size.

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6. Numerical Experiments

To verify the effectiveness of the algorithm, we conducted numerical experiments on a computer with Intel Core i5-8250U CPU @ 1.60GHz, 8.0 GB RAM, running MATLAB R2016b on Windows 10.

We randomly generated linear weighted complementarity problems of various sizes. For each problem size, we generated 10 instances and report the average results. The parameters were set as $\theta = 0.1$ and $\mu^0 = 1$. The termination condition was $\mu^k \leq 10^{-6}$.

Table 1 shows the numerical results for different problem sizes. The columns represent the problem dimension n, the average number of iterations (Iter), the average CPU time in seconds (Time), and the average duality gap (Gap).

Consider a specific LWCP in $\mathbb{R}^3$ with

$$M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \quad \omega = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Starting from $x^0 = (3,2,2)$ and $s^0 = (2,3,2)$, the algorithm converges to the optimal solution $x^* = (2.477, 1.477, 2.000)$ and $s^* = (1.615, 3.385, 1.500)$ after 8 iterations. The proximity measure δ and duality gap decrease monotonically throughout the iterations.

[Figure 1: see original paper]

The numerical results demonstrate that the algorithm is efficient and robust for solving linear weighted complementarity problems of various sizes.

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7. Conclusion

This paper proposes a modified full-Newton step feasible interior-point algorithm for solving a class of linear weighted complementarity problems. Based on an equivalent transformation of the central path, we compute the search direction using a modified Newton system. We prove that the iterates generated by the algorithm remain strictly feasible and converge to an optimal solution with polynomial-time complexity. The numerical experiments validate the theoretical results and demonstrate the practical effectiveness of the algorithm.

Future research directions include extending the algorithm to nonlinear weighted complementarity problems and developing infeasible interior-point variants.

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Note: Figure translations are in progress. See original paper for figures.