

A Functional Law of the Iterated Logarithm for Increments of Two-Parameter Brownian Motion Postprint

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Abstract

Addressing the problem of two-parameter Brownian motion and the law of the iterated logarithm for increments, this work employs large deviation principles for two-parameter Brownian motion and its increments' law of the iterated logarithm as tools to refine relevant results concerning Brownian motion and its increments' law of the iterated logarithm, extending them to the two-parameter setting and ultimately establishing the logarithmic law for increments of two-parameter Brownian motion. As a generalization of Brownian motion, two-parameter Brownian motion exhibits a series of probabilistic and analytic properties analogous to those of Brownian motion. Consequently, by leveraging prior research on the law of the iterated logarithm for Brownian motion and its increments and strengthening the conditions for the law of the iterated logarithm in the two-parameter case, a functional limit result for two-parameter Brownian motion under these enhanced conditions was derived—specifically, the logarithmic law for increments of two-parameter Brownian motion—and its validity was verified.

Full Text

Preamble

Chinese Library Classification Number

Abstract

This paper investigates the functional iterated logarithm law for increments of two-parameter Brownian motion. Using large deviation principles for two-parameter Brownian motion as the primary analytical tool, we improve upon existing results concerning the iterated logarithm law and extend them to the

two-parameter setting. Two-parameter Brownian motion represents a generalization of classical Brownian motion, possessing a series of probability and analytic properties that correspond to those of its one-parameter counterpart. Building upon previous research on the iterated logarithm law for Brownian motion and its increments, we strengthen the conditions required for the iterated logarithm law in the two-parameter case and obtain a functional limit result—specifically, a functional iterated logarithm law for increments of two-parameter Brownian motion. The validity of our results is rigorously verified.

Keywords: two-parameter Brownian motion; increments; triple logarithm law

Introduction

Brownian motion, also known as the Wiener process, is a stochastic process with continuous time parameter and continuous state space, representing one of the simplest and most fundamental stochastic processes in probability theory [1]. With advancements in science and technology, it has become increasingly apparent that real-world random phenomena are influenced by multiple factors rather than a single variable. For instance, weather patterns depend not only on latitude but also on atmospheric circulation and land-sea distribution. This realization has motivated researchers to extend single-parameter theory to more complex multi-parameter settings. Among multi-parameter Brownian motions, the two-parameter Brownian motion is the most representative case [2].

The iterated logarithm law for Brownian motion and two-parameter Brownian motion [3] has been extensively studied. Bi Qiuxiang et al. [4] proved that under certain assumptions, generalized Brownian motion satisfies the iterated logarithm law and established corresponding results. Utilizing large deviation principles for Brownian motion [5-8], researchers have obtained results for Brownian motion increments in Hölder norm. Strassen-type theorems have been established for two-parameter Brownian motion increments [9-10], yielding results on rectangular domains.

Xu Jie et al. [11] employed large deviation principles for two-parameter Brownian motion increments to derive continuity modulus results for a class of two-parameter Brownian processes. Addressing the logarithm law problem for two-parameter Brownian motion, this paper develops large deviation results for its increments and provides rigorous proofs.

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Main Results

We begin by establishing the necessary framework and preliminary results.

Lemma 2 [10]. Let W be a two-parameter Brownian motion. For any closed

set $F \subset C([0, 1]^2)$,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log P(\sqrt{\varepsilon}W \in F) \leq - \inf_{f \in F} I(f)$$

and for any open set $G \subset C([0, 1]^2)$,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log P(\sqrt{\varepsilon}W \in G) \geq - \inf_{f \in G} I(f)$$

where $I(f)$ is the rate function defined for absolutely continuous functions f with $f(0, 0) = 0$ and $\int_{[0, 1]^2} \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 dx dy < +\infty$.

The norm is defined as $\|f\| := \sup_{(x, y) \in [0, 1]^2} |f(x, y)|$, and the function space is $\mathcal{F} = \{f : f(0, 0) = 0, \|f\| < \infty\}$.

Lemma 4 [12] (Borel-Cantelli Lemma). If $\{A_n\}$ is a sequence of events with $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup_{n \rightarrow \infty} A_n) = 0$.

Lemma 5 [13]. Let $\{\xi_n\}_{n \geq 1}$ be a sequence of random variables. If there exists a subsequence $\{\xi_{n_k}\}$ such that $\limsup_{k \rightarrow \infty} \xi_{n_k} \leq \xi_0$ almost surely, then $\liminf_{n \rightarrow \infty} \xi_n \leq \xi_0$ almost surely.

We now state our main theorem. Let $a_u : (0, \infty) \rightarrow (0, \infty)$ be a non-decreasing continuous function satisfying: 1. $0 < a_u \leq u$ 2. $\frac{u}{a_u}$ is non-decreasing 3. $\lim_{u \rightarrow \infty} \frac{\log(u/a_u)}{\log \log \log u} = \infty$

Define $\gamma_u = \sqrt{2a_u^2 \log \log u}$ and the increment process $\Delta_u(x, y) = W(u + a_u x, u + a_u y) - W(u, u)$ for $(x, y) \in [0, 1]^2$.

Theorem. Under the above conditions, we have:

$$\liminf_{u \rightarrow \infty} \sup_{0 \leq x, y \leq 1} \left\| \frac{\Delta_u(\cdot, \cdot)}{\gamma_u} - f(\cdot, \cdot) \right\| = 0 \quad \text{a.s.}$$

and for any $f \in \mathcal{K}$,

$$\liminf_{u \rightarrow \infty} \inf_{0 \leq x, y \leq 1} \left\| \frac{\Delta_u(\cdot, \cdot)}{\gamma_u} - f(\cdot, \cdot) \right\| = 0 \quad \text{a.s.}$$

where $\mathcal{K} = \{f : I(f) \leq 1\}$ is the Strassen ball.

Proof Sketch

The proof proceeds via discretization and application of large deviation principles. Partition the domain using $s_i = ia_u$ and $t_j = ja_u$ for $i, j = 0, 1, \dots, \lfloor u/a_u \rfloor$. For any $\delta > 0$, define the closed set

$$A = \{g : \|g - f\| \geq \delta\}.$$

By Lemma 2, the large deviation principle yields:

$$P\left(\frac{\Delta_u}{\gamma_u} \in A\right) \leq \exp\left(-2 \inf_{g \in A} I(g) \log \log u\right).$$

For sufficiently large u , we have $a_u \geq u/(\log \log u)^M$ for some $M > 0$, which ensures that

$$\sum_{n=1}^{\infty} P\left(\frac{\Delta_{u_n}}{\gamma_{u_n}} \in A\right) < \infty$$

for the subsequence $u_n = e^n$. Applying Lemma 4 (Borel-Cantelli) gives:

$$\limsup_{n \rightarrow \infty} \left\| \frac{\Delta_{u_n}}{\gamma_{u_n}} - f \right\| \leq \delta \quad \text{a.s.}$$

The result is extended to the continuous limit using the monotonicity properties of a_u and the continuity of the sample paths. For any $u_n \leq u \leq u_{n+1}$, we have

$$\left\| \frac{\Delta_u}{\gamma_u} - \frac{\Delta_{u_n}}{\gamma_{u_n}} \right\| \leq C \left(\sqrt{\frac{a_{u_{n+1}}}{a_{u_n}}} - 1 \right) \rightarrow 0$$

as $n \rightarrow \infty$, completing the proof.

Conclusion

This paper extends results on the iterated logarithm law for Brownian motion increments to the two-parameter case. By strengthening the conditions on the increment window a_u , we have successfully generalized the functional iterated logarithm law to two-parameter Brownian motion. Future research directions include investigating the iterated logarithm law for two-parameter Brownian motion increments under Hölder norms and exploring other variants such as the Chung-type iterated logarithm law.

References

- [1] Xi' an University of Electronic Science and Technology, 2006: 38-39. [2] Generalized Brownian sheet iterated logarithm law and increments of generalized Brownian motion. Fujian Normal University. [3] Lu Chuanrong, Zhang Lixin. Sample Path Properties of Gaussian Processes. Science Press, 2001: 3-4. [4] Sample path properties of Gaussian processes, 2001: 92-110. [5] Generalized Brownian motion iterated logarithm law. Journal of Applied Probability and Statistics, 1988, 13(2): 96-102. [6] CHEN Bin. On Strassen' s iterated logarithm for Brownian motion increments in Hölder norm. *Asymptotic Analysis*, 1998, 3(2): 343-358. [7] GAO Fuqing. The functional law of iterated logarithm for multiparameter Wiener process. *Journal of Probability and Statistics*, 1998, 20(4): 321-324. [8] ZHANG Yongkui. Brownian motion increments under L^2

norm. *Journal of Hubei University (Natural Science Edition)*, 2007, 29(4): 342-345. [9] PARK W J. Strassen' s version of the law of the iterated logarithm for two-parameter Gaussian process. *Journal of Multivariate Analysis*, 1974, 4(1): 479-485. [10] Strassen-type theorems for increments of two-parameter Wiener process. [11] XU Jie. Quasi two-parameter Wiener process continuity modulus and functional limit theorems. *Journal of Mathematical Analysis and Applications*, 2016, 434(1): 501-515. [12] Generalized Brownian sheet iterated logarithm law. *Journal of Applied Probability and Statistics*, 2000, 16(4): 1-6. [13] WEI Qicai. Functional law of iterated logarithm for k-dimensional Brownian motion increments under Hölder norm. *Acta Mathematica Sinica, English Series*, 2000, 16(4): 637-654.

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