

Asymptotic Stability of the Zero Solution of Second-Order Integro-Differential Equations via Fixed Point Theory: Postprint

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Abstract

Without employing Lyapunov's direct method, this work investigates the asymptotic stability of a second-order differential-integral equation. When the differential-integral equation contains unbounded terms or the delay is infinite, the application of Lyapunov's direct method to address the asymptotic stability of the zero solution encounters significant difficulties. In contrast, this paper utilizes the fixed point theorem to derive necessary and sufficient conditions for the asymptotic stability of the zero solution of a class of neutral-type second-order differential-integral equations with infinite delay. The fixed point theorem resolves the asymptotic stability problem for the zero solution of second-order differential-integral equations, with its results not only eliminating previous stringent restrictions on infinite delay but also substantially reducing the constraints imposed on the function g .

Full Text

Abstract

This paper investigates the asymptotic stability of the zero solution of a class of second-order differential-integral equations without employing Lyapunov's direct method. When such equations contain unbounded terms or involve infinite delay, the application of Lyapunov's direct method encounters substantial difficulties. By utilizing fixed point theorems, we obtain necessary and sufficient conditions for the asymptotic stability of the zero solution for neutral second-order differential-integral equations with infinite delay. This approach not only resolves the asymptotic stability problem but also eliminates previous stringent restrictions on infinite delay and significantly relaxes the constraints imposed on the nonlinear function g .

1. Introduction

Lyapunov's direct method has long been the predominant technique for analyzing stability and boundedness of ordinary and functional differential equations. However, when equations feature unbounded terms or unbounded delays [?, ?, ?], or when the derivative of the delay fails to satisfy smallness conditions [?], this methodology faces severe limitations. Fixed point theorems offer a powerful alternative that can overcome these challenges while providing additional advantages for stability analysis [?]. Numerous researchers, notably Becker and Burton, have successfully applied fixed point methods to various stability problems [?, ?, ?, ?].

Burton [?] examined the stability of the zero solution for equations of the form $x' + b(t)g(x(t - \tau(t))) = 0$, obtaining sufficient stability conditions under the assumption that g is strictly increasing. Subsequent work [?] addressed asymptotic stability for equations with variable delay, requiring g to be an odd function and strictly monotone increasing in the Lipschitz space. Similar investigations [?] established asymptotic stability results under the conditions $g(0) = 0$ and strict monotonicity of g .

The present work extends these contributions by considering a broader class of second-order differential-integral equations with infinite delay, establishing stability criteria through fixed point theory that impose significantly weaker conditions on both the delay mechanism and the nonlinear function.

2. Problem Formulation and Main Results

We consider the neutral second-order differential-integral equation with infinite delay. Let C_φ denote the space of continuous functions with the supremum norm $\|\phi\| = \sup\{|\phi(s)| : s \in \mathbb{R}^+\}$. For a given continuous initial function φ defined on $(-\infty, 0]$, we define the solution $x(t; 0, \varphi)$ with initial condition $x(s) = \varphi(s)$ for $s \leq 0$.

We impose the following hypotheses:

(H1) The delay function $\tau(t)$ is twice differentiable with $\tau'(t)$ bounded and $\lim_{t \rightarrow \infty} (t - \tau(t)) = +\infty$.

(H2) The nonlinear function g satisfies $g(0) = 0$ and is Lipschitz continuous: there exists $L > 0$ such that $|g(x) - g(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$.

(H3) There exists a continuous function $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and a constant $\alpha \in (0, 1)$ such that for all $t \geq 0$:

$$\int_0^t e^{-\int_s^t a(u)du} |g(\phi(s - \tau(s)))| ds \leq \alpha \sup_{s \leq t} |\phi(s)|$$

(H4) The coefficient functions satisfy appropriate boundedness and integrability conditions ensuring the mapping is well-defined.

(H5) The contraction condition holds: there exists a constant $\alpha \in (0, 1)$ such that the mapping P defined below satisfies $\|P(\phi_1) - P(\phi_2)\| \leq \alpha \|\phi_1 - \phi_2\|$.

Theorem 2.1. Under assumptions (H1)-(H5), the zero solution of the equation is asymptotically stable if and only if the following condition holds:

$$\int_0^{\infty} a(s)ds = +\infty$$

3. Proof of Theorem 2.1

The proof employs the Banach fixed point theorem. Define the complete metric space $C_{\varphi}^0 \subset C_{\varphi}$ consisting of functions satisfying the initial condition $\phi(s) = \varphi(s)$ for $s \leq 0$ and $\lim_{t \rightarrow \infty} \phi(t) = 0$. Construct the mapping $P : C_{\varphi}^0 \rightarrow C_{\varphi}^0$ by:

$$[P(\phi)](t) = \varphi(0)e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_s^t a(u)du} \left(\int_{-\infty}^s g(\phi(u - \tau(u)))du \right) ds$$

We first verify that P maps C_{φ}^0 into itself. Through careful estimation using hypotheses (H1)-(H4), we establish that for any $\phi_1, \phi_2 \in C_{\varphi}^0$:

$$\|P(\phi_1) - P(\phi_2)\| \leq \alpha \|\phi_1 - \phi_2\|$$

where $\alpha \in (0, 1)$ is the contraction constant from (H5). Thus, P is a contraction mapping on the Banach space C_{φ}^0 .

By the Banach fixed point principle, P possesses a unique fixed point in C_{φ}^0 , which corresponds to the unique solution of the differential-integral equation. The stability properties follow from the contraction estimate and the integrability conditions, yielding both stability and asymptotic stability of the zero solution.

The necessity follows from considering a sequence $t_n \rightarrow \infty$ and applying the mean value theorem to show that if the zero solution is asymptotically stable, then the integral condition must hold.

4. Conclusion

This work demonstrates that fixed point theory provides an effective framework for establishing asymptotic stability of second-order differential-integral equations with infinite delay. The methodology eliminates the restrictive boundedness requirements on infinite delay terms that limit traditional Lyapunov approaches and substantially reduces the regularity constraints on the nonlinear function g . These results represent a significant generalization of previous work and open new avenues for stability analysis of functional differential equations with unbounded terms or infinite delays. The fixed point method is not limited to Banach's theorem but can be extended to other fixed point theorems such as Krasnoselskii's and Schauder's theorems.

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