Linear Complementarity Problem on the Monotone Extended Second Order Cone

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Abstract

In this paper, we study linear complementarity problems on monotone extended second-order cones. We demonstrate that the linear complementarity problem on a monotone extended second-order cone can be converted into a mixed complementarity problem on the nonnegative orthant. We prove that any point satisfying the FB equation is a solution of the converted problem. We also show that the semismooth Newton method can be used to solve the converted problem, and we provide a numerical example. Finally, we derive the explicit solution of a portfolio optimisation problem based on the monotone extended second-order cone.

Full Text

Preamble

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In this paper, we study linear complementarity problems on monotone extended second order cones. We demonstrate that the linear complementarity problem on the monotone extended second order cone can be converted into a mixed complementarity problem on the non-negative orthant. We prove that any point satisfying the FB equation is a solution of the converted problem. We also show that the semismooth Newton method can be used to solve the converted problem, and we provide a numerical example. Finally, we derive the explicit solution of a portfolio optimization problem based on the monotone extended second order cone.

Keywords: Complementarity problem • Monotone extended second order cone • Portfolio optimization

Introduction

The concept of complementarity and complementarity problems, first introduced by Karush in [18], is a cross-cutting area of research with wide-ranging applications in economics, finance, and other fields [2, 3, 8, 11]. Previous studies show that second-order cone programming has played a significant role in complementarity problems. The concept of extended second order cone (ESOC) was introduced by Németh and Zhang in [24] as a natural extension of the notion of second order cone. Sznajder calculated the Lyapunov rank (or bilinearity rank) of ESOC in [27] and proved the irreducibility of the ESOC. Ferreira and Németh found an efficient numerical method to project onto the ESOC [7]. Furthermore, Németh and his collaborators investigated the properties of ESOC and used it as a tool for solving various complementarity problems [21–25]. They also proposed an application to the optimization problem of portfolio allocation, called the mean- 2 norm (ML2N) model in [28]. The latter paper exhibits advantages of the mean- 2 norm (ML2N) model compared to the well-known mean-variance model (MV), developed by Markowitz in [19], and the mean-absolute deviation model (MAD), introduced in [15]. The application of the ESOC to solving general complementarity problems is based on determining its isotone projection sets, a concept which is an extension of the notion of isotone projection cones (see [14]) and was introduced in [20]. For the importance of isotone projections in applications, see also [13, 26].

The importance of the ESOC and ordered vector spaces in investigating and solving equilibrium problems in economics, finance, traffic equilibrium, and other fields motivated the introduction in [12] of another extension of the second order cone, namely the monotone extended second order cone (MESOC). In the latter paper, the Lyapunov rank of MESOC was determined, and it was also shown that the monotone extended second order cone can be used to investigate and solve mixed complementarity problems. Furthermore, Ferreira et al. found a numerical way to project onto MESOC [6] and suggested applying MESOC to portfolio optimization. In this paper, we will show how to solve the linear complementarity problem on MESOC and provide an explicit solution to a portfolio optimization problem on MESOC.

The structure of the paper is as follows: In Section 2, we introduce the main terminology and definitions. In Section 3, we convert the linear complementarity problem on MESOC to a mixed complementarity problem on the non-negative orthant. In Sections 4, 5, and 6, we introduce a numerical algorithm that can be used to solve the linear complementarity problem on MESOC, and in Section 7 we present a corresponding numerical example. Finally, in Section 8 we derive the explicit solution of the considered portfolio optimization problem.

2 Preliminaries

Let n $\geq$ 2 be an integer and ^{n} be the n-dimensional Euclidean space, whose elements are identified with column vectors of n components and endowed with the classical inner product $\langle\cdot,\cdot\rangle$: $^n \times {}^n \to$ defined by $\langle x, y\rangle = x y$. Two vectors x, y ^{n} are called perpendicular if $\langle x, y\rangle = 0$, which is denoted by x y.

If p, q are positive integers such that n = p + q, then for simplicity of notation, we identify the vector space × with $^+$ by identifying a pair of vectors (x, u) × , where x and u , with the vector (x , u) $^+$. Therefore, we call a pair of vectors (x, u) simply a vector. Through this identification, the inner product $\langle\cdot,\cdot\rangle$ in × becomes $\langle(x, u), (y, v)\rangle = \langle x, y\rangle + \langle u, v\rangle$ for any (x, u), (y, v) × .

In the literature, there are various ways of defining cones and various types of cones are used. However, in this paper, we consider only cones that are closed and convex sets. Therefore, for simplicity, we call a closed set K a cone if and only if $\alpha x + \beta y$ K for any x, y K and any $\alpha, \beta \geq 0$. A cone K is called proper if it has nonempty interior and K (−K) = {0}.

Let K be a cone. The dual of K is the cone defined by K* := {y ^{n}: $\langle x, y\rangle \geq 0$, x K}, and the complementarity set of K is the set defined by C(K) := {(x, y): x K, y K*, x y}.

Definition 1. The monotone extended second order cone (MESOC) is the proper cone defined by L := {(x, u) × : $x_1 \geq x_2 \geq \ldots \geq x \geq \|u\|$}.

Sometimes we also use the notation L(p, q) to denote that the MESOC is in × .

For the sake of completeness, we quote the following four results that will help us prove Theorem 5, which are Propositions 3.1, 3.2 in [12] and Propositions 4, 5 in [6].

Proposition 1. The dual of the monotone extended second order cone L is the proper cone defined by L* = {(x, u) × : Σ_1 x ≥ 0, j {1, ..., p−1}, Σ_1 x ≥ ‖u‖}.

From now on, p and q will always denote positive integers, while L will always denote the monotone extended second order cone and M its dual.

Proposition 2. Let (x, y, u, v) C(L). If u ≠ 0, v ≠ 0, then C(L) = {(x, u, y, v): (x, u) L, (y, v) M, $\langle x, y\rangle = \|u\|\|v\|$, and $\lambda > 0$ such that $v = -\lambda u$}.

Proposition 3. For arbitrary points (x, u), (y, v) × , we have: (i) (x, u) L if and only if x − ‖u‖e (ii) (y, v) M if and only if y − ‖v‖e ($_1$) *

Proposition 4. Let x, y and u, v {0}. Then we have the following equivalences: (i) (x, 0, y, 0) C(L) if and only if (x, y) C( ) (ii) (x, 0, y, v) C(L) if and only if x = 0, Σ_1 y ≥ ‖v‖ and (x, y) C( ) (iii) (x, u, y, 0) C(L) if and only if x ≥ ‖u‖ for all i, Σ_1 y = 0 and (x, y) C( ) (iv) (x, u, y, v) C(L) if and only if x = ‖u‖, $\langle y, e\rangle = \|v\|$, $\langle u, v\rangle = -\|u\|\|v\|$, and (x − ‖u‖e, y − ‖v‖e) C($_1$)

Below we list definitions of various types of complementarity problems.

Definition 2. Let F: $^n \rightarrow$ ^{n} be an arbitrary mapping and K ^{n} an arbitrary cone. The complementarity problem defined by K and F is CP(F, K) := {find x K such that (x, F(x)) C(K)}. If T $^{n\ n}$ is a constant matrix, r ^{n} is a constant vector and F(x) = Tx + r, then the problem CP(F, K) is called the linear complementarity problem defined by T, r, and K, denoted by LCP(T, r, K).

Definition 3. Let G: $\times$ $\rightarrow$, H: $\times$ $\rightarrow$, F: $\times$ $\rightarrow$ be arbitrary mappings and K an arbitrary cone. The mixed implicit complementarity problem defined by K, G, H and F is MiICP(G, H, F, K) := {find (x, u) $\times$ such that H(x, u) = 0, (F(x, u), G(x, u)) C(K)}.

Definition 4. Let G: $\times$ $\rightarrow$, H: $\times$ $\rightarrow$ be arbitrary mappings and K an arbitrary cone. Then the mixed complementarity problem defined by G, H and K is MiCP(G, H, K) := {find (x, u) $\times$ such that H(x, u) = 0 and (x, G(x, u)) C(K)}.

3 The Linear Complementarity Problem on the MESOC

Theorem 5. Let (x, u), (y, v) be arbitrary vectors with x, y and u, v . Consider the nonsingular block matrix:

T = [A B; C D]

where A , B , C and D are constant matrices. Then, for arbitrary vectors z and r, such that z = (x, u) and r = (y, v), the following statements hold:

(i) Let u = 0. Then z is a solution of LCP(T, r, L) if and only if x is a solution of LCP(A, y,), x = 0 and Σ $_1$ (Ax + y) $\geq$ ‖Cx + v‖.

(ii) Let Cx + Du + v = 0. Then z is a solution of LCP(T, r, L) if and only if x is a solution of MiCP(G, H,), x $\geq$ ‖u‖, and Σ $_1$ (Ax + Bu + y) = 0, where G and H are defined by G(x', u') = Ax' + Bu' + y and H(x', u') = 0.

(iii) Let u $\neq$ 0 $\neq$ Cx + Du + v. Then z is a solution of LCP(T, r, L) if and only if z is a solution of MiICP(G, H, F, $_1$), where F, G and H are defined by: F(x', u') = x' − ‖u'‖e, G(x', u') = Ax' + Bu' + y − ‖Cx' + Du' + v‖e , H(x', u') = u'e (Ax' + Bu' + y) + ‖u'‖(Cx' + Du' + v).

(iv) Define $\bar{z}$ = ($\bar{x}$, u) = (x − ‖u‖e, u) and let u $\neq$ 0 $\neq$ Cx + Du + v. Then z is a solution of LCP(T, r, L) if and only if $\bar{z}$ is a solution of MiCP($\bar{G}$, $\bar{H}$, $_1$), where $\bar{G}$ and $\bar{H}$ are defined by: $\bar{G}$(x', u') = A(x' + ‖u'‖e) + Bu' + y − ‖C(x' + ‖u'‖e) + Du' + v‖e , $\bar{H}$(x', u') = u'e (A(x' + ‖u'‖e) + Bu' + y) + ‖u'‖(C(x' + ‖u'‖e) + Du' + v).

(v) When u $\neq$ 0 $\neq$ Cx + Du + v, the problem of finding a solution z = (x, u) of the linear complementarity problem LCP(T, r, L) is converted to finding a vector z = (x, u) such that (α, β) C($_1$), where:

α = [x_1 − x_2, x_2 − x_3, …, x $_1$ − x , x − ‖u‖] β = [Σ $_1{}^1$(Ax + Bu + y) , Σ $_1{}^2$(Ax + Bu + y) , …, Σ $_1{}^{-1}$(Ax + Bu + y) , Σ $_1$ (Ax + Bu + y)]

Moreover, denote for any i = 1, 2, …, p−1 and any x', w' , u' : p (w') = Σ $^{-1}$ w' + ‖u'‖ p (w') = ‖u'‖

Let x'(w') be defined accordingly. Then the problem of finding a vector z = (x, u) such that (α, β) C($_1$) is equivalent to finding a solution of MiCP(Ĝ, Ĥ, $_1$), where:

Ĝ(w', u') = [Σ $_1{}^1$(Ax'(w') + Bu' + y) , Σ $_1{}^2$(Ax'(w') + Bu' + y) , …, Σ $_1{}^{-1}$(Ax'(w') + Bu' + y)] Ĥ(w', u') = u'e (Ax'(w') + Bu' + y) + ‖u'‖(Cx'(w') + Du' + v)

(vi) Let t = ‖u‖. Then z is a solution of LCP(T, r, L) if and only if x is a solution of MiCP(Ǧ, Ȟ, $_1$), where Ǧ and Ȟ are defined by:

Ǧ(w', u', t') = [Σ $_1{}^1$(Ax'(w', t') + Bu' + y) , Σ $_1{}^2$(Ax'(w', t') + Bu' + y) , …, Σ $_1{}^{-1}$(Ax'(w', t') + Bu' + y)] Ȟ(w', u', t') = [u'e (Ax'(w', t') + Bu' + y) + t'(Cx'(w', t') + Du' + v); t'^{2} − ‖u'‖^2]

where x'(w', t') is defined appropriately.

Proof.

(i) By the definition of the linear complementarity problem, z = (x, 0) is a solution of LCP(T, r, L) if and only if (x, 0, Ax + y, Cx + v) C(L), which by item (ii) in Proposition 4 is equivalent to x = 0, Σ $_1$ (Ax + y) ≥ ‖Cx + v‖ and (x, Ax + y) C( ). This is further equivalent to x being a solution of LCP(A, y,).

(ii) Let Cx + Du + v = 0. By definition, z = (x, u) is a solution of LCP(T, r, L) if and only if (x, u, Ax + Bu + y, 0) C(L), which by item (iii) of Proposition 4 is equivalent to x ≥ ‖u‖, e (Ax + Bu + y) = 0 and (x, Ax + Bu + y) C( ). We conclude that z = (x, u) is a solution of LCP(T, r, L) if and only if z = (x, u) is a solution of MiCP(G, H,).

(iii) By definition, if z = (x, u) is a solution of LCP(T, r, L), then (x, u, Ax + Bu + y, Cx + Du + v) C(L). From item (iv) of Proposition 4 and the equality case of the Cauchy inequality, we have that (x, u, Ax + Bu + y, Cx + Du + v) C(L) is equivalent to the existence of $\lambda > 0$ such that: x = ‖u‖, Cx + Du + v = −λu, e (Ax + Bu + y) = ‖Cx + Du + v‖ = λ‖u‖, (x − ‖u‖e, Ax + Bu + y − ‖Cx + Du + v‖e) C($_1$)

Using these relations, we conclude that (F(x, u), G(x, u)) C($_1$) and H(x, u) = ue (Ax + Bu + y) + ‖u‖(Cx + Du + v) = 0. Thus, z being a solution of LCP(T, r, L) is equivalent to z being a solution of MiICP(G, H, F, $_1$).

(iv) Let z̄ = (x̄, u) = (x − ‖u‖e, u). Then using the notation and conclusion in (iii), we have F(x, u) G(x, u). We also have: F(x, u) = x − ‖u‖e = x̄ G(x, u) = Ax + Bu + y − ‖Cx + Du + v‖e = A(x̄ + ‖u‖e) + Bu + y − ‖C(x̄ + ‖u‖e) + Du + v‖e = Ǧ(x̄, u)

Thus, x̄ G̃(x̄, u). From the proof of (iii) we get: 0 = H(x, u) = ue (Ax + Bu + y) + ‖u‖(Cx + Du + v) = ue (A(x̄ + ‖u‖e) + Bu + y) + ‖u‖(C(x̄ + ‖u‖e) + Du + v) = H̃(x̄, u)

Hence, z = (x, u) being a solution of LCP(T, r, L) is equivalent to z̄ = (x − ‖u‖e, u) being a solution of MiCP(G̃, H̃, $_1$).

(v) If z = (x, u) is a solution of LCP(T, r, L), we have (x, u) L and (Ax + Bu + y, Cx + Du + v) M. From Proposition 2, for any (x, u, y, v) C(L), we have: x = ‖u‖, Σ $_1$ y = ‖v‖, Σ $_1$ (x − x $_1$)y = 0 for all i = 1, ..., p−1, v = −λu for some λ > 0.

In our case, since (x, u, Ax + Bu + y, Cx + Du + v) C(L), we have: $x_1 - x_2$ Σ $_1{}^1$(Ax + Bu + y) $x_2 - x_3$ Σ $_1{}^2$(Ax + Bu + y) ... x $_1$ − x Σ $_1{}^{-1}$(Ax + Bu + y) x − ‖u‖ Σ $_1$ (Ax + Bu + y)

with x = ‖u‖, Cx + Du + v = −λu, and Σ $_1$ (Ax + Bu + y) = ‖Cx + Du + v‖.

Thus, the problem of finding a solution z = (x, u) of LCP(T, r, L) is converted to finding a vector z = (x, u) such that (α, β) C($_1$). Moreover, let w such that w = x − x $_1$ for any i = 1, 2, ..., p−1 and w = x − ‖u‖ = 0. Then we have x = x(w) where x (w) = Σ $^{-1}$ w + ‖u‖ for any i = 1, 2, ..., p−1 and x (w) = ‖u‖. Thus, the complementarity condition is equivalent to w Ĝ(w, u). We also have from the solution of (iv) that Ĥ(w, u) = ue (Ax(w) + Bu + y) + ‖u‖(Cx(w) + Du + v) = 0. Hence, the solution is equivalent to the solution of MiCP(Ĝ, Ĥ, $_1$).

(vi) Note that the function Ĥ(w, u) is a semismooth function and is not differentiable at u = 0. Thus, we need to reformulate this function to ensure it is differentiable everywhere. Let t = ‖u‖. Then, similarly to the proof of (v), for any (x, u, Ax + Bu + y, Cx + Du + v) C(L), we have the complementarity conditions with t replacing ‖u‖. Let ŵ such that ŵ = x − x $_1$ for any i = 1, 2, ..., p−1 and ŵ = x − t = 0. Then we have x = x(ŵ, t), where x (ŵ, t) = Σ $^{-1}$ ŵ + t for any i = 1, 2, ..., p−1 and x (w) = t. Thus, the condition is equivalent to ŵ G(ŵ, u, t). We also have from the solution of (v) that H(ŵ, u, t) = [ue (Ax(ŵ, t) + Bu + y) + t(Cx(ŵ, t) + Du + v); $t^2 - \|u\|^2$]. Hence, the solution is equivalent to the solution of MiCP(G, H, $_1$).

4 F-B Function

From the conclusion in Theorem 5, we have shown that the linear complementarity problem on monotone extended second order cones can be converted to a mixed complementarity problem on the non-negative orthant (as defined by Facchinei and Pang, see Subsection 9.4.2 in [5]). This transformation is important because the converted problem can be well studied using the Fischer-Burmeister function, introduced by Fischer in [9, 10]. For arbitrary numbers a and b, the Fischer-Burmeister function is defined as: $\phi(a, b) = \sqrt{(a^2 + b^2)} - (a + b)$.

From the definition, we can conclude the following property: ϕ(a, b) = 0   a ≥ 0, b ≥ 0 and ab = 0.

We also note that ϕ(a, b) is a continuously differentiable function on ^{2} {0}. Using this function, for any continuously differentiable functions G_1, G_2, ..., G where G = (G_1, G_2, ..., G), the mixed complementarity problem MiCP(G, H,) is equivalent to the following root-finding problem Φ(x) = 0, where: Φ(x) = [ϕ(x_1, G_1), ϕ(x_2, G_2), ..., ϕ(x , G)] .

Meanwhile, the natural merit function Ψ(x) := $\|\Phi(x)\|^2$ is also continuously differentiable and equals zero at a point x* if and only if x* is a solution of MiCP(G, H,). This is equivalent to finding the stationary point x* of the unconstrained problem min Ψ(x). De Luca et al. used this reformulation to propose an algorithm proven to be globally convergent and locally Q-quadratically convergent under considerably weaker regularity conditions than those required by the NE/SQP method in [4].

In our case, note that F_2 is not differentiable at u = 0, so we need to perform some relaxation.

5 Generalized Newton Method for Semismooth Functions

Let DF ^{n} denote the set of points at which F is differentiable. Our aim is to introduce several objects from nonsmooth analysis that provide generalizations of the classical differentiability concept. We start by defining the B-subdifferential, where B stands for "Bouligand," who introduced the concept.

Definition 5. Let F: U ^{n} → , where U is open and F is Lipschitz continuous for any x U. Then the B-differential of function F at x is given by: BF(x) := {G ^{n} : {x } DF with x → x, F(x) → G}.

Algorithm 1: Newton's Method for Nonsmooth Systems 1. Given F: ^{n} → ^{n} locally Lipschitz continuous and x ^{n}, set k = 0. 2. Unless the stopping criterion is satisfied, solve the system G(x)d = −F(x) and obtain d , where G(x) is an arbitrary element of F(x). 3. Set x $_1$ = x + d , k = k + 1, and return to step 1.

First, let us define the following matrices. Let D_1 = diag(d_{11}(x, u, t), ..., d $_1$, $_1$(x, u, t)) and D_2 = diag(d'$_{11}$(x, u, t), ..., d' $_1$, $_1$(x, u, t)), where: d = (√(x ^{2} + G ^{2}(x, u, t)) − x) / √(x ^{2} + G ^{2}(x, u, t)) d' = (√(x ^{2} + G ^{2}(x, u, t)) − G (x, u, t)) / √(x ^{2} + G ^{2}(x, u, t))

Note that we also have (d + 1)^2 + (d' + 1)^2 = 1. When x = 0 = G , (d , d') {(y, z): (y + 1)^2 + (z + 1)^2 = 1}.

Then the generalized Jacobian of the FB function is the set given by: Φ(x, u, t) [D_1 + D_2J G(x, u, t) D_2J , G(x, u, t); J , H(x, u, t) J H(x, u, t)]

For an arbitrary element in the set of generalized Jacobian when x ≠ 0 ≠ G (x, u, t), we have G Φ(x, u, t) with: (G) (x, u, t) = d e + d' J (G (x, u, t))

When x = 0 = G (x, u, t), we have: (G) (x, u, t) {d_{11}e + d'_{11}J G(x, u, t): (d_{11}, d'_{11}) Ball((−1, −1), 1)}.

6 Finding the Minimizer of the Merit Function

A point x is a solution to the mixed complementarity problem if it satisfies Ψ(x) = 0. Since Ψ(x) is a quadratic nonnegative function, if x solves Ψ(x) = 0 then x is a global minimizer of Ψ(x). Thus, finding a solution to the mixed complementarity problem is equivalent to finding a stationary point of Ψ(x).

Consider the following index sets: C = {i: v ≥ 0, H (u, v) ≥ 0, v H (u, v) = 0} (complementarity indices) R = {1, 2, …, n} C (residual indices) P = {i R: v > 0, H (u, v) > 0} (positive indices) N = {1, 2, …, q} (C P) (negative indices)

For any arbitrary vector z, denote by z the i-th coordinate of z where i S and S {C, P, N}.

Definition 6. For the general mixed complementarity problem MiCP(G, H), for arbitrary x , u and t , denote ũ = (u, t) . Then a point (x, u, t) is called FB-regular if JũG(x, u, t) is nonsingular and if for any nonzero vector z with z_C = 0, z_P > 0, z_N < 0, there exists a nonzero vector w such that w_C = 0, w_P ≥ 0, w_N ≤ 0 and: z (M(x, u, t)/JũG(x, u, t))w ≥ 0

where: M(x, u, t) = [J G(x, u, t) JũG(x, u, t); J H(x, u, t) JũH(x, u, t)] (+ +1) (+ +1)

and M(x, u, t)/JũH(x, u, t) is the Schur complement of JũH(x, u, t) in M(x, u, t).

In our case, for MiCP(G(ŵ, u, t), H(ŵ, u, t)), the Jacobians are given as:

JŵG(ŵ, u, t) = [LIA $_1$, $_1$UI] where UI is an upper triangular matrix of ones J , G(ŵ, u, t) = [LIB LIAp $_1$, e] JŵH(ŵ, u, t) = [tC* + ue A*] J , H(ŵ, u, t) = [tD + ue B + (Ax(ŵ, t) + Bu + y) eIq×q; Du + v + Cx(ŵ, t) + tCe + ue Ae −2u]

where C* and A* are defined appropriately.

If D is nonsingular, the Schur complement of D in M(ŵ, u, y) is: Π/D = Ã − $BD^{-1}C$

Proposition 6. The matrix M(x, u, t) is nonsingular for any z = (x, u, t) × × if the corresponding matrices Ã and D are nonsingular.

We can also conclude that the Jacobian Φ(ŵ, u, t) = [Ã B; C D] is nonsingular if Ã and D are nonsingular.

Proposition 7. Since the Jacobian is Φ(ŵ, u, t) = [D_1 + D_2JŵG(ŵ, u, t) D_2J , G(ŵ, u, t); J , H(ŵ, u, t) JŵH(ŵ, u, t)], it is nonsingular if and only if both matrices Ã and D are nonsingular for any vector z = (w, u, t) $^{+1}$.

Proof. For the generalized Jacobian Φ(ŵ, u, t) = [D_1 + D_2Ã D_2B; C D], we conclude that Φ(w, u, t) is nonsingular if and only if both the Schur complement of D and the submatrix $D_1 + D_2\tilde{A}$ are nonsingular, which is equivalent to both matrices $\tilde{A}$ and D being nonsingular.

The following theorem was introduced by Facchinei and Pang. For completeness, we quote Theorem 9.4.4 in [5] and provide a detailed proof.

Theorem 8. For arbitrary vectors ŵ $^{-1}$, u and t , z = (ŵ, u, t) is a solution of MiCP(G, H, $_1$) if and only if z is an FB-regular point of Ψ(x) and a stationary point of Φ(x).

Proof. First, suppose z = (ŵ, u, t) solves MiCP(G, H, $_1$). Then z is a stationary point and the global minimum of the associated merit function Φ(x). Moreover, since (ŵ, G(z)) C($_1$), we have ŵ = w_C. Thus, FB-regularity holds for ŵ with P = = N.

Conversely, if z = (ŵ, u, t) is a stationary point of Ψ(x), then Ψ(z) = 0, which implies: (Φ(ŵ, u, t)) Φ(ŵ, u, t) = $[D_1 + \tilde{A}\ D_2$ C ; B D_2 D]Φ(ŵ, u, t) = 0

Thus, for any x $^+$: x $[D_1 + \tilde{A}\ D_2$ C ; B D_2 D]Φ(ŵ, u, t) = 0

Consider x with x_C = 0, x_P > 0, x_N < 0. If z is not a solution of MiCP(G, H, $_1$), then {1, 2, …, p+q} C ≠ . Let y = D_2Φ(x), giving y_C = 0, y_P > 0, y_N < 0. By definition of D_1 and D_2, D_1Φ(x) and D_2Φ(x) have the same sign. Thus: x (D_1Φ) = x_C (D_1Φ)_C + x_P (D_1Φ)_P + x_N (D_1Φ)_N > 0

since x_{[1,…,p+q]C} ≠ 0, and x J_G(z) (D_1Φ)y ≥ 0. These inequalities contradict condition (11). Therefore, {1, 2, …, p+q} C = and z is a solution of MiCP(G, H, $_1$).

7 A Numerical Example

In this section, we provide a numerical example of the linear complementarity problem defined on the MESOC, which is a more general case satisfying item (iv) in Proposition 4. Consider the linear complementarity problem on MESOC L ^{3} × ^{2}. For any z = (x, u) ^{3} × ^{2}, the goal is to find z = (x, u) ^{3} × ^{2} such that (z, Tz + r) C(L).

By item (vi) in Theorem 5, the solution z = (x, u) of LCP(T, r, L) is equivalent to the solution of MiCP(G, H, $_1$). We have:

ŵ G(ŵ, u, t) ^{2}

where: G(ŵ, u, t) = [Σ $_1{}^1$(Ax(ŵ, t) + Bu + y) , Σ $_1{}^2$(Ax(ŵ, t) + Bu + y)] H(ŵ, u, t) = [ue (Ax(ŵ, t) + Bu + y) + t(Cx(ŵ, t) + Du + v); $t^2 - \|u\|^2$] x(ŵ, t) = $[\hat{w}_1 + \hat{w}_2 + t, \hat{w}_2 + t, t]$

To find the solution, we use the FB-based equation: Φ(ŵ, u, t) = $[\phi(\hat{w}_1, G_1(\hat{w}, u, t)), \phi(\hat{w}_2, G_2(\hat{w}, u, t)), H(\hat{w}, u, t)]$

Consider the example with: T = [A B; C D] = [[1, 2, 3; 4, 5, 6; 7, 8, 9], [1, 2; 3, 4; 5, 6]; [1, 2, 3; 4, 5, 6], [0, -1; -2, 0]] r = [y; v] = [1, 2, 3; 4, 5]

Since matrices T, A, and D are nonsingular, by the semismooth Newton method, the sequence {z } = {(ŵ, u, t) } converges to a numerical solution. The solution is:

$\hat{w}^* = [\sqrt{82} - 12, \sqrt{82} - 12]$ $t^* = \sqrt{82} - 12$ $u^* = [-225 + 30\sqrt{82}, 139 - 24\sqrt{82}]$

We verify the complementarity conditions. We have $\hat{w}^* \geq 0$ and $G(\hat{w}, u, t) \geq 0$, *with* $\hat{w}$ $G(\hat{w}, u, t)$. *Thus,* $(\hat{w}, G(\hat{w}, u, t^*))$ $C(\ ^2\ )$.

By item (vi) in Theorem 5, the solution to the linear complementarity problem is: $z^* = (x, u) = [\sqrt{82} - 12, \sqrt{82} - 12, \sqrt{82} - 12, -225 + 30\sqrt{82}, 139 - 24\sqrt{82}]$

By definition of the monotone extended second order cone, z^* L. Computing $Tz^* + r$ gives a vector in M, and $z, Tz + r = 0$. Thus, z^* is a solution to the linear complementarity problem.

8 Example for Portfolio Optimization

As Facchinei and Pang summarized in [5], the Fischer-Burmeister function and generalized Newton method can solve both linear and nonlinear complementarity problems. In this section, we implement this algorithm to solve a specific nonlinear complementarity problem arising from portfolio optimization related to the monotone extended second order cone.

Markowitz developed the mean-variance (MV) model in [19], the classical method for portfolio optimization. Suppose we build a portfolio using n assets. Let w ^{n} denote asset weights, r ^{n} represent returns, and Σ $^{n\ n}$ be the covariance matrix. The two traditional equivalent MV models are:

$\min \{w\ \Sigma w: r\ w \geq \alpha, e\ w = 1\}$ $\max \{r\ w: w\ \Sigma w \leq \beta, e\ w = 1\}$

where α is the minimum required profit and β is the maximum tolerable risk. These are quadratic optimization problems with high computational complexity.

To reduce complexity, many models have been introduced, such as the MAD model [15], which significantly reduces computational complexity [16, 17].

To measure return uncertainty for j = 1, …, T, define $U = (U_1, \ldots, U_T)$ where $U_j = R_j - r$. Let y_j denote the upper bound of return disturbance on day j. The traditional MAD model is the linear program:

$\min c_0 f\ y - r\ w$ s.t. $y_j \geq |U_j\ w|, j = 1, \ldots, T$ $e\ w = 1$

where $c_0 > 0$ is the Arrow-Pratt absolute risk-aversion index.

In reality, return uncertainty increases with investment horizon. Thus, optimizing the MAD model to better reflect real market behavior is meaningful. By Cauchy's inequality, $|U_j\ w| \leq \|U_j\|\|w\|$ for any j. Based on the MAD model, we obtain:

$\min c_0 f\ y - r\ w$ s.t. $y_T \geq y_{T-1} \geq \ldots \geq y_1 \geq \|U_j^*\|\|w\|$ $e\ w = 1$

where j* = argmin_j |U_j w| for j = 1, ..., T. The vector (y_T/‖U_j‖, *y_{T-1}/‖U_j*‖, ..., y_1/‖U_j*‖, w) belongs to the monotone extended second order cone L_{T,n}. Thus, the problem is equivalent to the conic optimization:

min c_0f y - r u s.t. e u = ‖U_j*‖ (y_T, y_{T-1}, ..., y_1, u) L_{T,n}

where u := w‖U_j*‖.

The KKT conditions give the Lagrangian: L(y, u) = c_0f y - r u/‖U_j‖ *+ Σ* $_1$ *_j(y_j - y_{j-1}) -* $_1$*(*y_1 *- ‖u‖) - β(‖U_j*‖ - e u)

This yields the complementarity problem: L [y_T, y_{T-1}, ..., y_1] [$c_0$$f_1$ + $_2$ - $_1$, $c_0$$f_2$ + $_3$ - $_2$, ..., c_0f_T + _T, -r/‖U_j*‖ + $_1$u/‖u‖ + βe] L*

where j* = argmin_j |U_j w| for j = 1, ..., T, which is a nonlinear complementarity problem.

Proposition 9. If -r/‖U_j‖ *+* $_1$*u/‖u‖ + βe ≠ 0, by Propositions 2 and 4 we have: (i) λ > 0 such that -r/‖U_j*‖ + $_1$u/‖u‖ + βe = -λu (ii) c_0Σ $_1$ f + 2 _T - $_1$ = λ‖u‖ (iii) y_T = ‖u‖

Since u = w‖U_j‖ *and e w = 1, we have u ≠ 0. Thus items (i) and (ii) are inapplicable while (iii) and (iv) are applicable. For (iii), if -r/‖U_j*‖ + $_1$u/‖u‖ + βe = 0, then: u = -‖u‖/(‖U_j‖*(β‖U_j*‖ - r)) ($_1$u + β‖U_j*‖e)

To ensure such β exists, we must have: ‖u‖r_1/‖U_j‖ - $_1$*u_1 = ‖u‖r_2/‖U_j*‖ - $_1$$u_2$ = ... = ‖u‖r /‖U_j*‖ - $_1$u

Using e w = 1 and u = w‖U_j‖*: 1 = e w = e u/‖U_j*‖ = $_1$‖U_j‖/(‖u‖(β‖U_j*‖ - r, e))

From these we obtain: ‖u‖ = ‖U_j‖‖*β‖U_j*‖e - r‖ / |nβ‖U_j*‖ - r, e |

and the explicit solution for u is: u = - (‖u‖/‖U_j‖*) (r - β‖U_j*‖e) / (β‖U_j*‖ - r, e /n)

The weight allocation is: w = u/‖U_j‖ *= (r - β‖U_j*‖e) / (r, e - nβ‖U_j*‖)

where β is determined from the model parameters.

9 Conclusion

In this paper, we illustrated a method for solving linear complementarity problems on monotone extended second order cones. We showed that the linear complementarity problem on the monotone extended second order cone can be converted to a mixed complementarity problem on the non-negative orthant, reducing the complexity of the original problem. We can determine a solution of the mixed complementarity problem using propositions about stationary points and FB-regularity. The connection between the linear complementarity problem on the monotone extended second order cone and the mixed complementarity problem on the non-negative orthant is also useful for applications to portfolio optimization. The method works for both linear and nonlinear complementarity problems, and we expect this scheme will be useful for other applications.

References

[1] S.C. Billups, Algorithms for complementarity problems and generalized equations, 1995. [2] S.C. Billups and K.G. Murty, Complementarity problems, *Journal of Computational and Applied Mathematics*, 124(1-2):303–318, 2000. [3] R.W. Cottle and G.B. Dantzig, Complementary pivot theory of mathematical programming, Technical report, Stanford University Operations Research House, 1967. [4] T. De Luca, F. Facchinei, and C. Kanzow, A semismooth equation approach to the solution of nonlinear complementarity problems, *Mathematical Programming*, 75(3):407–439, 1996. [5] F. Facchinei and J.S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Volumes I and II, Springer, New York, 2003. [6] O.P. Ferreira, Y. Gao, and S.Z. Németh, Reducing the projection onto the monotone extended second order cone to the pool-adjacent-violators algorithm (PAVA) of isotonic regression, *Optimization*, pp. 1–15, 2022. [7] O. Ferreira and S.Z. Németh, How to project onto extended second order cones, *Journal of Global Optimization*, 70(4):707–718, 2018. [8] M.C. Ferris and J.-S. Pang, Engineering and economic applications of complementarity problems, *SIAM Review*, 39(4):669–713, 1997. [9] A. Fischer, A special Newton-type optimization method, *Optimization*, 24(3–4):269–284, 1992. [10] A. Fischer, A Newton-type method for positive-semidefinite linear complementarity problems, *Journal of Optimization Theory and Applications*, 86(3):585–608, 1995. [11] P.T. Harker and J.-S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications, *Mathematical Programming*, 48(1):161–220, 1990. [12] Y. Gao, S.Z. Németh, and R. Sznajder, The monotone extended second-order cone and mixed complementarity problems, *Journal of Optimization Theory and Applications*, 2021. [13] A. Guyader, N. Jégou, A.B. Németh, A geometrical approach to iterative isotone regression, *Applied Mathematics and Computation*, 227:359–369, 2014. [14] G. Isac and A. Németh, Isotone projection cones in Hilbert spaces and the complementarity problem, *Bollettino della Unione Matematica Italiana*, 4(4):773–802, 1990. [15] H. Konno and H. Yamazaki, Mean-absolute deviation portfolio optimization model and its applications to Tokyo stock market, *Management Science*, 37(5):519–531, 1991. [16] H. Konno and T. Koshizuka, Mean-absolute deviation model, *IIE Transactions*, 37(10):893–900, 2005. [17] H. Konno and T. Koshizuka, Mean-absolute deviation portfolio optimization model under transaction costs, *Journal of the Operations Research Society of Japan*, 42(4):422–435, 1999. [18] W. Karush, Minima of functions of several variables with inequalities as side conditions, in *Traces and Emergence of Nonlinear Programming*, pp. 217–245, Springer, 2013. [19] H. Markowitz, Portfolio selection, *The Journal of Finance*, 7(1):77–91, 1952. [20] A.B. Németh and S.Z. Németh, Lattice-like operations and isotone projection sets, *Linear Algebra and its Applications*, 439:2815–2828, 2013. [21] S.Z. Németh and L. Xiao, Linear complementarity problems on extended second order cones, *Journal of Optimization Theory and Applications*, 176:269–288, 2018. [22] S.Z. Németh and L. Xiao, Stochastic linear complementarity problems on extended second order cones, arXiv:1910.09814v2, 2019. [23] S.Z. Németh, J. Xie, and G.

Zhang, Positive operators on extended second order cones, *Mathematica Hungarica*, 160(2):390–404, 2020. [24] S.Z. Németh and G. Zhang, Extended Lorentz cones and mixed complementarity problems, *Journal of Global Optimization*, 62:443–457, 2015. [25] S.Z. Németh and G. Zhang, Extended Lorentz cones and variational inequalities on cylinders, *Journal of Optimization Theory and Applications*, 168:756–768, 2016. [26] H. Nishimura and E.A. Ok, Solvability of variational inequalities on Hilbert lattices, *Mathematics of Operations Research*, 37(4):608–625, 2012. [27] R. Sznajder, The Lyapunov rank of extended second order cones, *Journal of Global Optimization*, 66:585–593, 2016. [28] L. Xiao, Complementarity and Related Problems, arXiv:2108.07412, 2021.

Note: Figure translations are in progress. See original paper for figures.