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Authors: Su Zhirong, Zhao Liyun, Su Zhirong

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Abstract

Cellular algebras are an algebraic structure that has attracted considerable attention in recent years, while regular semigroups are an important class of semigroups and constitute one of the main research areas in semigroup algebra theory. Based on a class of generalized cellular algebras—standardly based algebras—proposed by Du and Rui, the generalized cellularity of regular semigroup algebras is studied.

Full Text

Generalized Cellularity of Regular Semigroup Algebras

Zhirong Su[1],*, Liyun Zhao[1],[2]
[1],[2]College of Mathematics and Statistics, Guangdong University of Technology, Guangzhou 510520, China

Abstract

Cellular algebras are algebraic structures that have received considerable attention in recent years, while regular semigroups constitute an important class of semigroups and represent one of the main research areas in semigroup algebra theory. Based on the notion of standardly based algebras—a class of generalized cellular algebras introduced by Du and Rui—this paper investigates the generalized cellularity of regular semigroup algebras.

Keywords: cellular algebra, regular semigroup, standardly based algebra

1 Introduction

The emergence of cellular algebras was inspired by the study of multiplication properties of the Kazhdan-Lusztig basis for Hecke algebras in [?]. Through this basis, Graham and Lehrer [?] determined the parametrization of irreducible representations and resolved fundamental problems in representation theory. König and Xi [?] subsequently provided a new equivalent definition of cellular algebras. Du and Rui [?] introduced a class of generalized cellular algebras known as standardly based algebras. Regular semigroups form an important class of semigroups and constitute a major research area in semigroup algebra theory. East [?] investigated the cellularity of inverse semigroup algebras, while Wilcox [?] and Guo and Xi [?] studied the cellularity of twisted semigroup algebras, generalizing previous results. This paper extends these investigations by exploring the generalized cellularity of regular semigroup algebras, thereby broadening the scope of earlier findings.

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2 Preliminaries

This section establishes the foundational concepts concerning semigroups and standardly based algebras required for this work. Throughout the paper, R denotes an integral domain and $\mathbb{N}$ the set of natural numbers. For concepts and definitions not explicitly mentioned here, we refer the reader to [?, ?, ?].

We begin by recalling key definitions and results from semigroup theory. The set of all idempotents of a semigroup S is denoted by $E(S)$. Green' s relations $\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}$, and $\mathcal{D}$ play a fundamental role in semigroup theory [?]. For a semigroup S and elements $a, b \in S$, these relations are defined as:

$$a\mathcal{R}b \iff aS = bS, \quad a\mathcal{L}b \iff Sa = Sb, \quad a\mathcal{J}b \iff SaS = SbS, \quad \mathcal{H} = \mathcal{L}\cap\mathcal{R}, \quad \mathcal{D} = \mathcal{L}\vee\mathcal{R}.$$

When S is finite, the Green relations $\mathcal{J}$ and $\mathcal{D}$ coincide. For any $\mathcal{K}$-class K, we denote by $S/\mathcal{K}$ the set of all $\mathcal{K}$-classes of S. This yields a partially ordered set $(S/\mathcal{J}, \leq)$ where $J_a \leq J_b$ if and only if $a \in SbS$.

A semigroup S is called **simple** if it has no proper ideals. A semigroup with zero is called **0-simple** if $\{0\}$ and S itself are its only ideals. A **completely simple semigroup** is a simple semigroup containing a primitive idempotent. A **completely 0-simple semigroup** is a 0-simple semigroup containing a primitive idempotent.

Let G be a group, I and Λ non-empty sets, and $P = (p_{\lambda i})$ a $\Lambda \times I$ matrix over $G \cup \{0\}$ where each row and column contains at least one non-zero entry. The **Rees matrix semigroup** $\mathcal{M}(G; I, \Lambda; P)$ is defined on the set $(G \times I \times \Lambda) \cup \{0\}$ with multiplication:

$$(a, i, \lambda)(b, j, \mu) = \begin{cases} (ap_{\lambda j}b, i, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$

and 0 as the zero element. Under these assumptions, $\mathcal{M}(G; I, \Lambda; P)$ is a completely 0-simple semigroup. We may assume $p_{0,0}$ is the identity of G and $0 \in I \cap \Lambda$. For any non-zero element $a \in S$, the **principal factor** determined by a is $S_a = J_a \cup \{0\}$ with multiplication $\circ$ extended linearly from:

$$x \circ y = \begin{cases} xy & \text{if } xy \in J_a, \\ 0 & \text{if } xy \notin J_a. \end{cases}$$

This makes $(S_a, \circ)$ a semigroup with zero. Note that for a finite regular semigroup S, each principal factor is a completely 0-simple semigroup.

The **semigroup algebra** $R[S]$ is the R-module with basis S and multiplication extended linearly from S. If S has a zero element θ, the **contracted semigroup algebra** is $R_0[S] = R[S]/R\theta$. When S has no zero, we set $R_0[S] = R[S]$.

Lemma 2.1 [?]. Let S be a semigroup. 1. If $a, b \in S$ and $ab \in J_a$, then $ab\mathcal{R}a$. 2. For any $a, b \in S$, if $ab \in J_a$ then $ab\mathcal{L}b$.

Lemma 2.2 [?]. For any $a, b \in S$, the following are equivalent: 1. $a\mathcal{R}b \iff a = ba$. 2. $a\mathcal{L}b \iff a = ab$. 3. $a\mathcal{H}b \iff a = ba = ab$.

In a completely simple semigroup, all non-zero elements lie in the same $\mathcal{D}$-class, and any two maximal subgroups contained in it are isomorphic.

Definition 2.3 [?]. Let A be an associative algebra with identity over a commutative ring R with unity. Then A is a **standardly based algebra** with structure (Λ, M, N, C) if: 1. There exists a poset $(\Lambda, \leq)$ and, for each $\lambda \in \Lambda$, index sets $M(\lambda)$ and $N(\lambda)$ together with an injection $C : \coprod_{\lambda \in \Lambda} M(\lambda) \times N(\lambda) \to A$ mapping $(S, T) \mapsto C^{\lambda}_{S,T}$ such that the set $\mathfrak{B} = \{C^{\lambda}_{S,T} \mid \lambda \in \Lambda, S \in M(\lambda), T \in N(\lambda)\}$ forms an R-basis of A. 2. For any $a \in A$ and $C^{\lambda}_{S,T} \in \mathfrak{B}$, there exist constants $r_a(S', S) \in R$, independent of T, such that:

$$aC^{\lambda}_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^{\lambda}_{S',T} \pmod{A(< \lambda)}.$$

3. For any $a \in A$ and $C^{\lambda}_{S,T} \in \mathfrak{B}$, there exist constants $r'_a(T', T) \in R$, independent of S, such that:

$$C^{\lambda}_{S,T} a \equiv \sum_{T' \in N(\lambda)} r'_a(T', T) C^{\lambda}_{S,T'} \pmod{A(< \lambda)}.$$

Definition 2.4. Let R be a commutative ring with unity and S a semigroup (possibly with zero). The semigroup algebra $R[S]$ (or contracted semigroup algebra $R_0[S]$) is a **type-standardly based algebra** with structure (Λ, M, N, C) if it satisfies Definition 2.3 and the additional condition:

$$\operatorname{supp}(C^{\lambda}_{S,T}) \subseteq J \quad \text{for each } \lambda \in \Lambda, S \in M(\lambda), T \in N(\lambda).$$

Definition 2.5. The semigroup algebra $R[S]$ (or $R_0[S]$) is a **type-standardly based algebra** with structure (Λ, M, N, C) if it satisfies both Definition 2.4 and the condition:

$$\operatorname{supp}(C^{\lambda}_{S,T}) \subseteq H \quad \text{for each } \lambda \in \Lambda, S \in M(\lambda), T \in N(\lambda).$$

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3 Generalized Cellularity of Semigroup Algebras

Theorem 3.1. Let R be an integral domain and $\mathfrak{S}$ a semigroup. If $R[\mathfrak{S}]$ is a type-standardly based algebra with structure (Λ, M, N, C), then for any $a \in \mathfrak{S}$, $R[J_a]$ is also a type-standardly based algebra.

Proof. We define:

$$\Lambda_a = \{\lambda \in \Lambda \mid \operatorname{supp}(C^{\lambda}_{S,T}) \subseteq J_a \text{ for some } S \in M(\lambda), T \in N(\lambda)\},$$

$$M_a = \bigcup_{\lambda \in \Lambda_a} M(\lambda), \quad N_a = \bigcup_{\lambda \in \Lambda_a} N(\lambda), \quad C_a = \{C^{\lambda}_{S,T} \mid \lambda \in \Lambda_a, S \in M(\lambda), T \in N(\lambda)\}.$$

We must verify three conditions:

1. **Basis condition**: By assumption, $R[\mathfrak{S}]$ is a type-standardly based algebra, so C_a forms a basis for $R[J_a]$.
2. **Left multiplication**: For any $W \in J_a$ and $C^{\lambda}_{S,T} \in C_a$ with $\lambda \in \Lambda_a$, we have:

   $$W \circ C^{\lambda}_{S,T} \equiv \sum_{S' \in M(\lambda)} L_W(S', S) C^{\lambda}_{S',T} \pmod{R[J_a](< \lambda)}.$$

   Since $R[\mathfrak{S}]$ is type-standardly based, the coefficients $L_W(S', S)$ are independent of T, and each basis element on the right has support contained in J_a.
3. **Right multiplication**: Similarly, for any $W \in J_a$:

   $$C^{\lambda}_{S,T} \circ W \equiv \sum_{T' \in N(\lambda)} R_W(T', T) C^{\lambda}_{S,T'} \pmod{R[J_a](< \lambda)}.$$

   The coefficients $R_W(T', T)$ are independent of S, and all basis elements have support in J_a.

Thus $R[J_a]$ is a type-standardly based algebra with structure $(\Lambda_a, M_a, N_a, C_a)$.

Theorem 3.2. Let R be an integral domain and $\mathfrak{S} = \mathcal{M}(G; \Lambda, \Gamma; P)$ a Rees matrix semigroup. Let e be an idempotent of $\mathfrak{S}$. If $R_0[\mathfrak{S}]$ is a type-standardly based algebra with structure (I, M, N, C), then the group algebra $R[G_e]$ of the maximal subgroup containing e is also a type-standardly based algebra.

Proof. Assume $e = (H, 0, 0)$ and define:

$$K = \{\lambda \in I \mid \mathrm{supp}(C^{\lambda}_{S,T}) \subseteq G\}.$$

For $\lambda \in K$, set:

$$M'(\lambda) = \{S \in M(\lambda) \mid \exists T \in N(\lambda), \mathrm{supp}(C^{\lambda}_{S,T}) \subseteq G\},$$

$$N'(\lambda) = \{T \in N(\lambda) \mid \exists S \in M(\lambda), \mathrm{supp}(C^{\lambda}_{S,T}) \subseteq G\}.$$

Let $D = \{D^{\lambda}_{S,T} \mid \lambda \in K, S \in M'(\lambda), T \in N'(\lambda)\}$ where $D^{\lambda}_{S,T} = C^{\lambda}_{S,T}$. We verify the three conditions:

1. **Basis**: Since $R_0[\mathfrak{S}]$ is type-standardly based, D forms an R-basis for $R[G]$.
2. **Left action**: For $L \in G$ and $D^{\lambda}_{S,T} \in D$, we have:

   $$LD^{\lambda}_{S,T} = LC^{\lambda}_{S,T} \equiv \sum_{U \in M'(\lambda)} L_g(U, S) C^{\lambda}_{U,T} \pmod{R_0[\mathfrak{S}](< \lambda)}.$$

   By the type-standardly based property, the coefficients $L_g(U, S)$ are independent of T, and each $C^{\lambda}_{U,T}$ has support in G.
3. **Right action**: The argument is symmetric, yielding coefficients $R_g(V, T)$ independent of S.

Therefore $R[G_e]$ is a type-standardly based algebra with structure (K, M', N', D).

Theorem 3.3. Let R be an integral domain and $\mathfrak{S}$ a regular semigroup whose principal factors are $\mathfrak{S}_\alpha = \mathcal{M}(G_\alpha; \Lambda_\alpha, \Gamma_\alpha; P_\alpha)$ for $\alpha \in Y = \mathfrak{S}/\mathcal{J}$. Let E be the set of idempotents of $\mathfrak{S}$. Then $R[\mathfrak{S}]$ is a type-standardly based algebra if and only if for each $\alpha \in Y$ and each $e \in E_\alpha = E \cap \mathfrak{S}_\alpha$, the group algebra $R[G_e]$ is type-standardly based.

Proof. Assume $R[\mathfrak{S}]$ is type-standardly based with structure (I, M, N, C). For $\alpha \in Y$ and $e \in E_\alpha$, Theorem 3.2 shows that $R[G_e]$ is type-standardly based.

Conversely, suppose each $R[G_e]$ is type-standardly based. For each $\alpha \in Y$, fix $e_\alpha \in E_\alpha$. Then $J_\alpha = \mathfrak{S}_\alpha \setminus \{0\}$ is a $\mathcal{J}$-class. Define:

$$\mathfrak{T} = \bigcup_{\alpha \in Y} (\alpha, K_\alpha), \quad \text{where } K_\alpha = \{\lambda \in I_\alpha \mid \mathrm{supp}(C^{\lambda}_{S,T}) \subseteq G_\alpha\}.$$

For $(\alpha, \lambda) \in \mathfrak{T}$, set:

$$M_\alpha(\lambda) = \Lambda_\alpha \times M_\alpha, \quad N_\alpha(\lambda) = \Gamma_\alpha \times N_\alpha,$$

and define structure constants:

$$C^{(\alpha,\lambda)}_{\alpha;(x,S),(y,T)} = C^{\lambda}_{S,T} \text{ for } (x, S) \in M_\alpha(\lambda), (y, T) \in N_\alpha(\lambda).$$

We verify the three axioms:

1. **Partial order**: Define $(\alpha, \lambda) \leq (\beta, \mu)$ iff either $J_\alpha < J_\beta$ in $\mathfrak{S}/\mathcal{J}$, or $\alpha = \beta$ and $\lambda \leq \mu$ in I_α. This makes $(\mathfrak{T}, \leq)$ a poset.
2. **Left multiplication**: For $H \in \mathfrak{S}$ and a basis element $C^{(\alpha,\lambda)}_{\alpha;(x,S),(y,T)}$, we have:

   $$HC^{(\alpha,\lambda)}_{\alpha;(x,S),(y,T)} = HC^{\lambda}_{S,T} \equiv \sum_{U \in M_\alpha(\lambda)} L_H(U,S) C^{\lambda}_{U,T} \pmod{R[\mathfrak{S}](< (\alpha, \lambda))}.$$

   The coefficients $L_H(U,S)$ are independent of T by the type-standardly based property of $R[G_\alpha]$.
3. **Right multiplication**: The symmetric argument yields coefficients $R_H(V,T)$ independent of S.

Thus $R[\mathfrak{S}]$ is a type-standardly based algebra with structure $(\mathfrak{T}, M, N, C)$.

Corollary 3.4. Let R be an integral domain and $\mathfrak{S}$ a regular semigroup with principal factors $\mathfrak{S}_\alpha = \mathcal{M}(G_\alpha; \Lambda_\alpha, \Gamma_\alpha; P_\alpha)$. Then $R[\mathfrak{S}]$ is a type-standardly based algebra if and only if for each $\alpha \in Y = \mathfrak{S}/\mathcal{J}$, the group algebra $R[G_\alpha]$ is type-standardly based.

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Corresponding author: Zhirong Su, E-mail: 2112014026@mail2.gdut.edu.cn

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