Resonant Tunneling through Arbitrarily Shaped Double Barriers: Testing the Existence of a Minimum Spatial Scale

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Abstract

Is the three-dimensional space (real space) we humans live in continuous? In other words, is real space infinitely divisible? In the macroscopic world, the answer is undoubtedly affirmative, because everyday experiences—such as flowing water, running animals, birds in flight, and freely falling objects—all give the impression of continuous trajectories. In classical mechanics, using continuous coordinates to describe moving objects is considered self-evident.

In the quantum world, the situation changes dramatically. For example, when a particle' s motion is constrained, the energy of its stable states can only take certain discrete values. Some modern theories attempting to unify quantum mechanics and gravity predict the existence of a minimum length, typically taken as the Planck scale, which is approximately 1.6$×10^{-35}$ meters. However, experimental verification of this length is extremely challenging, as it is far smaller than the smallest length LIGO can measure (approximately 10^{-19} meters).

In this work, the authors investigate the quantum tunneling phenomenon in double-barrier systems of arbitrary shape and prove a theorem. This theorem demonstrates that if the separation between the two barriers can be varied continuously—that is, if real space is continuous—then by appropriately adjusting the barrier spacing, incident particles can completely penetrate the double-barrier system. This phenomenon is commonly known as resonant tunneling. Conversely, if real space is discontinuous, meaning there exists a non-zero minimum scale, then when the barrier dimensions exceed a certain upper limit, the resonant tunneling phenomenon ceases to occur. This work reveals a profound connection between quantum tunneling phenomena and the minimum scale in quantum gravity theories, opening a new avenue for testing the existence of a minimum scale.

Full Text

Penetration of Arbitrary Double Potential Barriers with Probability Unity: Implications for Testing the Existence of a Minimum Length

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Abstract. Quantum tunneling across double potential barriers is studied. With the assumption that real space is a continuum, it is rigorously proved that large barriers of arbitrary shapes can be penetrated by low-energy particles with a probability of unity, i.e., realization of resonant tunneling (RT), by simply tuning the inter-barrier spacing. The results are demonstrated by tunneling of electrons and protons, in which resonant and sequential tunneling are distinguished. The critical dependence of tunneling probabilities on barrier positions not only demonstrates the crucial role of phase factors, but also points to the possibility of ultrahigh accuracy measurements near resonance. By contrast, the existence of a nonzero minimum length puts upper bounds on the barrier size and particle mass, beyond which effective RT ceases. A scheme is suggested for dealing with the practical difficulties arising from the delocalization of particle position due to the uncertainty principle. This work opens a possible avenue for experimental tests of the existence of a minimum length based on atomic systems.

I. INTRODUCTION

The scenario of a minimum length (L_{min}) plays an essential role in the quantum theory of gravity [1-11]. Breakdown of Lorentz invariance may occur when real space approaches such a minimum length scale [12-20], which is generally taken to be the Planck length ($l_P = \sqrt{\hbar G/c^3} \sim 1.6 \times 10^{-35}$ m, where $\hbar$ is the reduced Planck' s constant, G is the gravitational constant, and c is the speed of light) [7]. Despite numerous efforts [1-20], the question remains open regarding the existence of such a minimum length [7, 21-23]. Since this length scale is well below the lower bound of spatial resolution achieved by state-of-the-art instruments such as LIGO ($\sim 10^{-19}$ m) [24, 25], experimental verification presents a great challenge. Here, we show the possibility of tackling this problem through investigations of quantum tunneling across double potential barriers.

Quantum tunneling [26] is a classically forbidden phenomenon in which a particle passes through a potential barrier higher than its own energy. In the early years of quantum mechanics, theories based on quantum tunneling explained experimental puzzles such as thermionic and field-induced emission of electrons from metal surfaces [26] and the alpha decay of heavy nuclei [26]. In subsequent decades, research on quantum tunneling of electrons in condensed matter has led to fruitful discoveries [27-33] and enabled important inventions such as the scanning tunneling microscope (STM) [34] and tunneling diodes [35-37]. Since the pioneering works by Tsu, Esaki, and Chang [31-33], double barriers have received considerable attention in studies of electron transport in heterostructures [37, 38].

Resonant tunneling (RT) typically occurs in double-barrier systems, where incident electrons may pass through the barriers without reflection, i.e., with a transmission probability of 100%. This behavior arises from coherent interference of electron waves that cancel reflected waves and enhance transmitted ones, analogous to resonant transmission through a Fabry-Perot etalon in optics. Typical inter-barrier spacing in RT-based devices is several tens of angstroms (Å), matching the de Broglie wavelengths of electrons. In recent decades, RT phenomena in mesoscopic and nanoscale structures have continued to attract research interest [39-43].

Historically, RT of electrons was inferred from negative differential resistance (NDR) observed in current-voltage (I-V) curves [32, 35, 37]. Later, an alternative mechanism for NDR was suggested: sequential tunneling, in which the phase memory of electron wave functions is lost due to inelastic scattering [37, 44-51]. It was argued that resonant (coherent) tunneling is a prerequisite for sequential tunneling [51]. The effects of external electric fields, inelastic scattering, and repulsive electron-electron interactions on RT were also studied [38, 52, 53]. Despite these efforts, consensus on the underlying physics has not been reached. Large discrepancies remain between theory and measured I-V curves (e.g., peak-to-valley ratio). This gap originates partly from the fact that calculations related to experiments adopt the simplest rectangular barriers (or their variants), which usually differ significantly from the true barriers experienced by electrons.

To resolve these puzzles, an exact theoretical description of the conditions for RT across double barriers is highly desirable. For the simplest rectangular double barriers, exact mathematical relations for energy and geometric conditions have been established [38, 54, 55]. For the more general and realistic situation where double barriers have arbitrary shapes, aside from semi-classical approaches [38], a full quantum-level description of RT conditions is still lacking. It is generally accepted that RT occurs when the energy of incident particles matches the energy levels of quasi-bound states within the potential well between the two barriers [37, 38, 44, 49, 51, 52]. In principle, this applies to electrons as well as massive particles like protons, atoms, and molecules. Recent simulations have shown RT of H and He atoms across small double barriers with barrier heights $E_b \sim 0.2$ eV [56, 57] and ~ 0.02 eV [58], respectively. However, when a particle tunnels across arbitrarily-shaped double barriers, it remains unclear how the level-matching condition can be achieved, and rigorous theoretical descriptions remain elusive.

In this paper, we revisit this topic in double-barrier systems consisting of equal barriers of arbitrary geometries. With the assumption of continuously variable inter-barrier spacing (equivalently, $L_{\min} = 0$), we rigorously prove that quantum tunneling through the double-barrier system with unity probability can always occur (i.e., RT) when the inter-barrier spacing is appropriately chosen. An exact mathematical relation for RT is established. In the presence of a nonzero $L_{\min}$ ($L_{\min} > 0$), the inter-barrier spacing varies discontinuously, which sets upper bounds for barrier heights and particle mass, beyond which effective RT ceases. The results are demonstrated for electrons, protons, and some typical bosons. A practically feasible scheme is therefore provided for experimental tests of the existence of a nonzero minimum length.

The rest of this paper is organized as follows. Section II presents analytic and numerical results on RT, with examples of typical particles like electrons, protons, and some bosons. The connection between RT and the continuity of real space is revealed. The constraints imposed by the existence of a minimum length, practical obstacles due to the uncertainty principle, and plausible solutions are presented. We conclude in Sec. III with discussions on the impacts and future opportunities inspired by this work.

II. RESULTS AND DISCUSSIONS

We begin in Part A of this section by performing general analyses on the transmission properties of quantum particles across double barriers of arbitrary shapes and prove a theorem that establishes the mathematical condition for resonant tunneling (RT). Part B provides analyses of two typical models—rectangular and parabolic double barriers. The quantum tunneling of electrons and protons are studied and compared, with emphasis on the differences between resonant and sequential tunneling. Based on the results of Parts A and B, we show in Part C the upper bounds of barrier heights for RT set by the Planck length. In Part D, the fundamental limits imposed by the uncertainty principle are studied, and a possible solution to position delocalization of incident particles is suggested.

[Figure 1: see original paper] **FIG. 1.** Schematics of resonant tunneling (RT) across double barriers. (a) Quantum interference of incident and reflected matter waves; (b) Modulation of energy levels of quasi-bound states between the two barriers by varying the inter-barrier spacing w; (c) RT spectrum as a function of w with a period of $\Delta = \lambda/2$.

A. GENERAL ANALYSES ON ARBITRARY DOUBLE BARRIERS

Generally, double-barriers consist of two identical or different single barriers, referred to as homo-structured and hetero-structured, respectively. The double-barrier considered here is homo-structured in one-dimensional space, as shown in Fig. 1(a), with barrier height E_b and barrier width a for each. Our analyses are based on the transfer matrix method, a powerful technique for studying transmission properties in finite systems [31, 59-62]. For propagation of a quantum particle across a single barrier $V(x)$, the transmitted and reflected amplitudes $(A_L, B_L; A_R, B_R)$ of the wave functions (ψ_L, ψ_R) may be related by a transfer matrix (denoted by M) as follows [38, 56, 59-62]:

$$\begin{pmatrix} A_R \\ B_R \end{pmatrix} = M \begin{pmatrix} A_L \\ B_L \end{pmatrix} \equiv \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} A_L \\ B_L \end{pmatrix}$$

The incoming wave function (with incident energy E) is $\psi_L = A_L e^{ikx} + B_L e^{-ikx}$, and the outgoing wave function is $\psi_R = A_R e^{ikx} + B_R e^{-ikx}$, where $k = \sqrt{2mE}/\hbar$ and m is the particle mass. The determinant $|M| = 1$ for systems where time-reversal symmetry is preserved, and the transmission coefficient is given by [56] $T = |m_{11}|^{-2} = |m_{22}|^{-2}$. In general, the matrix elements m_{ij} $(i, j = 1, 2)$ are complex numbers and obey the conjugate relations [56, 59-62] $m_{11}^* = m_{22}$ and $m_{12}^* = m_{21}$. For a homo-structured double-barrier with inter-barrier spacing w, the following theorem holds:

Theorem. For any $E < E_b$, the transmission coefficient (tunneling probability) across a homo-structured double-barrier $T_{DB}(E; w) = 1$ at $w = w_n = \frac{(2n-1)\pi - \theta - 2ka}{2k}$, where $\theta = \arg(m_{11}^2)$ and n (referred to as resonance number) belongs to the integers.

Proof. The updated transfer matrix for a single barrier $V(x)$ translated by a distance $L = a + w$, $V(x - L)$, is given by [57, 63]

$$M(L) = \begin{pmatrix} m_{11} e^{ikL} & m_{12} e^{-ikL} \\ m_{21} e^{ikL} & m_{22} e^{-ikL} \end{pmatrix} = \begin{pmatrix} m_{11} e^{ik(a+w)} & m_{12} e^{-ik(a+w)} \\ m_{21} e^{ik(a+w)} & m_{22} e^{-ik(a+w)} \end{pmatrix}$$

The transfer matrix for the double-barrier $(U(x) = V(x) + V(x - L))$ is therefore [56]

$$M_{DB} = M(L) \cdot M = \begin{pmatrix} m_{11} e^{ik(a+w)} & m_{12} e^{-ik(a+w)} \\ m_{21} e^{ik(a+w)} & m_{22} e^{-ik(a+w)} \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

The diagonal matrix element describing transmission properties, $(M_{DB})_{11}$, is explicitly calculated to be

$$(M_{DB})_{11} = m_{11}^2 + m_{12} m_{21} e^{-i2k(a+w)}$$

Let $Z = m_{11}^2 = |Z|e^{i\theta}$, $\phi = 2k(a + w)$, $m_{12}m_{21} = |m_{12}|^2 = R$, with i the imaginary unit, and the angle $\theta = \arg(Z)$. The determinant $|M| = 1 = |m_{11}|^2 - |m_{12}|^2 = |Z| - R$ gives $|Z| = 1 + R$. Then

$$(M_{DB})_{11} = (1 + R)e^{i\theta} + Re^{-i\phi} = e^{i\theta} + Re^{i\theta}(e^{-i(\phi+\theta)} + 1)$$

When $e^{-i(\phi+\theta)} = -1$, i.e., $\phi + \theta = (2n - 1)\pi$ with n being integers, one has $(M_{DB})_{11} = e^{i\theta}$. It follows that the transmission coefficient $T_{DB}(E; w) = |(M_{DB})_{11}|^2 = 1$, which corresponds to RT. Using the condition $\phi+\theta = (2n-1)\pi$, we have $2k(a + w) + \theta = (2n - 1)\pi$, and consequently $w = \frac{(2n-1)\pi-\theta-2ka}{2k} \equiv w_n$. This completes the proof of the theorem.

It should be stressed that the proof inherently includes the precondition that the inter-barrier spacing w varies continuously ($L_{\min} = 0$) such that the angle ϕ can take any desired value to satisfy the RT condition. The theorem points to the possibility of penetration of arbitrarily large (but finite) potential barriers by low-energy particles with unity probability. For a quantum particle with incident energy E, it can completely tunnel across a homo-structured double barrier of height E_b when the inter-barrier spacing equals w_n described above, even in the case $E \ll E_b$.

In addition, one sees that the barrier separations (w_n) for RT are solely determined by the parameters (θ, a) describing transmission through single barriers. Physically, the onset of RT is due to the presence of quasi-bound states between the two barriers whose energy levels match that of incident particles [37, 38, 44, 49, 51, 52]. A direct consequence is that any quasi-bound energy levels ($E \leq E_b$) can be realized within the potential well formed by the two barriers via simply tuning the inter-barrier spacing, as illustrated in Fig. 1(b). Moreover, from its mathematical expression, one sees that $(M_{DB})_{11}$ is a periodic function of w with period $\Delta = \pi/k$. For a fixed E, the tunneling probability $T[E; w]$ displays periodic variations with w, showing comb-like structures with resonance peaks positioned at $L_n = a + w_n$, and the distance between any two neighboring peaks is $\Delta = w_n - w_{n-1} = \pi/k$ (Fig. 1(c)). The value of Δ is just half the de Broglie wavelength of the incident matter wave, indicating the key role of phase factors and quantum interference. Finally, the mathematical expression of w_n implies that there could be infinitely many resonance peaks in free space. The results may be readily extended to two- or three-dimensional systems where interaction potentials along the propagation direction are equivalently described by effective double barriers.

B. TUNNELING ACROSS TYPICAL DOUBLE BARRIERS: RESONANT TUNNELING VERSUS SEQUENTIAL TUNNELING

For homo-structured rectangular double barriers, analytic expressions for w_n are available, enabling in-depth understanding of RT physics. The matrix element

m_{11} describing transmission across a single rectangular barrier (barrier height V_0) may be expressed as follows (Appendix A):

$$m_{11} = 2\gamma e^{-ika}[i(k^2 - \beta^2)\sinh(\beta a) + 2\beta k \cosh(\beta a)]$$

where $k = \sqrt{2mE}/\hbar$, $\beta = \sqrt{2m(V_0 - E)}/\hbar$, and $\gamma = \frac{1}{2\beta k}$. Equation (4) may be reduced to

$$m_{11} = 2\gamma e^{-ika} \times \sigma e^{i\alpha} = 2\gamma\sigma e^{i(\alpha - ka)}$$

where $\sigma = \sqrt{A^2 + B^2}$, $A = (k^2 - \beta^2)\sinh(\beta a)$, $B = 2\beta k \cosh(\beta a)$, and the angle $\alpha = \arctan\left(\frac{B}{A}\right)$. Therefore, $m_{11}^2 = 4\gamma^2\sigma^2 e^{i2(\alpha - ka)}$. Using the theorem stated above, the angle $\theta = 2(\alpha - ka)$, and then $\theta + 2ka = 2\alpha$. The inter-barrier spacing is given by $w_n = \frac{(2n-1)\pi - 2\alpha}{2k}$. It follows that $2kw_n = (2n-1)\pi - 2\alpha$, and one arrives at the equality:

$$\tan(2kw_n) = \frac{2\delta \tanh(\beta a)}{1 - \delta^2 \tanh^2(\beta a)}$$

where $\delta \equiv \frac{k}{\beta}$. Alternatively, this equality can be obtained by direct calculation of the squared norm of the diagonal element $|(M_{DB})_{11}|^2$, a function of inter-barrier spacing w: the minimum of $|(M_{DB})_{11}|^2$ leads to RT (Appendix B). The equality for $\tan(2kw_n)$ is in line with Ref. [55], which was derived differently.

In the special case when the incident energy is half the barrier height ($E = 0.5V_0$), $\beta = k$, the angle $\alpha = 0$, and one obtains a simplified relation $2kw_n = (2n-1)\pi$, and $w_n = \frac{(n-1/2)\pi}{k} = \left(n - \frac{1}{2}\right)\frac{\lambda}{2}$, where $\lambda = 2\pi/k$ is the de Broglie wavelength. In another special case when $k \ll \beta$ and $\beta a \gg 1$, i.e., the incident energy is far below the barrier height, one has $\alpha \cong -\frac{\pi}{2}$ and $kw_n \cong n\pi - \frac{\pi}{4}$, $w_n \cong \frac{(n-1/4)\pi}{k}$. In both situations, the value of w_n is independent of the barrier width.

[Figure 2: see original paper] **FIG. 2.** Schematics of rectangular (a) and parabolic double barriers (b). The inter-barrier spacing for RT (w_n) as a function of resonance number (n_{RT}) for electrons (c) and protons (d) across the double barriers, at incident energy $E = 0.5$ eV.

The results are applicable to electrons and other massive quantum particles. However, demonstrating the quantum interference effects leading to RT would be much more challenging due to large differences in particle masses and corresponding de Broglie wavelengths. Here, we perform systematic investigations on RT characteristics of electrons and protons through two model systems: rectangular and parabolic double barriers (Fig. 2). Compared to analytic expressions for rectangular barriers, transfer matrices for parabolic barriers are evaluated numerically [56, 57]. Figures 2(c)-(d) show the calculated w_n for RT of electrons and protons across the two types of double barriers, as a function of resonance numbers (n_{RT}). For the same n_{RT}, the w_n for electrons is much larger than that for protons due to their smaller mass. The different geometries of the potential barriers (rectangular vs. parabolic) are reflected by slight differences in w_n. Despite these differences, the overall comparable magnitudes of the two sets of w_n indicate that rectangular double barriers may serve as approximations for qualitative description of some smoothly varying double barriers with regular geometries.

At fixed energy E, the tunneling probability varies periodically with inter-barrier spacing w. We have further studied such characteristics for electrons and protons tunneling through rectangular double barriers. Figures 3(a-b) show electron transmission at varying w for $E = 0.03$ eV and 0.5 eV. The effects of incident energy on the tunneling spectrum $T(E; w)$ are clearly seen. Higher energy not only results in a smaller oscillation period ($\tau = \pi/k$), but also a smaller peak-to-valley ratio. The resonance number can extend to very large integers, as long as perturbations from the environment are negligible and wave function coherence is maintained. To show the role of coherence, we have studied the energy-dependent tunneling probability $P(E)$ of electrons at a fixed inter-barrier spacing ($w \sim 10\mu$m). For resonant (coherent) tunneling, the quantity $P(E)$ ($= T(E; w)$) drops quickly with small deviations from the resonant energy level E_{RT}. For sequential tunneling, in which phase coherence is destroyed in a two-step process, the quantity $P(E)$ is simply the product of transmission coefficients across each single barrier: $P(E) = T_1(E) \times T_2(E) = T_1^2(E)$. Around the resonant energy E_{RT} (Figs. 3(c-d)), the probability of sequential tunneling (P_{ST}) changes smoothly with energy and is significantly smaller than unity at low incident energies.

[Figure 3: see original paper] **FIG. 3.** Tunneling spectrum of electrons across the rectangular double-barrier shown in Fig. 2(a). RT at resonance level $E_{RT} =$ 0.03 eV (a) and 0.5 eV (b), as a function of inter-barrier spacing w. Panels (c-d): Energy dependence of tunneling probability at fixed w, around $E_{RT} = 0.03$ eV (panel c, $w = 100008.49$ Å) and 0.5 eV (panel d, $w = 100991.45$ Å). The data lines labeled by P_1, P_{ST}, and P_{RT} correspond to tunneling through a single barrier, sequential tunneling, and resonant tunneling through double barriers, respectively.

For protons, more radical differences emerge. Shown in Fig. 4(a) is the tunneling spectrum of protons at $E = E_b/2 = 0.5$ eV. The periodically repeated isolated lines imply much narrower resonant peaks compared to electrons. The enlarged structure of one resonant peak is shown in Fig. 4(b). Around the RT peaks, the squared norm of the transfer matrix element may be expressed as follows (Appendix C):

$$|(M_{DB})_{11}|^2 \cong 1 + \sinh^2(2ka) \times (k\Delta w)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta w}$$

where Δw is the deviation from the peak position w_n. When $\Delta|M_{11}|^2_{\Delta w} = 1$,

$T(E; w) = 0.5$, and one has

$$|\Delta w| = \frac{1}{k \sinh(2ka)}$$

It follows that the term $2|\Delta w|$ is the full width at half maximum (FWHM) of the resonant peaks. For the double-barrier considered here (Fig. 2(a)), it turns out that $|\Delta w| \cong 4.235 \times 10^{-15}$ Å. When the deviation $\Delta w \sim 10^{-13}$ Å, the tunneling probability drops quickly to $T(E; w) \sim 10^{-3}$, in good agreement with the results presented in Fig. 4(b). In general, given that w is an approximate value to w_n, one can determine the significant digits of w by designating a deviation Δw such that $|w - w_n| \leq \Delta w$ and $T(E; w) \geq 1 - \delta_P$, where δ_P ($0 < \delta_P < 1$) is the tolerance of decrease in tunneling probability at which significant tunneling (effective RT, events measurable in experiment) is maintained. Furthermore, using the proof of the theorem, we find that for arbitrary homo-structured double barriers, the deviation Δw at tolerance δ_P may be given by (Appendix D):

$$\Delta w = \frac{\sqrt{\delta_P}}{2k\sqrt{R(1+R)}}$$

where $R = |m_{12}|^2$, k and δ_P are defined as above. Here we focus on the case $\delta_P = 0.5$, which yields the FWHM ($= 2\Delta w$).

Such ultrahigh sensitivity on tunneling parameters is also found for RT energies. Figure 4(c) compares tunneling of protons across single and double barriers at a fixed inter-barrier spacing ($w \sim 20$ Å). Near resonance, $P(E)$ descends drastically from 1 to $\sim 10^{-11}$ with a tiny shift of $\varepsilon = 10^{-10}$ eV from E_{RT}. At the vicinity of E_{RT}, the dependence of $|(M_{DB})_{11}|^2$ on deviation ΔE is given by (Appendix C):

$$|(M_{DB})_{11}|^2 \cong 1 + \sinh^2(2ka) \times \left(\frac{w\Delta E}{2E}\right)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta E}$$

The FWHM on the energy scale is obtained when $\Delta|M_{11}|^2_{\Delta E} = 1$, and

$$|\Delta E| = \frac{2E}{w \cdot k \sinh(2ka)}$$

In our case, $|\Delta E| \approx 4.235 \times 10^{-16}$ eV. When the energy broadening $\Delta E = \varepsilon = 10^{-10}$ eV, $|\frac{\varepsilon}{\Delta E}| = 2 \times 10^5$, $|(M_{DB})_{11}|^2 \cong 2.23 \times 10^{11}$, $P(E) = |(M_{DB})_{11}|^{-2} \approx 10^{-11.35}$, which compares well with numerical results. Without resonance, the probability of two-step tunneling (i.e., sequential tunneling) decreases by more than 25 orders of magnitude (Fig. 4(c)). The sharp contrast distinguishes RT from sequential tunneling.

For the more general case, the allowed energy broadening ΔE may be calculated as follows (Appendix D):

$$\Delta E = \frac{2E}{k(a+w)}\sqrt{\frac{\delta_P}{R(1+R)}}$$

In the case $R \gg 1$ (large reflection), for instance tunneling through large barriers or by massive particles, Eq. (9) reduces to $|\Delta E| \approx \frac{2E}{k(a+w)R}$. For a single barrier $V(x)$, the reflection and tunneling probabilities (Appendix D) are related to R by $|r|^2 = |m_{12}|^2|m_{11}|^{-2} \equiv RT_1(E)$, subject to $|r|^2 + |t|^2 = 1$. It is straightforward that $R = |t|^{-2} - 1 \cong |t|^{-2} = T_1(E)^{-1}$ when $R \gg 1$, where $T_1(E)$ is the tunneling probability across a single barrier.

As seen from Fig. 4(c), in the absence of phase coherence, incident protons will be nearly completely reflected by a single barrier. In contrast, when the inter-barrier spacing equals w_n and phase coherence is maintained, protons penetrate the two barriers with unity probability. These effects are schematically illustrated in Fig. 5. The key role of quantum interference is demonstrated. Experimental verifications may be carried out using atomically thin membranes, which have potential applications as proton sieve filters. Generally, the variation step (Δl) of the inter-barrier spacing w_n required by RT should be of the order of magnitude of Δw studied above, and no less than the minimum length (i.e., $\Delta l \sim \Delta w \geq L_{\min}$) such that effective RT can be reached by tuning the inter-barrier spacing. This is the topic of the next subsection.

[Figure 4: see original paper] **FIG. 4.** Tunneling spectrum of protons across the rectangular double-barrier shown in Fig. 2(a). Resonance at $E_{RT} = 0.5$ eV (a). Panels (b-c): Variations of tunneling probability with respect to small deviations from RT parameters: (b) Inter-barrier spacing $w = (n_w - n_p) \times \Delta l + w_n$, $n_p = 596$ and $\Delta l = 10^{-15}$ Å; (c) Incident energy in the vicinity of E_{RT}, for $w = 20.137016632763302$ Å and energy deviation $\varepsilon = 10^{-10}$ eV. All digits of w are meaningful.

[Figure 5: see original paper] **FIG. 5.** Schematics of quantum tunneling of protons across a single barrier (upper panel) and double barriers (lower panel) in the presence of RT. The probability of reflection is denoted by P_{Ri} and tunneling by P_{Ti}, $i = 1, 2$.

C. UPPER BOUNDS OF RT BARRIERS SET BY THE PLANCK LENGTH

The critical dependence of tunneling probabilities on barrier positions not only demonstrates the crucial role of phase factors, but also points to the possibility of ultrahigh accuracy measurements near resonance. As shown above, a deviation of $|\Delta w| \cong 4.235 \times 10^{-15}$ Å leads to a 50% drop in $P(E)$ for protons across a rectangular double-barrier. Such deviation is several orders of magnitude below the smallest length scale sensed by LIGO [24, 25]. Even smaller $|\Delta w|$ is expected for heavier particles or larger barriers. As mentioned above, to have measurable RT within some tolerance δ_P, an upper bound of deviation from the exact peak position w_n is given by $\Delta w = |w - w_n|$. Suppose the elementary variation step of distance is Δl; if real space is a continuum ($L_{\min} = 0$), then Δl can be arbitrarily small and in principle w_n can always be reached by a finite number of operations. In this case, the theorem stated above always holds. Conversely, if a nonzero $L_{\min}$ exists ($L_{\min} > 0$), to have significant RT the variation step should satisfy $\Delta w \geq \Delta l \geq L_{\min}$. Let $n = \lfloor \log_{10}(\Delta w) \rfloor$, then it is feasible to set $\Delta l = 10^n$ such that $w = N \times \Delta l$, with N being an integer and $w \mod \Delta l = 0$. Consequently, the existence of a minimum length leads to the inequality $|\Delta w| \geq L_{\min}$, which therefore puts upper bounds on particle mass, barrier height, and barrier width, above which RT will cease.

In the special case of tunneling across rectangular double barriers at $E = 0.5E_b$, realization of RT requires that

$$|\Delta w| = \frac{1}{k \sinh(2ka)} \geq L_{\min}$$

which may be rewritten as

$$\chi \sinh(\chi) \leq \frac{a}{L_{\min}}$$

where $\chi = 2ka$, $k = \sqrt{2mE}/\hbar = \sqrt{mV_0}/\hbar$. The upper bound of the term mV_0 is therefore determined for a given barrier width a. Assuming the minimum length is identical to the Planck length ($L_{\min} = l_P$), the upper bounds of barrier height ($V_{\max}$) for electrons and protons are calculated and shown in Fig. 6. At barrier widths of $a = 1$ Å, 5 Å, and 10 Å, $V_{\max}$ is ~ 5652.72 eV, 239.42 eV, and 61.32 eV for electrons, and ~ 3.08 eV, 0.13 eV, and 0.03 eV for protons, respectively. $V_{\max}$ decreases rapidly with increasing barrier width. For instance, when the barrier width increases to $a = 20$ Å and 30 Å, $V_{\max}$ is respectively ~ 15.70 eV and 7.08 eV for electrons, which may be feasible for experimental tests using metal-insulator-metal double barriers. The same variation trend is found for electrons and protons, with the magnitude of $V_{\max}$ scaled by a factor of $\eta = \frac{m_e}{m_p}$, where m_e and m_p are the masses of electron and proton, respectively. This is due to the conjugate relation that the particle mass times barrier height (mV_0) is constant at fixed barrier width.

For the general case of tunneling through arbitrary double barriers, the constraint imposed on particle mass and barrier size due to a nonzero minimum length is given by (Appendix D):

$$\frac{\sqrt{\delta_P}}{2\sqrt{2mE}\sqrt{R(1+R)}} \geq L_{\min}$$

where $R = |m_{12}|^2$ and δ_P $(0 < \delta_P < 1)$ has the same meaning as above. Provided that $L_{\min} = l_P$ and $\delta_P = 0.5$, the inequality reduces to

$$\frac{1}{2\sqrt{2mE} \cdot R(1+R)} \geq l_P$$

Since parameter R is generally an increasing function of barrier size (Appendix D), the upper bounds on barrier size for RT are therefore determined by the Planck length.

[Figure 6: see original paper] **FIG. 6.** Calculated upper bounds ($V_{\max}$) of barrier height for rectangular double barriers set by the Planck length for electrons (a) and protons (b), as a function of barrier width. Values of $V_{\max}$ at barrier widths of 6-10 Å are highlighted in the insets.

D. FUNDAMENTAL LIMITS PUT BY THE UNCERTAINTY PRINCIPLE AND POSSIBLE SOLUTION

For a group of incident particles, given that the standard deviation of the energy distribution, σ_E, is approximately the term ΔE for $\Delta|M_{11}|^2_{\Delta E} = 1$ and $P(E) = 0.5$. The narrow window of energy dispersion implies that particle momenta distribute dominantly within a narrow interval with small standard deviation (Δp). As a consequence of the uncertainty principle, the standard deviation of position, Δx, is expected to be large. Table I lists the energy and momentum broadening, and estimated standard position deviations of protons when $P(E) = 0.5$ at $E = 0.5E_b$, for rectangular double barriers with $E_b = 1$, 0.5, 0.2, and 0.1 eV, and $w \sim 20$ Å. It is clearly seen that energy broadening increases significantly with decreasing barrier height, resulting in reduced standard deviations of position. For $E_b = 1$ eV, the requirement of ultrahigh monochromaticity of incident energies leads to a very small Δp and consequently a quite large Δx ($\sim 1.53 \times 10^4$ m), which is practically very challenging, if not impossible, for experimental tests. Much smaller Δx (9.25×10^{-6} m) is found when E_b decreases to 0.1 eV.

It should be stressed that the constraint on standard deviations of particle momentum and position imposed by the uncertainty principle does not exclude the possibility that a subgroup of particles with ultrahigh monochromaticity coexists with another subgroup having large energy broadening. The reason is that, by definition, the standard deviation of some physical quantity for a single particle is the statistical average over a large number of events, which may be equivalently evaluated by statistical results from many identical particles within a small time interval. Indeed, this is in line with the impossibility of measuring the quantum state of a single system [64]. Therefore, a possible recipe for the practical difficulty of position delocalization is to have a much larger standard deviation of kinetic energy distribution than that required for the half-drop of $P(E)$, i.e., $\sigma_E \gg \Delta E$, such that the standard deviation of momentum, σ_p, is much larger than Δp corresponding to ΔE. Consider two microcanonical ensembles containing N weakly interacting identical bosons that follow kinetic energy distributions $g_1(E)$ and $g_2(E)$ respectively: $N = \int g_1(E)dE = \int g_2(E)dE$. In addition, they have the same average kinetic energies: $\langle E\rangle = \int Eg_1(E)dE = \int Eg_2(E)dE$. The key difference is the standard deviation of kinetic energies: $\sigma_{E,2} \gg \sigma_{E,1} \sim \Delta E$, i.e., the energy broadening of the first group of particles (distribution described by $g_1(E)$) is approximately the energy deviation for the half-drop of $P(E)$, while being much smaller than that of the second group. The distribution function of the mixed $2N$-particle ensemble is $g(E) = g_1(E)+g_2(E)$, with standard deviation $\sigma_E = \sqrt{\sigma_{E,1}^2 + \sigma_{E,2}^2} \gg \Delta E$. Therefore, mixing the two groups of identical particles drastically increases energy broadening and reduces position uncertainty, while maintaining a sufficient number of particles for resonant transmission. The modifications introduced by this procedure are illustrated in Fig. 7. For the general case of $P(E) = 1 - \delta_P$, the energy broadening ΔE is given by Eq. (9) and can be similarly analyzed. In weakly interacting dilute atomic gases, two-body collisions dominate the interactions, which simply exchange particle momenta and therefore keep kinetic energy distributions unchanged.

In practice, the first group of particles may be prepared using Bose-Einstein condensates [65-67], in which the momenta of all involved bosons are expected to have approximately the same value–condensation in momentum space. The second group of particles may be prepared at temperatures slightly above the critical temperature T_c of phase transition from normal states to new quantum states like superconductivity, superfluidity, or Bose-Einstein condensation. As an example, RT of some typical bosons (Cooper pairs of superconducting Nb, ^{4}He, ^{7}Li, ^{23}Na, ^{87}Rb) across rectangular double barriers is studied and related parameters are presented in Table II. The effects of energy broadening through mixing identical bosons from different ensembles are evidenced by significantly reduced standard position deviations. Nevertheless, preparation of the first group of particles with ultrahigh monochromaticity remains challenging even with state-of-the-art techniques. Another challenge for experimental tests may be accelerating the condensates as a whole to desired incident velocities while maintaining condensation [68, 69].

[Figure 7: see original paper] **FIG. 7.** Schematic diagram for the kinetic energy distribution $f(E)$ of identical particles in microcanonical ensembles: Particle groups of high monochromaticity ($f(E) = g_1(E)$), low monochromaticity ($f(E) = g_2(E)$), and their superposition ($f(E) = g(E) = g_1(E) + g_2(E)$).

Table I. Parameters describing RT of protons across rectangular double barriers at $E = 0.5E_b$. With deviation ΔE or $|\Delta w|$ from resonance parameters, the tunneling probability drops from 1 to 0.5. The corresponding momentum broadening Δp and minimum standard deviation of particle positions Δx_m are calculated using $\Delta p\Delta x_m \geq \hbar/2$. In all cases the barrier width $a = 1$ Å.

E_b (eV)	w (Å)	ΔE (eV)	$\lvert\Delta w\rvert$ (Å)	Δp (kg • m/s)	Δx_m (m)
1.0	20.1370	2.103×10^{-16}	4.235×10^{-15}	3.443×10^{-39}	1.53×10^{4}
0.5	20.1370	1.320×10^{-12}	5.328×10^{-11}	3.056×10^{-35}	5.39×10^{-4}
0.2	20.1370	2.671×10^{-9}	2.690×10^{-7}	9.778×10^{-32}	9.25×10^{-6}
0.1	20.1370	1.101×10^{-7}	2.220×10^{-5}	5.700×10^{-30}	9.25×10^{-6}

Table II. Similar to Table I but for RT of some typical bosons with incident energy $E = 0.5E_b$. In all cases the barrier width $a = 1$ Å, and barrier height $E_b = 0.01V_{\max}$, with $V_{\max}$ being the upper bound set by the Planck length. Cooper pairs of electrons are represented by $e^- \ldots e^-$. The energy broadening and resulting uncertainties of momenta and positions of mixed particle groups are displayed in the lower lines of the same columns. The broadening parameter of energy is chosen such that $\sigma_E \gtrsim k_B T_c$, with k_B the Boltzmann constant and T_c the phase transition temperatures.

Boson	E_b (eV)	w (Å)	$\lvert\Delta w\rvert$ (Å)	ΔE (eV)	Δp (kg • m/s)	Δx_m (m)	σ_E (eV)	σ_p (kg • m/s)	σ_x (m)
$e^- \ldots e^-$ (in Nb)	3.16×10^{-3}	7.69×10^{-3}	3.16×10^{-3}	1.41×10^{-12}	3.83×10^{-16}	1.43×10^{-37}	1×10^{-3}	1.43×10^{-37}	5.19×10^{-7}
^{4}He	4.39×10^{-3}	3.16×10^{-3}	2.19×10^{-16}	1.02×10^{-28}	2.82×10^{-10}	5×10^{-4}	1.87×10^{-25}	2.45×10^{-4}	
^{7}Li	1.34×10^{-3}	3.16×10^{-3}	6.67×10^{-17}	1.43×10^{-37}	8.07×10^{-4}	1×10^{-10}	6.54×10^{-32}	6.49×10^{-5}	
^{23}Na	3.53×10^{-4}	3.16×10^{-3}	1.76×10^{-17}	1.43×10^{-37}	2.15×10^{-31}	1×10^{-10}	2.15×10^{-31}	8.13×10^{-31}	

III. CONCLUSIONS

To summarize, we have studied quantum tunneling across double barriers and arrived at a theorem that leads to several physical consequences. First, by tuning the inter-barrier spacing, it is possible for low-energy particles to completely penetrate arbitrary finite-sized potential barriers via resonant tunneling (RT). This result points to the possibility of significant tunneling of massive quantum particles across large barriers under mild conditions. Second, it is possible to construct any desired quasi-bound energy levels within the quantum well formed by the two barriers through adjustment of inter-barrier spacing. Third, for RT of quantum particles, it is possible to detect tiny variations of energy levels and positions of involved potential barriers with unprecedented accuracy. Finally, the critical dependence on inter-barrier spacing (consequently the phase difference) demonstrates again the vital role of the phase factor of wave functions, which has manifested itself in remarkable phenomena such as the Aharonov-Bohm effect [70].

Demonstration of the above results involves two key factors: (i) continuity of real space and (ii) energy monochromaticity of incident particles. The first is determined by whether a nonzero minimum length ($L_{\min}$) exists, and the second is affected by the uncertainty principle. Provided that $L_{\min} = 0$, distances in real space change continuously and RT can always be realized at given incident energies. Conversely, the existence of a nonzero $L_{\min}$ will set constraints (upper bounds) on particle mass, barrier height, and barrier width, beyond which no RT is expected. Meanwhile, to surmount the practical difficulty (position delocalization of incident particles) due to the uncertainty principle, we suggest a plausible scheme in which high- and low-monochromaticity flows of identical particle groups are mixed. Potential applications of Bose-Einstein condensates in this scheme are discussed. This work reveals the deep connection between two seemingly different branches of quantum physics—quantum tunneling and quantum gravity—and opens a possible avenue for testing the existence of a minimum length.

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APPENDIX A: MATRIX ELEMENT m_{11} FOR TUNNELING ACROSS SINGLE RECTANGULAR BARRIER

Within the transfer matrix method, we derive the diagonal matrix element m_{11} that describes transmission across a single rectangular barrier. For a quantum particle with incident energy E tunneling through a rectangular barrier with height V_0 and width a, the transfer matrix may be given by [56]:

$$M = \frac{1}{2\beta k}\begin{pmatrix}(\beta + ik)e^{-(ik-\beta)a} & (\beta - ik)e^{(ik+\beta)a}\\(\beta - ik)e^{-(ik+\beta)a} & (\beta + ik)e^{(ik-\beta)a}\end{pmatrix}$$

where $k = \sqrt{2mE}/\hbar$, $\beta = \sqrt{2m(V_0 - E)}/\hbar$, and $\gamma = \frac{1}{2\beta k}$. The first diagonal term is:

$$m_{11} = \frac{(\beta + ik)^2 e^{-(ik-\beta)a} - (\beta - ik)^2 e^{-(ik+\beta)a}}{2\beta k}$$

which can be reduced to

$$m_{11} = -2i\gamma e^{-ika}[(\beta^2 - k^2)\sinh(\beta a) + 2i\beta k\cosh(\beta a)]$$

and finally

$$m_{11} = 2\gamma e^{-ika}[i(k^2 - \beta^2)\sinh(\beta a) + 2\beta k\cosh(\beta a)] \tag{A1}$$

APPENDIX B: DEDUCTION OF ALTERNATIVE RT CONDITION

In this appendix, we deduce the resonant tunneling (RT) condition for homo-structured rectangular double-barriers.

For a double-barrier (DB) consisting of single rectangular barriers with height V_0 and barrier widths a and b, the diagonal element M_{11} of the transfer matrix M may be expressed as follows [57]:

$$|M_{11}|^2 = 1+\frac{(\beta^2+k^2)^2}{4\beta^2k^2}[\sinh^2(\beta b)+\sinh^2(\beta a)]+2\left[\frac{(\beta^2+k^2)^2}{4\beta^2k^2}\right]^2\sinh^2(\beta b)\sinh^2(\beta a)-\frac{16k^4\beta^4}{(\beta^2+k^2)^2}\sinh(\beta b)\sinh(\beta \tag{B1}$$

where $k = \sqrt{2mE}/\hbar$, $\beta = \sqrt{2m(V_0 - E)}/\hbar$, and E is the energy of the incident particle.

In the case of homo-structured DB, $a = b$, then

$$|M_{11}|^2 = 1+\frac{(\beta^2+k^2)^2}{4\beta^2k^2}\times[2\sinh^2(\beta a)]+2\left[\frac{(\beta^2+k^2)^2}{4\beta^2k^2}\right]^2\sinh^4(\beta a)-\frac{16k^4\beta^4}{(\beta^2+k^2)^2}\sinh^2(\beta a)\times\{[[(\beta^2+k^2)^2-8k^2\beta$$

For a given E, $|M_{11}|^2 = T(E;w)$ is a function of inter-barrier spacing w. The minimum of $|M_{11}|^2$ gives the maximum transmission coefficient $T(E;w)$, i.e., resonant tunneling (RT). The RT condition can be established by $\frac{\partial|M_{11}|^2}{\partial w} = 0$. It follows that

$$[[(\beta^2+k^2)^2-8k^2\beta^2]\cosh(2\beta a)-(\beta^2+k^2)^2]\times(-2k)\sin(2kw)-4k\beta(\beta^2-k^2)\sinh(2\beta a)\times(2k)\cos(2kw) = 0$$

and consequently

$$\tan(2kw) = \frac{4k\beta(\beta^2 - k^2)\sinh(2\beta a)}{(\beta^2 + k^2)^2 - [(\beta^2 + k^2)^2 - 8k^2\beta^2]\cosh(2\beta a)} \tag{B3}$$

By dividing both numerator and denominator by $\beta^2 k^2$, Eq. (B3) becomes

$$\tan(2kw) = \frac{4\delta\sinh(2\beta a)}{(\delta^2 + 4) - (\delta^2 - 4)\cosh(2\beta a)}$$

where $\delta \equiv \frac{k}{\beta}$. Recalling that $\sinh(2\beta a) = 2\sinh(\beta a)\cosh(\beta a)$ and $\cosh(2\beta a) = 2\cosh^2(\beta a) - 1$, one has

$$\tan(2kw) = \frac{2\delta\sinh(\beta a)\cosh(\beta a)}{\delta^2 + (1 - \delta^2)\cosh^2(\beta a)} = \frac{\delta\tanh(\beta a)}{\operatorname{sech}^2(\beta a)} \tag{B4}$$

Using the equality $\operatorname{sech}^2(\beta a) = 1 - \tanh^2(\beta a)$, Eq. (B4) finally reduces to

$$\tan(2kw) = \frac{\delta\tanh(\beta a)}{1 - \delta^2\tanh^2(\beta a)}$$

APPENDIX C: DEPENDENCE OF TUNNELING ON SMALL POSITION AND ENERGY CHANGES

In this appendix, we deduce mathematical expressions describing the dependence of the squared norm of diagonal transfer matrix element $|(M_{DB})_{11}|^2$ on slight deviations from peak positions and incident energies at resonant tunneling (RT), for the special case when incident energy is half the barrier height (V_0) of a homo-structured rectangular double-barrier (single barrier width: a). The inverse of $|(M_{DB})_{11}|^2$ then describes tunneling behavior dependence on small position and energy changes.

In general, $|(M_{DB})_{11}|^2 \equiv f(E; w)$ is a function of incident energy E and inter-barrier spacing w. Near RT, $f(E; w)$ can be expressed as functions of small deviations from RT parameters using Taylor series, considering that $|(M_{DB})_{11}|^2 = 1$ and $\frac{\partial|(M_{DB})_{11}|^2}{\partial w} = 0$, $\frac{\partial|(M_{DB})_{11}|^2}{\partial E} = 0$ at the RT point.

I. For constant E, dependence on deviation (Δw) from RT positions (w_n)

$$|(M_{DB})_{11}|^2 \equiv f(E; w) \cong 1 + \frac{1}{2}\frac{\partial^2|(M_{DB})_{11}|^2}{\partial w^2} \times (\Delta w)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta w}$$

Using expressions for rectangular double barriers (Appendix B), one has

$$\frac{\partial^2|(M_{DB})_{11}|^2}{\partial w^2} = \frac{(\beta^2+k^2)^2}{4\beta^4 k^2}\sinh^2(\beta a)\left[g(\beta,k)\times(-2k)\times\sin(2kw)+h(\beta,k)\times(2k)\times\cos(2kw)\right] \tag{C2}$$

where $k = \sqrt{2mE}/\hbar$, $\beta = \sqrt{2m(V_0 - E)}/\hbar$, $g(\beta,k) \equiv [(\beta^2+k^2)^2 - 8\beta^2k^2]\cosh(2\beta a) - (\beta^2+k^2)^2$, and $h(\beta,k) = -4\beta k(\beta^2-k^2)\sinh(2\beta a)$.

The condition $\frac{\partial|(M_{DB})_{11}|^2}{\partial w} = 0$ gives $g(\beta,k)\sin(2kw) = h(\beta,k)\cos(2kw)$, and then $\tan(2kw) = \frac{h(\beta,k)}{g(\beta,k)}$. Using this,

$$\frac{\partial^2|(M_{DB})_{11}|^2}{\partial w^2} = \frac{(\beta^2+k^2)^2}{4\beta^4 k^2}\sinh^2(\beta a)\cdot h(\beta,k)\sin(2kw)[\cot^2(2kw)+1]$$

For rectangular double barriers, we have the general relation $2kw = (2n-1)\pi - 2\alpha$, and $\alpha = \arctan\left[\frac{2\beta k}{(k^2-\beta^2)}\tanh(\beta a)\right]$. Consequently, $\sin(2kw) = \sin(2\alpha) = \frac{2\tan\alpha}{1+\tan^2\alpha}$. Finally,

$$\frac{\partial^2|(M_{DB})_{11}|^2}{\partial w^2} = \frac{(\beta^2+k^2)^2}{4\beta^4 k^2}\sinh^2(2\beta a)(1+\tan^2\alpha) \tag{C6}$$

It is clear that $\frac{\partial^2|(M_{DB})_{11}|^2}{\partial w^2} > 0$ holds for all allowed incident energies E, proving that $|(M_{DB})_{11}|^2$ reaches its minimum and its reciprocal gives the maximum transmission probability, i.e., 1.

When incident energy is half the barrier height ($\beta = k$), we have $\alpha = 0$ and $\frac{\partial^2|(M_{DB})_{11}|^2}{\partial w^2} = 2k^2\sinh^2(2ka)$, therefore

$$|(M_{DB})_{11}|^2 \cong 1 + \frac{1}{2}\frac{\partial^2|(M_{DB})_{11}|^2}{\partial w^2}\times(\Delta w)^2 = 1 + \sinh^2(2ka)\times(k\Delta w)^2 \tag{C7}$$

II. For constant w, dependence on deviation (ΔE) from RT energies (E_{RT})

$$|(M_{DB})_{11}|^2 \equiv f(E;w) \cong 1 + \frac{1}{2}\frac{\partial^2|(M_{DB})_{11}|^2}{\partial E^2}\times(\Delta E)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta E}$$

Compared to $\frac{\partial^2|(M_{DB})_{11}|^2}{\partial w^2}$, computation of $\frac{\partial^2|(M_{DB})_{11}|^2}{\partial E^2}$ is much more complicated. Alternatively, we directly consider the dependence of $|(M_{DB})_{11}|^2$ on energy deviation (ΔE) to second order. For the special case $\beta = k$, the expression reduces to

$$|(M_{DB})_{11}|^2 = 1+2\sinh^2(ka)+2\sinh^4(ka)+\sinh^2(ka)[\cosh(2ka)+1]\times\cos(2kw)$$

Recalling that $2kw = (2n-1)\pi$ for $\beta = k$, the term $\cos(2kw)$ may be expressed by Taylor series around the RT point with respect to Δk to second order:

$$\cos(2kw) \cong -1 + (2w)^2(\Delta k)^2 = -1 + 2(w\Delta k)^2 \tag{C10}$$

Substitution yields

$$|(M_{DB})_{11}|^2 \cong 1 + \sinh^2(2ka)(w\Delta k)^2 \tag{C11}$$

Using $k = \sqrt{2mE}/\hbar$, we have $\Delta k = \frac{\sqrt{2m}}{\hbar}\frac{\Delta E}{2\sqrt{E}} = \frac{k}{2E}\Delta E$, and finally

$$|(M_{DB})_{11}|^2 \cong 1 + \sinh^2(2ka)\left(\frac{w\Delta E}{2E}\right)^2 \tag{C12}$$

APPENDIX D: GENERALIZED CONSTRAINTS ON BARRIER SIZE DUE TO A MINIMUM LENGTH

In this appendix, we deduce the constraint on barrier size (barrier height, barrier width) for effective resonant tunneling (RT) in the presence of a nonzero minimum length ($L_{\min}$). Effective RT implies that, given deviation Δw when $|w-w_n| \leq \Delta w$, the inequality $T(E;w) \geq 1-\delta_P$ holds, where δ_P $(0 < \delta_P < 1)$ is the tolerance of decrease in tunneling probability at which significant tunneling is measurable. Based on the theorem proof, we have

$$T_{DB}(E;w) = |(M_{DB})_{11}|^2 = |e^{i\theta}[1+R(e^{-i(\phi+\theta)}+1)]|^{-2} = |1+R(e^{-i(\phi+\theta)}+1)|^{-2}$$

Then Δw is determined by the equality

$$|1 + R(e^{-i(\phi+\theta)} + 1)|^{-2} = 1 - \delta_P$$

Equivalently,

$$|1 + R(e^{-i(\phi+\theta)} + 1)|^2 = \frac{1}{1-\delta_P}$$

It follows that

$$|1+R+R(\cos(\phi+\theta)-i\sin(\phi+\theta))|^2 = (1+R+R\cos(\phi+\theta))^2+R^2\sin^2(\phi+\theta) = \frac{1}{1-\delta_P}$$

which reduces to

$$2R(1+R)[1+\cos(\phi+\theta)] = \frac{\delta_P}{1-\delta_P}$$

Consequently,

$$\cos(\phi+\theta) = -1 + \frac{\delta_P}{2R(1+R)(1-\delta_P)} \tag{D8}$$

For a homo-structured double-barrier system, the two parameters $\theta = \arg(m_{11}^2)$ and $R = |m_{12}|^2$ are solely determined by a single barrier $V(x)$. The tunable parameter is $\phi = 2k(a+w)$, varied via inter-barrier spacing w by a small magnitude Δw. Near RT, $|\Delta w| \ll w_n$. Around $\phi+\theta = (2n-1)\pi$, i.e., the RT points, expanding $\cos(\phi+\theta)$ using Taylor series to second order yields

$$\cos(\phi+\theta) \cong -1 + \frac{(\Delta\phi)^2}{2} \tag{D9}$$

where $\Delta\phi = \pm 2k\Delta w$. Comparing Eq. (D8) and (D9) gives

$$2k\Delta w = \sqrt{\frac{\delta_P}{R(1+R)(1-\delta_P)}} \tag{D10}$$

Finally,

$$\Delta w = \frac{\sqrt{\delta_P}}{2k\sqrt{R(1+R)(1-\delta_P)}} \tag{D11}$$

To achieve effective RT, the existence of $L_{\min}$ requires that

$$\Delta w \geq L_{\min} \implies \frac{\sqrt{\delta_P}}{2k\sqrt{R(1+R)(1-\delta_P)}} \geq L_{\min} \tag{D12}$$

For a single barrier $V(x)$, the reflection coefficient is given by [59-62]

$$|r|^2 = \frac{|m_{12}|^2}{|m_{11}|^2} = RT_1(E) = R|t|^2 \tag{D13}$$

where $R = |m_{12}|^2$ and $T_1(E) = |t|^2$ is the transmission coefficient across $V(x)$ at energy E. Conservation of probability current gives $|r|^2 + |t|^2 = 1$. Qualitatively, $|r|^2$ increases with barrier width a and barrier height E_b, indicating that R is an increasing function of barrier size parameters a and E_b: $R = R(a, E_b)$. Larger barrier size results in larger R. Substituting k with $\sqrt{2mE}/\hbar$, inequality (D12) becomes

$$\frac{\hbar\sqrt{\delta_P}}{2\sqrt{2mE}\sqrt{R(1+R)(1-\delta_P)}} \geq L_{\min} \tag{D14}$$

This is the constraint imposed on particle mass, barrier height, and barrier width due to the minimum length.

In the case $\delta_P = 0.5$, the FWHM ($= 2\Delta w$) is obtained. With $L_{\min} = l_P$,

$$\frac{\hbar}{2\sqrt{2mE} \cdot R(1+R)} \geq l_P \tag{D15}$$

For fixed particle mass m and incident energy E, inequality (D15) sets upper bounds on R and consequently on barrier size of $V(x)$: the barrier width a and barrier height E_b.

Furthermore, we derive the constraint on incident energy broadening using Eq. (D9). With $w(= w_n)$ and a fixed, $\Delta\phi = 2\Delta k(a + w)$. Using $k = \sqrt{2mE}/\hbar$, we have $\Delta k = \frac{\sqrt{2m}}{\hbar}\frac{\Delta E}{2\sqrt{E}} = \frac{k}{2E}\Delta E$, so $2\Delta k = k \times \frac{\Delta E}{E}$, and $\Delta\phi = k(a + w) \times \frac{\Delta E}{E}$. It follows that

$$(\Delta\phi)^2 = \frac{\delta_P}{R(1+R)(1-\delta_P)} \implies \Delta\phi = k(a+w) \times \left|\frac{\Delta E}{E}\right| = \sqrt{\frac{\delta_P}{R(1+R)(1-\delta_P)}} \tag{D17}$$

Finally,

$$\Delta E = \frac{E}{k(a+w)}\sqrt{\frac{\delta_P}{R(1+R)(1-\delta_P)}} \tag{D18}$$

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