

The Relationship Between Endograph Metric and Γ -Convergence on Fuzzy Sets

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Abstract

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Full Text

Preamble

Relations of Endograph Metric and Γ -Convergence on Fuzzy Sets

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Abstract

This paper demonstrates that the endograph metric and Γ -convergence are compatible on a large class of fuzzy sets in \mathbb{R}^m .

Keywords: Endograph metric; Γ -convergence; Hausdorff metric; compatible

1. Introduction

We demonstrate that the endograph metric and Γ -convergence are compatible on a large class of fuzzy sets in \mathbb{R}^m . The results in this paper improve the corresponding results in [?, ?].

2. Preliminaries

In this section, we recall and provide some basic notions and fundamental results related to fuzzy sets and convergence structures on them. Readers may refer to [?, ?, ?, ?] for related content.

Throughout this paper, we suppose that X is a nonempty set and d is a metric on X . For simplicity, we also use X to denote the metric space (X, d) . The metric d on $X \times [0, 1]$ is defined as follows: for $(x, \alpha), (y, \beta) \in X \times [0, 1]$,

$$d((x, \alpha), (y, \beta)) = d(x, y) + |\alpha - \beta|.$$

Throughout this paper, we suppose that the metric on $X \times [0, 1]$ is d . For simplicity, we also use $X \times [0, 1]$ to denote the metric space $(X \times [0, 1], d)$.

A fuzzy set u in X can be seen as a function $u : X \rightarrow [0, 1]$. A subset S of X can be seen as a fuzzy set in X . If there is no confusion, the fuzzy set corresponding to S is often denoted by χ_S ; that is,

$$\chi_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \in X \setminus S. \end{cases}$$

For simplicity, for $x \in X$, we will use \hat{x} to denote the fuzzy set $\chi_{\{x\}}$ in X .

In this paper, if we want to emphasize a specific metric space X , we will write the fuzzy set corresponding to S in X as $S_F(X)$, and the fuzzy set corresponding to $\{x\}$ in X as $\hat{x}_F(X)$.

The symbol $F(X)$ is used to denote the set of all fuzzy sets in X . For $u \in F(X)$ and $\alpha \in [0, 1]$, let $\{u > \alpha\}$ denote the set $\{x \in X : u(x) > \alpha\}$, and let $[u]_\alpha$ denote the α -cut of u , i.e.,

$$[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0, 1], \\ \{u > 0\}, & \alpha = 0, \end{cases}$$

where \bar{S} denotes the topological closure of S in (X, d) .

The symbols $K(X)$ and $C(X)$ are used to denote the set of all nonempty compact subsets of X and the set of all nonempty closed subsets of X , respectively.

Let $F_{USC}(X)$ denote the set of all upper semi-continuous fuzzy sets $u : X \rightarrow [0, 1]$, i.e.,

$$F_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \cup \{\emptyset\} \text{ for all } \alpha \in [0, 1]\}.$$

Define

$$\begin{aligned} F_{USCB}(X) &:= \{u \in F_{USC}(X) : [u]_0 \in K(X) \cup \{\emptyset\}\}, \\ F_{USCG}(X) &:= \{u \in F_{USC}(X) : [u]_\alpha \in K(X) \cup \{\emptyset\} \text{ for all } \alpha \in (0, 1]\}. \end{aligned}$$

Clearly, $F_{USCB}(X) \subseteq F_{USCG}(X) \subseteq F_{USC}(X)$. Define

$$\begin{aligned} F_{CON}(X) &:= \{u \in F(X) : \text{for all } \alpha \in (0, 1], [u]_\alpha \text{ is connected in } X\}, \\ F_{USCCON}(X) &:= F_{USC}(X) \cap F_{CON}(X), \end{aligned}$$

$$F_{USCGCON}(X) := F_{USCG}(X) \cap F_{CON}(X).$$

Let $u \in F_{CON}(X)$. Then $[u]_0 = \bigcup_{\alpha > 0} [u]_\alpha$ is connected in X . If $u = \chi_\emptyset$, then $[u]_0 = \emptyset$ is connected in X . If $u \neq \chi_\emptyset$, then there is an $\alpha \in (0, 1]$ such that $[u]_\alpha \neq \emptyset$. Note that $[u]_\beta \supseteq [u]_\alpha$ when $\beta \in [0, \alpha]$. Hence $\bigcup_{0 < \beta < \alpha} [u]_\beta$ is connected, and thus $[u]_0 = \bigcup_{0 < \beta < \alpha} [u]_\beta$ is connected.

$$F_{CON}(X) = \{u \in F(X) : \text{for all } \alpha \in [0, 1], [u]_\alpha \text{ is connected in } X\}.$$

Let $F_{USC}^1(X)$ denote the set of all normal and upper semi-continuous fuzzy sets $u : X \rightarrow [0, 1]$, i.e.,

$$F_{USC}^1(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0, 1]\}.$$

We introduce some subclasses of $F_{USC}^1(X)$, which will be discussed in this paper. Define

$$\begin{aligned} F_{USCB}^1(X) &:= F_{USC}^1(X) \cap F_{USCB}(X), \\ F_{USCG}^1(X) &:= F_{USC}^1(X) \cap F_{USCG}(X), \\ F_{USCCON}^1(X) &:= F_{USC}^1(X) \cap F_{CON}(X), \\ F_{USCGCON}^1(X) &:= F_{USCG}(X) \cap F_{CON}(X). \end{aligned}$$

Clearly,

$$F_{USCB}^1(X) \subseteq F_{USCGCON}^1(X) \subseteq F_{USCG}^1(X) \subseteq F_{USCCON}^1(X) \subseteq F_{USC}^1(X).$$

Let (X, d) be a metric space. We use H to denote the Hausdorff distance on $C(X)$ induced by d , i.e., $H(U, V) = \max\{H^*(U, V), H^*(V, U)\}$ for arbitrary $U, V \in C(X)$, where $H^*(U, V) = \sup_{u \in U} d(u, V) = \sup_{u \in U} \inf_{v \in V} d(u, v)$. If there is no confusion, we also use H to denote the Hausdorff distance on $C(X \times [0, 1])$ induced by d .

Remark 2.1. ρ is said to be a metric on Y if ρ is a function from $Y \times Y$ into \mathbb{R} satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be a metric space. ρ is said to be an extended metric on Y if ρ is a function from $Y \times Y$ into $\mathbb{R} \cup \{+\infty\}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be an extended metric space.

We can see that for arbitrary metric space (X, d) , the Hausdorff distance H on $K(X)$ induced by d is a metric. So the Hausdorff distance H on $K(X \times [0, 1])$ induced by d on $X \times [0, 1]$ is a metric. In these cases, we call the Hausdorff distance the Hausdorff metric.

The Hausdorff distance H on $C(X)$ induced by d on X is an extended metric, but probably not a metric, because $H(A, B)$ could be equal to $+\infty$ for certain metric space X and $A, B \in C(X)$. Clearly, if H on $C(X)$ induced by d is not

a metric, then H on $C(X \times [0, 1])$ induced by d is also not a metric. So the Hausdorff distance H on $C(X \times [0, 1])$ induced by d on $X \times [0, 1]$ is an extended metric but probably not a metric.

In the cases that the Hausdorff distance H is an extended metric, we call the Hausdorff distance the Hausdorff extended metric.

We can see that H on $C(\mathbb{R}^m)$ is an extended metric but not a metric, and then the same is H on $C(\mathbb{R}^m \times [0, 1])$.

In this paper, for simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric.

For $u \in F(X)$, define

$$\text{end } u := \{(x, t) \in X \times [0, 1] : u(x) \geq t\},$$

$$\text{send } u := \{(x, t) \in X \times [0, 1] : u(x) \geq t\} \cap ([u]_0 \times [0, 1]).$$

$\text{end } u$ and $\text{send } u$ are called the endograph and the sendograph of u , respectively.

We can define the endograph metric H_{end} on $F_{USC}(X)$ as usual. For $u, v \in F_{USC}(X)$,

$$H_{\text{end}}(u, v) := H(\text{end } u, \text{end } v),$$

where H is the Hausdorff metric on $C(X \times [0, 1])$ induced by d on $X \times [0, 1]$.

Rojas-Medar and Román-Flores [?] introduced the Kuratowski convergence of a sequence of sets in a metric space.

Let (X, d) be a metric space. Let C be a set in X and $\{C_n\}$ a sequence of sets in X . $\{C_n\}$ is said to Kuratowski converge to C according to (X, d) ,

$$C = \liminf_{n \rightarrow \infty} C_n = \limsup_{n \rightarrow \infty} C_n,$$

where

$$\liminf_{n \rightarrow \infty} C_n = \{x \in X : x = \lim_{n \rightarrow \infty} x_n, x_n \in C_n\},$$

$$\limsup_{n \rightarrow \infty} C_n = \{x \in X : x = \lim_{k \rightarrow \infty} x_{n_k}, x_{n_k} \in C_{n_k}\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} C_k.$$

In this case, we'll write $C = \lim_{n \rightarrow \infty}^{(K)} C_n$ according to (X, d) . If there is no confusion, we will not emphasize the metric space (X, d) and write $\{C_n\}$ Kuratowski converges to C or $C = \lim_{n \rightarrow \infty}^{(K)} C_n$ for simplicity.

We can define the Γ -convergence of a sequence of fuzzy sets on $F_{USC}(X)$ as usual.

Let $u, u_n, n = 1, 2, \dots$, be fuzzy sets in $F_{USC}(X)$. $\{u_n\}$ is said to Γ -converge to u , denoted by $u = \lim_{n \rightarrow \infty}^{(\Gamma)} u_n$, if $\text{end } u = \lim_{n \rightarrow \infty}^{(K)} \text{end } u_n$ according to $(X \times [0, 1], d)$.

3. Main Results

We need the following conclusions.

Theorem 3.1. [?] Suppose that C, C_n are sets in $C(X)$, $n = 1, 2, \dots$. Then $H(C_n, C) \rightarrow 0$ implies that $\lim_{n \rightarrow \infty}^{(K)} C_n = C$.

Lemma 3.2. (Lemma 2.1 in [?]) Let (X, d) be a metric space, and $C_n, n = 1, 2, \dots$, be a sequence of sets in X . Suppose that $x \in X$. Then: (i) $x \in \liminf_{n \rightarrow \infty} C_n$ if and only if $\lim_{n \rightarrow \infty} d(x, C_n) = 0$, (ii) $x \in \limsup_{n \rightarrow \infty} C_n$ if and only if there is a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ such that $\lim_{k \rightarrow \infty} d(x, C_{n_k}) = 0$.

Proof. Here we give a detailed proof. Readers who think that this conclusion is obvious can skip this proof.

- (i) Assume that $x \in \liminf_{n \rightarrow \infty} C_n$. Then there is a sequence $\{x_n, n = 1, 2, \dots\}$ in X such that $x_n \in C_n$ for $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} d(x, x_n) = 0$. Since $d(x, C_n) \leq d(x, x_n)$, thus $\lim_{n \rightarrow \infty} d(x, C_n) = 0$.

Conversely, assume that $\lim_{n \rightarrow \infty} d(x, C_n) = 0$. For each $n = 1, 2, \dots$, we can choose an x_n in C_n such that $d(x, x_n) \leq d(x, C_n) + 1/n$. Hence $\lim_{n \rightarrow \infty} d(x, x_n) = 0$. So $x \in \liminf_{n \rightarrow \infty} C_n$.

- (ii) Assume that $x \in \limsup_{n \rightarrow \infty} C_n$. Then there is a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and $x_{n_k} \in C_{n_k}$ for $k = 1, 2, \dots$ such that $\lim_{k \rightarrow \infty} d(x, x_{n_k}) = 0$. Since $d(x, C_{n_k}) \leq d(x, x_{n_k})$, thus $\lim_{k \rightarrow \infty} d(x, C_{n_k}) = 0$.

Conversely, assume that there is a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ such that $\lim_{k \rightarrow \infty} d(x, C_{n_k}) = 0$. For each $k = 1, 2, \dots$, we can choose an x_{n_k} in C_{n_k} such that $d(x, x_{n_k}) \leq d(x, C_{n_k}) + 1/k$. Hence $\lim_{k \rightarrow \infty} d(x, x_{n_k}) = 0$. So $x \in \limsup_{n \rightarrow \infty} C_n$.

Theorem 3.3. (Theorem 5.19 in [?]) Let u be a fuzzy set in $F_{USCG}^1(X)$ and let $u_n, n = 1, 2, \dots$, be fuzzy sets in $F_{USCG}^1(X)$. Then the following are equivalent: (i) $H_{\text{end}}(u_n, u) \rightarrow 0$; (ii) $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ holds a.e. on $\alpha \in (0, 1)$; (iii) $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ for all $\alpha \in (0, 1) \setminus P_0(u)$; (iv) There is a dense subset P of $(0, 1) \setminus P_0(u)$ such that $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ for $\alpha \in P$; (v) There is a countable dense subset P of $(0, 1) \setminus P_0(u)$ such that $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ for $\alpha \in P$.

Theorem 3.4. (Theorem 6.2 in [?]) Let $u, u_n, n = 1, 2, \dots$ be fuzzy sets in $F_{USC}(\mathbb{R}^m)$. Then the following are equivalent: (i) $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$; (ii) $\lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha = [u]_\alpha$ holds a.e. on $\alpha \in (0, 1)$; (iii) $\lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha = [u]_\alpha$ holds for all $\alpha \in (0, 1) \setminus P(u)$; (iv) There is a dense subset P of $(0, 1) \setminus P(u)$ such that $\lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha = [u]_\alpha$ holds for $\alpha \in P$; (v) There is a countable dense subset P of $(0, 1) \setminus P(u)$ such that $\lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha = [u]_\alpha$ holds for $\alpha \in P$.

Proposition 3.5. Let C be a nonempty compact set in \mathbb{R}^m and for $n = 1, 2, \dots$ let C_n be a nonempty connected and closed set in \mathbb{R}^m . Then $H(C_n, C) \rightarrow 0$ if and only if $\lim_{n \rightarrow \infty}^{(K)} C_n = C$.

Proof. From Theorem 3.1, we have that $H(C_n, C) \rightarrow 0 \Rightarrow \lim_{n \rightarrow \infty}^{(K)} C_n = C$.

Now we show that $\lim_{n \rightarrow \infty}^{(K)} C_n = C \Rightarrow H(C_n, C) \rightarrow 0$. We prove by contradiction. Assume that $\lim_{n \rightarrow \infty}^{(K)} C_n = C$ but $H(C_n, C) \not\rightarrow 0$. Then $H^*(C, C_n) \not\rightarrow 0$ or $H^*(C_n, C) \not\rightarrow 0$. We split the proof into two cases.

Case (i) $H^*(C, C_n) \not\rightarrow 0$. In this case, there is an $\varepsilon > 0$ and a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ such that $H^*(C, C_{n_k}) > \varepsilon$. Thus for each $k = 1, 2, \dots$ there exists $x_k \in C$ such that $d(x_k, C_{n_k}) > \varepsilon$. Since C is compact, there is a subsequence $\{x_{k_l}\}$ of $\{x_k\}$ which converges to $x \in C$. Then there is a $L(\varepsilon)$ such that $d(x, x_{k_l}) < \varepsilon/2$ for all $l \geq L$. Hence $d(x, C_{n_{k_l}}) - d(x, x_{k_l}) > \varepsilon/2$. By Lemma 3.2 (i), this contradicts $x \in C = \liminf_{n \rightarrow \infty} C_n$.

Case (ii) $H^*(C_n, C) \not\rightarrow 0$. In this case, there is an $\varepsilon > 0$ and a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ such that $H^*(C_{n_k}, C) > \varepsilon$. Thus we have the following: (a) for each $k = 1, 2, \dots$ there exists $x_{n_k} \in C_{n_k}$ such that $d(x_{n_k}, C) > \varepsilon$.

Pick $y \in C$. Since $C = \liminf_{n \rightarrow \infty} C_n$, we can find a sequence $\{y_n\}$ satisfying that $y_n \in C_n$ for $n = 1, 2, \dots$ and $\{y_n\}$ converges to y . Hence there is a $N(\varepsilon)$ such that for all $n \geq N$, $d(y_n, y) < \varepsilon$. Since $d(y_n, C) \leq d(y_n, y)$, we have the following: (b) for all $n \geq N$, $d(y_n, C) < \varepsilon$.

Let $k \in \mathbb{N}$ with $n_k \geq N$. Define a function f_k from C_{n_k} to \mathbb{R} given by $f_k(z) = d(z, C)$ for each $z \in C_{n_k}$. Then f_k is a continuous function on C_{n_k} . Since C_{n_k} is a connected set in \mathbb{R}^m , $f_k(C_{n_k})$ is a connected set in \mathbb{R} . Combined this fact with the above clauses (a) and (b), we obtain that there exists $z_{n_k} \in C_{n_k}$ such that $d(z_{n_k}, C) = \varepsilon$.

From this and the compactness of C , the set $\{z_{n_k}, n_k \geq N\}$ is bounded in \mathbb{R}^m , and thus $\{z_{n_k}, n_k \geq N\}$ has a cluster point z . By continuity, $d(z, C) = \varepsilon$, which contradicts $z \in \limsup_{n \rightarrow \infty} C_n = C$.

Remark 3.6. Proposition 3.5 may be known; however, we can't find this conclusion in the references that we can obtain. So we give a proof here.

Let A be a nonempty compact set in \mathbb{R}^m and B a nonempty closed set in \mathbb{R}^m . If $H(A, B) < +\infty$, then B is bounded and hence a compact set in \mathbb{R}^m . Clearly, if B is compact in \mathbb{R}^m , then $H(A, B) < +\infty$. So $H(A, B) < +\infty$ if and only if B is a compact set in \mathbb{R}^m .

From the above fact we know that for C and $C_n, n = 1, 2, \dots$, satisfying the assumptions of Proposition 3.5, if $H(C_n, C) \rightarrow 0$ (by Proposition 3.5, $\lim_{n \rightarrow \infty}^{(K)} C_n = C$), then clearly there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $H(C_n, C) < +\infty$ and thus for all $n \geq N$, C_n is compact.

The following Proposition 3.7 is an immediate consequence of Proposition 3.5, Theorem 3.3, and Corollary 3.4.

Theorem 3.7. Let u be a fuzzy set in $F_{USCG}^1(\mathbb{R}^m)$ and for $n = 1, 2, \dots$, let u_n be a fuzzy set in $F_{USCCON}^1(\mathbb{R}^m)$. Then $H_{\text{end}}(u_n, u) \rightarrow 0$ as $n \rightarrow \infty$ if and only

if $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$.

Proof. The proof is routine. By letting $X = \mathbb{R}^m$ in Theorem 3.3 we have: (i) $H_{\text{end}}(u_n, u) \rightarrow 0$ if and only if $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ holds a.e. on $\alpha \in (0, 1)$.

By Theorem 3.4, we have: (ii) $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$ if and only if $[u]_\alpha = \lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha$ holds a.e. on $\alpha \in (0, 1)$.

Since for each $\alpha \in (0, 1]$ and $n \in \mathbb{N}$, $[u]_\alpha \in K(\mathbb{R}^m)$, $[u_n]_\alpha \in C(\mathbb{R}^m)$ and $[u_n]_\alpha$ is connected in \mathbb{R}^m . Thus by Proposition 3.5, for each $\alpha \in (0, 1]$, $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ if and only if $[u]_\alpha = \lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha$. Combined this fact with the above clauses (i) and (ii), we have that $H_{\text{end}}(u_n, u) \rightarrow 0$ if and only if $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$.

Remark 3.8. Theorem 3.7 would be false if \mathbb{R}^m were replaced by a general metric space X .

Here we mention that for $u \in F_{USCG}^1(\mathbb{R})$ and a sequence $\{u_n\}$ in $F_{USCCON}^1(\mathbb{R})$, $H(u_n, u) \rightarrow 0$ does not imply that there is an N satisfying that for all $n \geq N$, $u_n \in F_{USCGCON}^1(\mathbb{R})$.

The following Example 3.9 is such an example, which shows that there exists a $u \in F_{USCG}^1(\mathbb{R})$ and a sequence $\{u_n\}$ in $F_{USCCON}^1(\mathbb{R})$ such that: (i) $H(u_n, u) \rightarrow 0$, (ii) for each $n = 1, 2, \dots$, $u_n \notin F_{USCGCON}^1(\mathbb{R})$.

Example 3.9. Let $u = \hat{1}_F(\mathbb{R}) \in F_{USC}^1(\mathbb{R}) \subset F_{USCG}^1(\mathbb{R})$. For $n = 1, 2, \dots$, define $u_n \in F_{USC}(\mathbb{R})$ as follows:

$$u_n(t) = \begin{cases} 1, & t = 1, \\ \frac{1}{n}, & t \neq 1. \end{cases}$$

Then for $n = 1, 2, \dots$,

$$[u_n]_\alpha = \begin{cases} \{1\}, & \alpha \in (1/n, 1], \\ \mathbb{R}, & \alpha \in [0, 1/n]. \end{cases}$$

So for each $n = 1, 2, \dots$, $u_n \in F_{USCCON}^1(\mathbb{R})$ but $u_n \notin F_{USCGCON}^1(\mathbb{R})$. It can be seen that $H_{\text{end}}(u, u_n) = 1/n \rightarrow 0$.

Theorem 9.2 in [?] discusses the compatibility of the endograph metric H_{end} and the Γ -convergence.

Theorem 3.10. (Theorem 9.2 in [?]) Let u be a fuzzy set in $F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_F(\mathbb{R}^m)\}$ and for $n = 1, 2, \dots$, let u_n be a fuzzy set in $F_{USCGCON}(\mathbb{R}^m)$. Then $H_{\text{end}}(u_n, u) \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$.

The following Corollary 3.11 is an immediate corollary of Theorem 9.2 in [?].

Corollary 3.11. Let u be a fuzzy set in $F_{USCG}^1(\mathbb{R}^m)$ and for $n = 1, 2, \dots$, let u_n be a fuzzy set in $F_{USCGCON}^1(\mathbb{R}^m)$. Then $H_{\text{end}}(u_n, u) \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$.

We can see that Theorem 3.7 is an improvement of Corollary 3.11. Corollary 3.11 is the normal fuzzy set case of Theorem 9.2 in [?], which is an important special case of Theorem 9.2 in [?].

In the following, we give an improvement of Theorem 9.2 in [?].

For a subset S in $(X \times [0, 1], d)$, we still use d to denote the induced metric on S by d .

For a set S in X , we use \bar{S} to denote the topological closure of S in (X, d) ; for a set S in $X \times [0, 1]$, we use \bar{S} to denote the topological closure of S in $(X \times [0, 1], d)$. The readers can judge the meaning of \bar{S} according to the context.

For $D \subseteq X \times [0, 1]$ and $\alpha \in [0, 1]$, define $D_\alpha := \{x \in X : (x, \alpha) \in D\}$.

For a nonempty set D in $X \times [0, 1]$, define

$$f_D := \inf\{\alpha : (x, \alpha) \in D\},$$

$$S_D := \sup\{\alpha : (x, \alpha) \in D\}.$$

Proposition 3.12. Let D be a nonempty set in $X \times [0, 1]$ with $D_r \subseteq D_t$ for $f_D \leq t \leq r \leq 1$. Then D_{f_D} is connected in (X, d) if and only if D is connected in $(X \times [0, 1], d)$.

Proof. Sufficiency. We proceed by contradiction. Assume that D_{f_D} is connected in X . If D is not connected in $X \times [0, 1]$, then there exist two nonempty sets A and B in $X \times [0, 1]$ such that $A \cup B = D$, $A \cap B = \emptyset$ and $\bar{B} \cap A = \emptyset$.

Note that $D_{f_D} \times \{f_D\} \subseteq D$ and $D_{f_D} \times \{f_D\}$ is connected. Hence $D_{f_D} \times \{f_D\} \subseteq A$ or $D_{f_D} \times \{f_D\} \subseteq B$. Without loss of generality, we suppose that $D_{f_D} \times \{f_D\} \subseteq A$.

Pick $(x, \alpha) \in B$. Set $\gamma = \inf\{\beta : (x, \beta) \in B\}$. If $(x, \gamma) \in B$, we affirm that $\gamma > f_D$. Otherwise $\gamma = f_D$ and $(x, f_D) \in B$. Note that $(x, f_D) \in A$. Thus $A \cap \bar{B} \neq \emptyset$, which is a contradiction. Hence $(x, \xi) \in A$ for $\xi \in [f_D, \gamma)$, and therefore $(x, \gamma) = \lim_{\xi \rightarrow \gamma^-} (x, \xi) \in \bar{A}$. So $A \cap \bar{B} \neq \emptyset$. This is a contradiction.

If $(x, \gamma) \in A$, then there is a sequence $\{(x, \gamma_n)\}$ in B such that $\lim_{n \rightarrow \infty} \gamma_n = \gamma$. Hence $(x, \gamma) = \lim_{n \rightarrow \infty} (x, \gamma_n) \in \bar{B}$. Thus $A \cap \bar{B} \neq \emptyset$. This is a contradiction.

Necessity. Assume that D is connected in $X \times [0, 1]$. Define a function $f : (D, d) \rightarrow (D_{f_D}, d)$ given by $f(x, \alpha) = x$ for each $(x, \alpha) \in D$. Clearly for $(x, \alpha), (y, \beta) \in X \times [0, 1]$, $d(f(x, \alpha), f(y, \beta)) = d(x, y) \leq d((x, \alpha), (y, \beta))$. Then f is continuous and hence $D_{f_D} = f(D)$ is connected in X .

Corollary 3.13. Let E be a nonempty set in $X \times [0, 1]$ with $E_r \supseteq E_t$ for $0 \leq t \leq r \leq S_E$. Then E_{S_E} is connected in (X, d) if and only if E is connected in $(X \times [0, 1], d)$.

Proof. Let $D = \{(x, 1-\alpha) : (x, \alpha) \in E\}$. Then D is a nonempty set in $X \times [0, 1]$ with $D_r \subseteq D_t$ for $f_D \leq t \leq r \leq 1$. Hence by Proposition 3.12, D_{f_D} is connected in (X, d) if and only if D is connected in $(X \times [0, 1], d)$.

Define $f : (E, d) \rightarrow (D, d)$ as follows: $f(x, \alpha) = (x, 1 - \alpha)$ for $(x, \alpha) \in E$. Observe that $E_{S_E} = D_{f_D}$, so to verify the desired result it suffices to show that D is connected in $(X \times [0, 1], d)$ if and only if E is connected in $(X \times [0, 1], d)$, which follows from the fact that f is an isometry and $f(E) = D$.

The desired conclusion can also be proved in a similar manner as that of Proposition 3.12.

We use d_m to denote the Euclidean metric on \mathbb{R}^m . We also use $\mathbb{R}^m \times [0, 1]$ to denote the metric space $(\mathbb{R}^m \times [0, 1], d_m)$. \mathbb{R}^1 is written as \mathbb{R} .

Remark 3.14. Here we mention that $D_{S_D} = \emptyset$ is possible when D is connected in $X \times [0, 1]$ and satisfies the assumption in Proposition 3.12. Let $X = \mathbb{R}$ and define $D \subset \mathbb{R} \times [0, 1]$ by putting

$$D_\alpha = \begin{cases} \emptyset, & \alpha = 1, \\ (0, 1 - \alpha], & \alpha \in [0, 1). \end{cases}$$

Then D is such an example. Similarly, $E_{f_E} = \emptyset$ is possible when E is connected in $X \times [0, 1]$ and satisfies the assumption in Corollary 3.13.

Let $u \in F(X)$ and $0 \leq r \leq t \leq 1$. Define

$$\text{end}_r^t u := \text{end } u \cap ([u]_r \times [r, t]).$$

For simplicity, we write $\text{end}_r^1 u$ as $\text{end}_r u$. We can see that $\text{end}_0 u = \text{end } u$.

Clearly, for $u \in F_{USCG}(X)$ and $r \in (0, 1]$, $\text{end}_r u$ is a compact set in $X \times [0, 1]$.

Corollary 3.15. Let $u \in F(X)$. (i) For r, t with $0 \leq r \leq t \leq 1$, $\text{end}_r^t u$ is connected in $X \times [0, 1]$ if and only if $[u]_r$ is connected in X . (ii) X is connected if and only if $\text{end } u$ is connected in $X \times [0, 1]$.

Proof. If $\text{end}_r^t u \neq \emptyset$, then, by Proposition 3.12, the conclusion in (i) is true. Thus it suffices to consider the case when $\text{end}_r^t u = \emptyset$. In this case, $[u]_r = \emptyset$, and so clearly the conclusion in (i) is true. Note that $f_{\text{end } u} = 0$ and $(\text{end } u)_0 = X$. So (ii) follows immediately from Proposition 3.12.

The following Examples 3.16 and 3.17 give some connected sets in $\mathbb{R} \times [0, 1]$. Proposition 3.12, Corollary 3.13 and Corollary 3.15 are used to show the connectedness of these sets.

Example 3.16. Let $D \subset \mathbb{R} \times [0, 1]$ be defined by putting

$$D_\alpha = \begin{cases} [1 - \alpha^2, 1] \cup [3, 4 - \alpha^2], & \alpha \in (0.5, 1], \\ [0, 4], & \alpha \in [0, 0.5]. \end{cases}$$

We can see that $D = A \cup B \cup C$, where

$$A = [0, 4] \times [0, 0.5],$$

$$B = \bigcup_{\alpha \in [0.5, 1]} [1 - \alpha^2, 1] \times \{\alpha\},$$

$$C = \bigcup_{\alpha \in [0.5, 1]} [3, 4 - \alpha^2] \times \{\alpha\}.$$

Clearly A is connected in $\mathbb{R} \times [0, 1]$. By Corollary 3.13, B is connected in $\mathbb{R} \times [0, 1]$. By Proposition 3.12, C is connected in $\mathbb{R} \times [0, 1]$. $A \cap B = [1 - 0.5^2, 1] \times \{0.5\} \neq \emptyset$. $A \cap C = [3, 4 - 0.5^2] \times \{0.5\} \neq \emptyset$. Thus D is connected in $\mathbb{R} \times [0, 1]$.

Here we mention that there is no $u \in F(\mathbb{R})$ satisfying $D = \text{send } u$ because $D_1 \not\subseteq D_{0.9}$.

Example 3.17. Let $u \in F_{USC}^1(\mathbb{R})$ be defined by putting:

$$[u]_\alpha = \begin{cases} [0, 1] \cup [3, 4], & \alpha \in (0.6, 1], \\ [0, 4], & \alpha \in [0, 0.6]. \end{cases}$$

We can see that $[0, 1] \cup [3, 4]$ is not connected, while $[0, 4]$ and \mathbb{R} are connected. So $u \in F_{USC}^1(\mathbb{R}) \setminus F_{USCCON}(\mathbb{R})$. By Corollary 3.15, $\text{end } u$ is connected in $\mathbb{R} \times [0, 1]$; $\text{end}_r u$ is connected in $\mathbb{R} \times [0, 1]$ if and only if $r \in [0, 0.6]$.

We say that a sequence $\{u_n, n = 1, 2, \dots\}$ in $F_{USC}(\mathbb{R}^m)$ satisfies the connectedness condition if for each $\varepsilon > 0$, there is a $\delta \in (0, \varepsilon]$ and $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N$, $\text{end}_\delta u_n$ is connected in $\mathbb{R}^m \times [0, 1]$.

We will use the following conclusion. Let (Y, ρ) be a metric space, $x, y \in Y$ and $W \subseteq Y$. Then

$$\rho(x, W) = \inf_{z \in W} \rho(x, z) \leq \inf_{z \in W} \{\rho(x, y) + \rho(y, z)\} = \rho(x, y) + \rho(y, W).$$

Theorem 3.18. Let $u \in F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_F(\mathbb{R}^m)\}$, and let $\{u_n, n = 1, 2, \dots\}$ be a fuzzy set sequence in $F_{USC}(\mathbb{R}^m)$ which satisfies the connectedness condition. Then $H_{\text{end}}(u_n, u) \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$.

Proof. From Theorem 3.1, $H_{\text{end}}(u_n, u) \rightarrow 0 \Rightarrow \lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$.

Now we show that $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u \Rightarrow H_{\text{end}}(u_n, u) \rightarrow 0$. We prove by contradiction. Assume that $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$ but $H_{\text{end}}(u_n, u) \not\rightarrow 0$. Then $H^*(\text{end } u, \text{end } u_n) \not\rightarrow 0$ or $H^*(\text{end } u_n, \text{end } u) \not\rightarrow 0$. We split the proof into two cases.

Case (i) $H^*(\text{end } u, \text{end } u_n) \not\rightarrow 0$. In this case, there is an $\varepsilon > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $H^*(\text{end } u, \text{end } u_{n_k}) > \varepsilon$. Thus for each $k = 1, 2, \dots$ there exists $(x_k, \alpha_k) \in \text{end } u$ such that $d_m((x_k, \alpha_k), \text{end } u_{n_k}) > \varepsilon$. Note that for each $k = 1, 2, \dots$, $\alpha_k > \varepsilon$. So $\{(x_k, \alpha_k), k = 1, 2, \dots\} \subseteq \text{end}_\varepsilon u$.

Since $\text{end}_\varepsilon u$ is compact, there is a subsequence $\{(x_{k_l}, \alpha_{k_l})\}$ of $\{(x_k, \alpha_k)\}$ which converges to $(x, \alpha) \in \text{end}_\varepsilon u \subset \text{end } u$. Then there is a $L(\varepsilon)$ such that

$d_m((x, \alpha), (x_{k_l}, \alpha_{k_l})) < \varepsilon/2$ for all $l \geq L$. Hence, by the triangle inequality, $d_m((x, \alpha), \text{end } u_{n_{k_l}}) - d_m((x, \alpha), (x_{k_l}, \alpha_{k_l})) > \varepsilon/2$. By Lemma 3.2 (i), this contradicts $(x, \alpha) \in \text{end } u = \liminf_{n \rightarrow \infty} \text{end } u_n$.

Case (ii) $H^*(\text{end } u_n, \text{end } u) \not\rightarrow 0$. In this case, there is an $\varepsilon > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $H^*(\text{end } u_{n_k}, \text{end } u) > \varepsilon$.

Take $y \in \mathbb{R}^m$ with $u(y) > 0$. Set $\mu = u(y)$. Since $\{u_n\}$ satisfies the connectedness condition, there is a $\xi \in (0, \min\{\mu, \varepsilon\})$ and $N_1 \in \mathbb{N}$ such that $\text{end}_\xi u_n$ is connected in $\mathbb{R}^m \times [0, 1]$ for all $n \geq N_1$.

Firstly we show the conclusions in the following clauses (I), (II) and (III): (I) For each $n_k, k = 1, 2, \dots$, there exists $(x_{n_k}, \alpha_{n_k}) \in \text{end}_\xi u_{n_k}$ with $d_m((x_{n_k}, \alpha_{n_k}), \text{end}_\xi u) > \xi$. (II) There is an $N_2 \in \mathbb{N}$ such that for each $n \geq N_2$, there exists $(y_n, \beta_n) \in \text{end}_\xi u_n$ with $d_m((y_n, \beta_n), \text{end}_\xi u) < \xi$. (III) Set $N_3 := \max\{N_1, N_2\}$. For each $n_k \geq N_3$, there exists $(z_{n_k}, \gamma_{n_k}) \in \text{end}_\xi u_{n_k}$ such that $d_m((z_{n_k}, \gamma_{n_k}), \text{end}_\xi u) = \xi$.

To show (I), let $k \in \mathbb{N}$. From the assumption, there exists $(x_{n_k}, \alpha_{n_k}) \in \text{end } u_{n_k}$ such that $d_m((x_{n_k}, \alpha_{n_k}), \text{end}_\xi u) \geq d_m((x_{n_k}, \alpha_{n_k}), \text{end } u) > \varepsilon > \xi$. Clearly $\alpha_{n_k} > \varepsilon$, and then $(x_{n_k}, \alpha_{n_k}) \in \text{end}_\varepsilon u_{n_k} \subset \text{end}_\xi u_{n_k}$. Thus (I) is true.

As $(y, \mu) \in \text{end } u$ and $\text{end } u = \liminf_{n \rightarrow \infty} \text{end } u_n$, we can find a sequence $\{(y_n, \beta_n)\}$ satisfying that $(y_n, \beta_n) \in \text{end } u_n$ for $n = 1, 2, \dots$ and $\{(y_n, \beta_n)\}$ converges to (y, μ) . Hence there is an N_2 such that for all $n \geq N_2$, $d_m((y_n, \beta_n), (y, \mu)) < \min\{\mu - \xi, \xi\}$. Let $n \in \mathbb{N}$ with $n \geq N_2$. Then $\beta_n > \xi$, and therefore $(y_n, \beta_n) \in \text{end}_\xi u_n$. Note that $(y, \mu) \in \text{end}_\xi u$. So $d_m((y_n, \beta_n), \text{end}_\xi u) \leq d_m((y_n, \beta_n), (y, \mu)) < \xi$. Thus (II) is true.

To show (III), let $k \in \mathbb{N}$ with $n_k \geq N_3$. Define a function f_k from $(\text{end}_\xi u_{n_k}, d_m)$ to \mathbb{R} as follows: $f_k(z, \zeta) = d_m((z, \zeta), \text{end}_\xi u)$ for $(z, \zeta) \in \text{end}_\xi u_{n_k}$. By the triangle inequality, $|f_k(z, \zeta) - f_k(z', \zeta')| \leq d_m((z, \zeta), (z', \zeta'))$ for $(z, \zeta), (z', \zeta') \in \text{end}_\xi u_{n_k}$. Thus f_k is a continuous function on $\text{end}_\xi u_{n_k}$. Note that $\text{end}_\xi u_{n_k}$ is a connected set in \mathbb{R}^m . Thus $f_k(\text{end}_\xi u_{n_k})$ is a connected set in \mathbb{R} ; that is, $f_k(\text{end}_\xi u_{n_k})$ is an interval. Combined this fact with the above clauses (I) and (II), we obtain that there exists $(z_{n_k}, \gamma_{n_k}) \in \text{end}_\xi u_{n_k}$ with $d_m((z_{n_k}, \gamma_{n_k}), \text{end}_\xi u) = \xi$. Thus (III) is true.

Now using (III), we can obtain a contradiction. From the construction and the compactness of $\text{end}_\xi u$, the set $\{(z_{n_k}, \gamma_{n_k}), n_k \geq N_3\}$ is bounded in $\mathbb{R}^m \times [0, 1]$, and thus $\{(z_{n_k}, \gamma_{n_k}), n_k \geq N_3\}$ has a cluster point (z, γ) . So $(z, \gamma) \in \limsup_{n \rightarrow \infty} \text{end } u_n = \text{end } u$. As $(z_{n_k}, \gamma_{n_k}) \in \text{end}_\xi u_{n_k}$, we have that $\gamma_{n_k} \geq \xi$ and therefore $\gamma \geq \xi$. Thus $(z, \gamma) \in \text{end } u \cap (X \times [\xi, 1]) = \text{end}_\xi u$. But, by construction, $d_m((z, \gamma), \text{end}_\xi u) = \xi$, which is a contradiction.

Corollary 3.19. Let u be a fuzzy set in $F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_F(\mathbb{R}^m)\}$ and for $n = 1, 2, \dots$, let u_n be a fuzzy set in $F_{USCCON}(\mathbb{R}^m)$. Then $H_{\text{end}}(u_n, u) \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$.

Proof. Let $\{u_n : n \in \mathbb{N}\}$ be a sequence in $F_{USCCON}(\mathbb{R}^m)$. Then, by clause (i) of Corollary 3.15, for each $n \in \mathbb{N}$ and $r \in (0, 1]$, $\text{end}_r u_n$ is a connected set in $\mathbb{R}^m \times [0, 1]$. Hence $\{u_n : n \in \mathbb{N}\}$ satisfies the connectedness condition. Thus the desired result follows immediately from Theorem 3.18.

Here we mention that for $u \in F_{USC}^1(\mathbb{R})$ which satisfies the connectedness condition, $H(u_n, u) \rightarrow 0$ does not imply that there is an N such that for each $n \geq N$, $u_n \in F_{USCCON}(\mathbb{R})$.

The following Example 3.20 is such an example, which shows that there exists a $u \in F_{USC}^1(\mathbb{R})$ and a sequence $\{u_n\}$ in $F_{USC}^1(\mathbb{R})$ such that: (i) $H(u_n, u) \rightarrow 0$, (ii) $\{u_n\}$ satisfies the connectedness condition, (iii) for each $n = 1, 2, \dots$, $u_n \notin F_{USCCON}(\mathbb{R})$.

Example 3.20. For $n = 1, 2, \dots$, let u_n be the fuzzy set u given in Example 3.17; that is, $u_n = u$ is a fuzzy set in $F_{USC}^1(\mathbb{R})$ defined by putting:

$$[u]_\alpha = \begin{cases} [0, 1] \cup [3, 4], & \alpha \in (0.6, 1], \\ [0, 4], & \alpha \in [0, 0.6]. \end{cases}$$

We have the following conclusions: (i) $H(u_n, u) \rightarrow 0$ since $u_n = u$ for $n = 1, 2, \dots$; (ii) $\{u_n\}$ satisfies the connectedness condition because $\text{end}_r u$ is connected in $\mathbb{R} \times [0, 1]$ when $r \in [0, 0.6]$ (see Example 3.17); (iii) for each $n = 1, 2, \dots$, $u_n \notin F_{USCCON}(\mathbb{R})$, as $u \notin F_{USCCON}(\mathbb{R})$ (see Example 3.17).

Remark 3.21. Theorem 9.2 in [?], which is Theorem 3.10 in this paper, is a corollary of Corollary 3.19 since $F_{USCGCON}(\mathbb{R}^m) \subseteq F_{USCCON}(\mathbb{R}^m)$. Theorem 3.7 is a corollary of Corollary 3.19, as $F_{USCG}^1(\mathbb{R}^m) \subseteq F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_F(\mathbb{R}^m)\}$ and $F_{USCCON}^1(\mathbb{R}^m) \subseteq F_{USCCON}(\mathbb{R}^m)$. Corollary 3.19 is a corollary of Theorem 3.18. Theorem 3.18 improves Theorem 9.2 in [?] and Theorem 3.7.

Now we give an improvement of Theorem 3.18.

Let $u \in F(X)$. Define $S(u) := \sup\{u(x) : x \in X\}$. Clearly $[u]_{S(u)} = \emptyset$ is possible.

Let $\{u_n\}$ be a fuzzy set sequence in $F_{USC}(\mathbb{R}^m)$ and $\{u_{n_k}\}$ a subsequence of $\{u_n\}$. If there is a $u \in F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_F(\mathbb{R}^m)\}$ such that $u = \lim_{n \rightarrow \infty}^{(\Gamma)} u_n$, then we can define

$$\begin{aligned} A_1 &:= \{\xi > 0 : \text{for each } n_k, H^*(\text{end } u, \text{end } u_{n_k}) > \xi\}, \\ A_2 &:= \{\xi > 0 : \text{for each } n_k, H^*(\text{end } u_{n_k}, \text{end } u) > \xi\}. \end{aligned}$$

Let $\{u_n\}$ be a fuzzy set sequence in $F_{USC}(\mathbb{R}^m)$ and let u be a fuzzy set in $F_{USC}(\mathbb{R}^m) \setminus \{\emptyset_F(\mathbb{R}^m)\}$. We call the pair $\{u_n\}, u$ a weak connectedness compact pair if one of the following (i) and (ii) holds: (i) $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n$ does not exist; (ii) $u = \lim_{n \rightarrow \infty}^{(\Gamma)} u_n$ and both (ii-1) and (ii-2) are true: (ii-1) for each $\{u_{n_k}\}$ with $A_1 \neq \emptyset$, there exists a $\xi \in A_1$ such that $\text{end}_\xi u$ is compact; (ii-2) for each $\{u_{n_k}\}$

with $A_2 \neq \emptyset$, there exists a $\xi \in A_2$ with $\xi < S(u)$, $\text{end}_\xi u$ is compact, and there is an $N(\xi) \in \mathbb{N}$ such that $\text{end}_\xi u_{n_k}$ is connected in $\mathbb{R}^m \times [0, 1]$ for all $n_k \geq N$.

Theorem 3.22. Let $u \in F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_F(\mathbb{R}^m)\}$, and let $\{u_n, n = 1, 2, \dots\}$ be a fuzzy set sequence in $F_{USC}(\mathbb{R}^m)$. If the pair $\{u_n\}, u$ is a weak connectedness compact pair, then $H_{\text{end}}(u_n, u) \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$.

Proof. The proof is similar to that of Theorem 3.18.

From Theorem 3.1, $H_{\text{end}}(u_n, u) \rightarrow 0 \Rightarrow \lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$.

Now we show that $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u \Rightarrow H_{\text{end}}(u_n, u) \rightarrow 0$. We prove by contradiction. Assume that $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$ but $H_{\text{end}}(u_n, u) \not\rightarrow 0$. Then $H^*(\text{end } u, \text{end } u_n) \not\rightarrow 0$ or $H^*(\text{end } u_n, \text{end } u) \not\rightarrow 0$. We split the proof into two cases.

Case (i) $H^*(\text{end } u, \text{end } u_n) \not\rightarrow 0$. In this case, $A_1 \neq \emptyset$. Since the pair $\{u_n\}, u$ is a weak connectedness compact pair, there is a $\xi \in A_1$ with $\text{end}_\xi u$ compact. So we can prove that there is a contradiction in a similar manner to that in case (i) of the proof of Theorem 3.18.

Case (ii) $H^*(\text{end } u_n, \text{end } u) \not\rightarrow 0$. In this case, $A_2 \neq \emptyset$. Since the pair $\{u_n\}, u$ is a weak connectedness compact pair, there is a $\xi \in A_2$ and $N_1 \in \mathbb{N}$ such that $\xi < S(u)$, $\text{end}_\xi u$ is compact, and $\text{end}_\xi u_{n_k}$ is connected in $\mathbb{R}^m \times [0, 1]$ for all $n_k \geq N_1$.

Firstly we show the conclusions in the following clauses (I), (II) and (III): (I) For each $n_k, k = 1, 2, \dots$, there exists $(x_{n_k}, \alpha_{n_k}) \in \text{end}_\xi u_{n_k}$ with $d_m((x_{n_k}, \alpha_{n_k}), \text{end}_\xi u) > \xi$. (II) There is an $N_2 \in \mathbb{N}$ such that for each $n \geq N_2$, there exists $(y_n, \beta_n) \in \text{end}_\xi u_n$ with $d_m((y_n, \beta_n), \text{end}_\xi u) < \xi$. (III) Set $N_3 := \max\{N_1, N_2\}$. For each $n_k \geq N_3$, there exists $(z_{n_k}, \gamma_{n_k}) \in \text{end}_\xi u_{n_k}$ such that $d_m((z_{n_k}, \gamma_{n_k}), \text{end}_\xi u) = \xi$.

Note that $\xi \in A_2$; that is, for each $u_{n_k}, k = 1, 2, \dots$, $H^*(\text{end } u_{n_k}, \text{end } u) > \xi$. Let $k \in \mathbb{N}$. From this, there exists $(x_{n_k}, \alpha_{n_k}) \in \text{end } u_{n_k}$ such that $d_m((x_{n_k}, \alpha_{n_k}), \text{end}_\xi u) \geq d_m((x_{n_k}, \alpha_{n_k}), \text{end } u) > \xi$. Clearly $\alpha_{n_k} > \xi$, and then $(x_{n_k}, \alpha_{n_k}) \in \text{end}_\xi u_{n_k}$. Thus (I) is true.

Since $\xi < S(u)$, we can take $y \in \mathbb{R}^m$ with $u(y) > \xi > 0$. Set $u(y) = \mu$. As $(y, \mu) \in \text{end } u$ and $\text{end } u = \liminf_{n \rightarrow \infty} \text{end } u_n$, we can find a sequence $\{(y_n, \beta_n)\}$ satisfying that $(y_n, \beta_n) \in \text{end } u_n$ for $n = 1, 2, \dots$ and $\{(y_n, \beta_n)\}$ converges to (y, μ) . Hence there is an N_2 such that for all $n \geq N_2$, $d_m((y_n, \beta_n), (y, \mu)) < \min\{\mu - \xi, \xi\}$. Let $n \in \mathbb{N}$ with $n \geq N_2$. Then $\beta_n > \xi$, and therefore $(y_n, \beta_n) \in \text{end}_\xi u_n$. Note that $(y, \mu) \in \text{end}_\xi u$. So $d_m((y_n, \beta_n), \text{end}_\xi u) \leq d_m((y_n, \beta_n), (y, \mu)) < \xi$. Thus (II) is true. (III) follows from (I), (II) and the connectedness of $\text{end}_\xi u_{n_k}$ when $n_k \geq N_3$. The proof of (III) is the same as that of clause (III) in the proof of Theorem 3.18.

Now using (III) and the compactness of $\text{end}_\xi u$, we can have a contradiction. The proof is the same as the counterpart in the proof of Theorem 3.18.

Example 3.23. Let u be a fuzzy set in $F_{USC}^1(\mathbb{R}) \setminus \{\emptyset_F(\mathbb{R}^m)\}$ defined by

$$[u]_\alpha = \begin{cases} \{1\}, & \alpha \in (0.6, 1], \\ (-\infty, -1] \cup [1, +\infty), & \alpha \in [0, 0.6]. \end{cases}$$

For $n = 1, 2, \dots$, let $u_n = u$. We have the following conclusions: (i) $H(u_n, u) \rightarrow 0$ since $u_n = u$ for $n = 1, 2, \dots$; (ii) the pair $\{u_n\}, u$ is a weak connectedness compact pair; (iii) $\{u_n\}$ does not satisfy the connectedness condition because $\text{end}_r u$ is not connected in $\mathbb{R} \times [0, 1]$ when $r \in [0, 0.6]$. Clearly each subsequence $\{u_{n_k}\}$ of $\{u_n\}$ does not satisfy the connectedness condition; (iv) $u \notin F_{USCG}(\mathbb{R})$.

Remark 3.24. Let $\{u_n\}$ be a fuzzy set sequence in $F_{USC}(\mathbb{R}^m)$ satisfying the connectedness condition and let u be a fuzzy set in $F_{USCG}(\mathbb{R}^m) \setminus \{\emptyset_F(\mathbb{R}^m)\}$. Then clearly the pair $\{u_n\}, u$ is a weak connectedness compact pair. So Theorem 3.18 is a corollary of Theorem 3.22. Theorem 3.22 is an improvement of Theorem 3.18.

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Note: Figure translations are in progress. See original paper for figures.

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