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On the size of graphs without repeated cycle lengths (postprint)

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Abstract

In 1975, P. Erd {o}s proposed the problem of determining the maximum number $f(n)$ of edges in a graph with n vertices in which any two cycles are of different lengths. In this paper, it is proved that

$$f(n)geq\ n + frac1073t + frac73$$

for $t = 1260r + 169$, $(rgeq\ 1)$ and $n\ geq\ frac21194t^2 + 87978t + frac159574$. Consequently,$ liminf sb {n to infty} {f(n)-n over sqrt n} geq sqrt {2 + frac{7654}{19071}},$ which is better than the previous bounds $ sqrt 2$ Y. Shi, Discrete Math. 71(1988), 57-71 , $ sqrt {2.4}$ C. Lai, Australas. J. Combin. 27(2003), 101-105 . The conjecture $ lim_{n rightarrow infty} {f(n)-n over sqrt n}= sqrt {2.4}$ is not true.

Full Text

Preamble

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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph with n vertices in which any two cycles have different lengths. In this paper, we prove that for $t = 1260r + 169$ ($r \geq 1$) and $n \geq 2119$, $f(n) \geq n + \frac{4t^2+87978t+15957}{19071}$. Consequently, $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{\frac{4}{19071}}$, which improves upon the previous bounds of 2 (see Shi, Discrete Math. 71(1988), 57-71) and $2 + \frac{2}{5}$ (see Lai, Australas. J. Combin. 27(2003), 101-105). This result disproves Conjecture 4 of Lai.

Key words: Graph, cycle, number of edges.
AMS 2000 MSC: 05C38, 05C35.

Introduction

Let $f(n)$ denote the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdős posed the problem of determining $f(n)$ (see Bondy and Murty [1], p.247, Problem 11). Shi [15] established the following lower bound:

Theorem 1 (Shi [15]). $f(n) \geq n + \lfloor (\sqrt{8n-23}+1)/2 \rfloor$ for $n \geq 3$.

Additional related results were obtained by Chen, Lehel, Jacobson, and Shreve [3], Jia [5], Lai [6-8], and Shi [16,18-20]. Boros, Caro, Füredi, and Yuster [2] proved an upper bound:

Theorem 2 (Boros, Caro, Füredi and Yuster [2]). For n sufficiently large, $f(n) < n + 1.98\sqrt{n}$.

Lai [9] improved the lower bound:

Theorem 3 (Lai [9]). $f(n) \geq n + \sqrt{2n(1-o(1))}$.

Lai [6,9] proposed the following conjectures:

Conjecture 4 (Lai [9]). $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = 2$.

Conjecture 5 (Lai [6]). $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq 2 + \frac{2}{5}$.

Markström [13] raised the following problem:

Problem 6 (Markström [13]). Determine the maximum number of edges in a Hamiltonian graph on n vertices with no repeated cycle lengths.

Results for the maximum number of edges in a 2-connected graph on n vertices with distinct cycle lengths can be found in [2,3,15]. Survey articles on this problem appear in Tian [21], Zhang [24], and Lai and Liu [10]. The progress on all 50 problems from [1] can be found in Locke [12].

A related topic concerns the Entringer problem, which asks which simple graphs have exactly one cycle of each length ℓ for $3 \leq \ell \leq v$ (see Problem 10 in [1]). This problem was posed in 1973 by R.C. Entringer. For developments on this topic, see [4,11,13,14,17,22,23].

In this paper, we construct a graph G with no two cycles of the same length, yielding the following result:

Theorem 7. Let $t = 1260r + 169$ $(r \geq 1)$. Then $f(n) \geq n + \frac{4t^2+87978t+15957}{19071}$ for $n \geq 2119$. Consequently, Conjecture 4 is false.

Proof of Theorem 7

Proof. Let $t = 1260r + 169$ with $r \geq 1$, and let $n_t = 2119$. We shall show that for $n \geq n_t$, there exists a graph G on n vertices with $n + \frac{4t^2+87978t+15957}{19071}$ edges such that all cycles in G have distinct lengths.

We construct the graph G from several subgraphs B_i defined for indices i belonging to the following sets: - $0 \leq i \leq 20t$ - $27t \leq i \leq 28t + 64$ - $29t - 734 \leq i \leq 29t + 267$ - $30t - 531 \leq i \leq 30t + 57$ - $31t - 741 \leq i \leq 31t + 58$ - $32t - 740 \leq i \leq 32t + 57$ - $33t - 741 \leq i \leq 33t + 57$ - $34t - 741 \leq i \leq 34t + 52$ - $35t - 746 \leq i \leq 35t + 60$ - $36t - 738 \leq i \leq 36t + 60$ - $37t - 738 \leq i \leq 37t + 799$ - $i = 20t + j$ for $1 \leq j \leq \frac{t-7}{6}$ - $i = 20t + \frac{t-1}{6}$ - $i = 20t + \frac{t-1}{3}$ - $i = 20t + \frac{t-1}{2}$ - $i = 20t + \frac{2t-2}{3}$ - $i = 21t - 2$ - $i = 21t - 1$ - $i = 21t + 2j + 1$ for $0 \leq j \leq t - 1$ - $i = 21t + 2j$ for $0 \leq j \leq \frac{t-3}{2}$ - $i = 23t + 2j + 1$ for $0 \leq j \leq \frac{t-3}{2}$ - $i = 26t$

All these subgraphs share a single common vertex x, and their vertex sets are otherwise pairwise disjoint.

Construction of the subgraphs:

For $1 \leq i \leq \frac{t-7}{6}$, let B_{20t+i} consist of a cycle $C_{20t+i} = xa^i_1a^i_2 \cdots a^i_{\frac{62t-8}{3}+2i}$ and a path $a^i_1a^i_2 \cdots a^i_{\frac{59t-5}{3}}a^i_{\frac{61t-1}{3}}$. By construction, B_{20t+i} contains exactly three cycles of lengths $20t + i$, $20t + \frac{t-1}{6} + i - 1$, and $20t + \frac{2t-2}{3} + 2i - 1$.

For $1 \leq i \leq \frac{t-7}{6}$, let $B_{20t+\frac{t-1}{6}+i}$ consist of a cycle $C_{20t+\frac{t-1}{6}+i} = xb^i_1b^i_2 \cdots b^i_{\frac{62t-5}{3}+2i}$ and a path $b^i_1b^i_2 \cdots b^i_{10t-1}b^i_{\frac{61t-1}{3}}$. This subgraph contains exactly three cycles of lengths $20t + \frac{t-1}{6} + i$, $20t + \frac{t-1}{3} + i$, and $20t + \frac{2t-2}{3} + 2i$.

For $0 \leq i \leq t - 1$, let $B_{21t+2i+1}$ consist of a cycle $C_{21t+2i+1} = xu^i_1u^i_2 \cdots u^i_{25t+2i-1}$ and a path $u^i_1u^i_2 \cdots u^i_{\frac{19t+2i-1}{2}}u^i_{\frac{23t+2i+1}{2}}$. This subgraph contains exactly three cycles of lengths $21t + 2i + 1$, $23t + 2i$, and $25t + 2i$.

For $0 \leq i \leq \frac{t-3}{2}$, let B_{21t+2i} consist of a cycle $C_{21t+2i} = xv^i_1v^i_2 \cdots v^i_{25t+2i}$ and a path $v^i_1v^i_2 \cdots v^i_{9t+i-1}v^i_{12t+i}$. This subgraph contains exactly three cycles of lengths $21t + 2i$, $22t + 2i + 1$, and $25t + 2i + 1$.

For $i = \frac{t-1}{2}$, B_{21t+2i} is simply a cycle of length $22t - 1$.

For $0 \leq i \leq \frac{t-3}{2}$, let $B_{23t+2i+1}$ consist of a cycle $C_{23t+2i+1} = xw^i_1w^i_2 \cdots w^i_{26t+2i+1}$ and a path $w^i_1w^i_2 \cdots w^i_{\frac{21t+2i-1}{2}}w^i_{\frac{25t+2i+1}{2}}$. This subgraph contains exactly three cycles of lengths $23t + 2i + 1$, $24t + 2i + 2$, and $26t + 2i + 2$.

For $i = \frac{t-1}{2}$, $B_{23t+2i+1}$ is simply a cycle of length $24t$.

For $58 \leq i \leq t - 742$, let $B_{27t+i-57}$ consist of a cycle $C_{27t+i-57} = xy^i_1y^i_2 \cdots y^i_{132t+11i+893}$ and ten paths sharing the common vertex x, with their other endpoints on the cycle $C_{27t+i-57}$: - $y^i_1y^i_2 \cdots y^i_{\frac{17t-1}{2}}$ - $y^i_2y^i_3 \cdots y^i_{\frac{19t-1}{2}}$ - $y^i_3y^i_4 \cdots y^i_{\frac{19t-1}{2}}$ - $y^i_4y^i_5 \cdots y^i_{\frac{21t-1}{2}}$ - $y^i_5y^i_6 \cdots y^i_{\frac{21t-1}{2}}$ - $y^i_6y^i_7 \cdots y^i_{\frac{23t-1}{2}}$ - $y^i_7y^i_8 \cdots y^i_{\frac{23t-1}{2}}$ -

$y_8^i y_9^i \cdots y_{\frac{25t-1}{2}}^i$ - $y_9^i y_{10}^i \cdots y_{\frac{25t-1}{2}}^i$ - $y_{10}^i y_{11}^i \cdots y_{\frac{27t-1}{2}}^i y_{\frac{37t-115}{2}+i} y_{\frac{57t-103}{2}+2i} y_{\frac{77t+315}{2}+3i} y_{\frac{97t+313}{2}+4i} y_{\frac{117t+313}{2}+5i} y_{\frac{137t+311}{2}+6i} y_{\frac{157t+}{2}}$

A cycle with d chords contains exactly $\binom{d+2}{2}$ distinct cycles. Therefore, $B_{27t+i-57}$ contains cycles of the following 66 lengths: $27t+i-57$, $28t+i+7$, $29t+i+210$, $30t+i$, $31t+i+1$, $32t+i$, $33t+i$, $34t+i-5$, $35t+i+3$, $36t+i+3$, $37t+i+742$, $38t+2i-51$, $38t+2i+216$, $40t+2i+209$, $40t+2i$, $42t+2i$, $42t+2i-1$, $44t+2i-6$, $44t+2i-3$, $46t+2i+5$, $46t+2i+744$, $48t+3i+158$, $49t+3i+215$, $50t+3i+209$, $51t+3i-1$, $52t+3i-1$, $53t+3i-7$, $54t+3i-4$, $55t+3i-1$, $56t+3i+746$, $59t+4i+157$, $59t+4i+215$, $61t+4i+208$, $61t+4i-2$, $63t+4i-7$, $63t+4i-5$, $65t+4i-2$, $65t+4i+740$, $69t+5i+157$, $70t+5i+214$, $71t+5i+207$, $72t+5i-8$, $73t+5i-5$, $74t+5i-3$, $75t+5i+739$, $80t+6i+156$, $80t+6i+213$, $82t+6i+201$, $82t+6i-6$, $84t+6i-3$, $84t+6i+738$, $90t+7i+155$, $91t+7i+207$, $92t+7i+203$, $93t+7i-4$, $94t+7i+738$, $101t+8i+149$, $101t+8i+209$, $103t+8i+205$, $103t+8i+737$, $111t+9i+151$, $112t+9i+211$, $113t+9i+946$, $122t+10i+153$, $122t+10i+952$, and $132t+11i+894$.

Finally, B_0 is a path with one endpoint at x and length $n-n_t$, while each remaining B_i is simply a cycle of length i.

This construction yields $f(n) \geq n + \frac{107}{3}$ for $n \geq n_t$, completing the proof.

From Theorem 7, we obtain $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{\frac{4}{19071}}$, which improves upon the previous bounds of 2 (see [15]) and $2+\frac{2}{5}$ (see [9]). Therefore, Conjecture 4 is false. Combining this with the upper bound of Boros, Caro, Füredi, and Yuster (Theorem 2), we have:

$$1.98 \geq \limsup_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{\frac{4}{19071}}.$$

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