Non-uniform Berry-Esseen Estimates for Supercritical Branching Processes in Random Environments

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Abstract

Let Z_n be a supercritical branching process in an i.i.d. environment. In this paper, we obtain two non-uniform Berry-Esseen bounds for Z_n. This result extends the Berry-Esseen bound of Grama et al. [Stochastic,Process.,Appl.,127(4),1255-1281,2017] to the non-uniform case. Finally, we discuss applications of these results to interval estimation.

Full Text

Abstract

Let (Z_n) be a supercritical branching process in an independent and identically distributed random environment. We establish nonuniform Berry-Esseen bounds for the process (Z_n), which refine the Berry-Esseen bound of Grama et al. [Stochastic Process. Appl., 127(4), 1255-1281, 2017]. We also discuss an application of our result to constructing confidence intervals for the criticality parameter.

Keywords: Branching processes, Random environment, Nonuniform Berry-Esseen bounds

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1. Introduction

Branching processes in random environments have been extensively studied since the pioneering work of Smith and Wilkinson [1]. Consider a branching process $(Z_n)_{n\geq 0}$ in an i.i.d. random environment $\xi = (\xi_0, \xi_1, ...)$. The process evolves such that, given the environment ξ_n, the offspring distribution at generation n is $p(\xi_n) = \{p_k(\xi_n) = P(X_{n,i} = k \mid \xi_n) : k \in \mathbb{N}\}$, where $(X_{n,i})_{i\geq 1}$ are i.i.d. random variables representing the number of offspring of the i-th individual in generation n.

Let $m_n = E_\xi X_{n,i}$ denote the conditional mean offspring number in generation n, and define $\Pi_n = \prod_{i=0}^{n-1} m_i$ with $\Pi_0 = 1$. The process $W_n = Z_n/\Pi_n$ forms a nonnegative martingale with respect to the filtration $\mathcal{F}_n = \sigma\{\xi, X_{k,i}, 0 \leq k \leq n-1, i \geq 1\}$. Under suitable conditions, W_n converges almost surely to a limit W with $E[W] \leq 1$.

The asymptotic behavior of Z_n is governed by the random walk $S_n = \log \Pi_n = \sum_{i=0}^{n-1} X_i$, where $X_i = \log m_i$. Let $\mu = E[X]$ and $\sigma^2 = \mathrm{Var}(X)$. The process is classified as subcritical ($\mu < 0$), critical ($\mu = 0$), or supercritical ($\mu > 0$). In the supercritical case, $\mu > 0$ and the process grows exponentially at rate μ.

Previous work has established central limit theorems and Berry-Esseen bounds for $\log Z_n$. Grama et al. [9] proved a uniform Berry-Esseen bound:

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq x\right) - \Phi(x)\right| \leq \frac{C}{\sqrt{n}},$$

under moment conditions $E[X^{2+\delta}] < \infty$ and $E[Z_1^p] < \infty$ for some $p > 1$ and $\delta \in (0, 1]$. However, uniform bounds do not capture the precise rate of convergence in the tails.

Nonuniform Berry-Esseen bounds provide refined estimates that depend on x, typically of the form:

$$\left| P\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq x\right) - \Phi(x)\right| \leq \frac{C}{(1 + |x|^{1+\delta'})},$$

for $\delta' \in (0, \delta)$. Such bounds are crucial for constructing confidence intervals and understanding moderate deviations.

2. Model and Main Results

2.1 Branching Process in Random Environment

Let $\xi = (\xi_0, \xi_1, ...)$ be an i.i.d. sequence of random variables representing the environment. The branching process $(Z_n)_{n\geq 0}$ is defined recursively by:

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}, \quad n \geq 0,$$

where, conditioned on ξ, the random variables $(X_{n,i})_{i\geq 1}$ are i.i.d. with distribution $p(\xi_n)$. The conditional mean offspring number is $m_n = E_\xi X_{n,i}$, and we define:

$$\Pi_0 = 1, \quad \Pi_n = \prod_{i=0}^{n-1} m_i, \quad n \geq 1.$$

The normalized process $W_n = Z_n/\Pi_n$ is a nonnegative martingale converging a.s. to W with $E[W] \leq 1$.

Let $X_i = \log m_i$ and define the random walk:

$$S_0 = 0, \quad S_n = \log \Pi_n = \sum_{i=0}^{n-1} X_i, \quad n \geq 1.$$

Assume $\mu = E[X] \in (0, \infty)$ and $\sigma^2 = \mathrm{Var}(X) \in (0, \infty)$. The criticality parameter μ determines the exponential growth rate of Z_n.

We impose the following moment conditions:

- **(A1)** There exists $\delta \in (0, 1]$ such that $E[|X|^{2+\delta}] < \infty$.
- **(A2)** There exists $p > 1$ such that $E[Z_1^p] < \infty$.
- **(A3)** There exists $\lambda_0 > 0$ such that $E[e^{\lambda_0 X}] = E[m_0^{\lambda_0}] < \infty$ (Cramér' s condition).
- **(A4)** There exists $p > 1$ such that $E[Z_1^p] < \infty$.

2.2 Main Results

Our first result establishes a nonuniform Berry-Esseen bound under conditions (A1) and (A2).

Theorem 1. Assume (A1) and (A2) hold. Then for any $\delta' \in (0, \delta)$, there exists a constant $C > 0$ such that for all $x \in \mathbb{R}$ and $n \geq 1$,

$$\left| P\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{1 + |x|^{1+\delta'}}. \tag{2.3}$$

This refines the uniform bound of Grama et al. [9] by providing x-dependent convergence rates. The bound (2.3) is particularly useful for moderate deviations where $|x| = o(\sqrt{n})$.

Under stronger exponential moment conditions, we obtain an even sharper bound:

Theorem 2. Assume (A3) and (A4) hold. Then there exist constants $C, c > 0$ such that for all $x \in \mathbb{R}$ and $n \geq 1$,

$$\left| P\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \leq x \right) - \Phi(x) \right| \leq C(1 + x^2) \exp\left(-\frac{cx^2}{1 + |x|/\sqrt{n}} \right). \tag{2.4}$$

This bound provides exponential decay in the tails and is valid for $|x| = o(\sqrt{n})$. The result improves upon Theorem 1.1 in Grama et al. [9] by giving an explicit nonuniform estimate.

Theorem 3 (Confidence Interval). Under (A1) and (A2), let $\kappa_n \in (0,1)$ satisfy $|\log \kappa_n| = o(\log n)$. Then for the interval:

$$\left[\frac{\log Z_n}{\sqrt{n}} - \sigma\Phi^{-1}(1-\kappa_n/2), \frac{\log Z_n}{\sqrt{n}} + \sigma\Phi^{-1}(1-\kappa_n/2)\right],$$

we have:

$$P\left(\mu \in \left[\frac{\log Z_n}{n} - \frac{\sigma}{\sqrt{n}}\Phi^{-1}(1-\kappa_n), \frac{\log Z_n}{n} + \frac{\sigma}{\sqrt{n}}\Phi^{-1}(1-\kappa_n)\right]\right) = 1-\kappa_n+o(1).$$

This provides a practical method for constructing asymptotic confidence intervals for the criticality parameter μ.

3. Preliminary Lemmas

We establish several technical lemmas concerning the martingale limit W and its logarithm.

Lemma 2. Under (A1) and (A2), for any $q \in (1, 1+\delta)$, we have $E[|\log W|^q] < \infty$ and $\sup_n E[|\log W_n|^q] < \infty$.

Proof. Using the decomposition $\log Z_n = S_n + \log W_n$ from (2.1) and moment estimates for W_n, we obtain the uniform bound:

$$E[|\log W_n|^q] \leq CE[W] + E[|\log W|^q \mathbf{1}_{\{W\leq 1\}}] < \infty.$$

Lemma 3. Under (A1) and (A2), there exists $\gamma \in (0,1)$ such that:

$$E[|\log W_n - \log W|] \leq C\gamma^n.$$

This exponential convergence is crucial for controlling the remainder term in the decomposition of $\log Z_n$.

Lemma 4. Under (A1) and (A2), for any $x \in \mathbb{R}$ and $\delta' \in (0, \delta)$:

$$\left|P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right) - \Phi(x)\right| \leq \frac{C}{1+|x|^{1+\delta'}}. \tag{4.18}$$

The proof follows from classical nonuniform Berry-Esseen bounds for i.i.d. sums (see Bikelis [11] and Chen & Shao [12]).

Lemma 5. Under (A1) and (A2), for any $x > 0$:

$$P\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \geq x\right) \leq C\exp\left(-\frac{cx^2}{1+|x|/\sqrt{n}}\right). \tag{4.19}$$

4. Proof of Main Results

The key identity is the decomposition:

$$\log Z_n = S_n + \log W_n, \tag{2.1}$$

where $S_n = \sum_{i=0}^{n-1} X_i$ is the random walk of environmental means and $W_n = Z_n/\Pi_n$ is the normalized population size.

4.1 Proof of Theorem 1

We analyze the probability:

$$P\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x\right) = P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} + \frac{\log W_n}{\sigma\sqrt{n}} \le x\right).$$

Let $m = \lfloor n/2 \rfloor$ and define $V_n = \log W_n$. Then:

$$P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x, \frac{\log Z_n - n\mu}{\sigma\sqrt{n}} > x\right) \le P(|V_n - V_m| > \alpha_n) + P\left(Y_n \le x + \alpha_n, Y_n > x\right),$$

where $Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$ and $\alpha_n = n^{-1/2}$.

Using Lemma 3, we bound:

$$P(|V_n - V_m| > \alpha_n) \le \frac{E|V_n - V_m|}{\alpha_n} \le C\gamma^m\sqrt{n} \le \frac{C}{1 + |x|^{1+\delta'}}.$$

The main term is controlled by Lemma 4, yielding:

$$\left|P\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x\right) - \Phi(x)\right| \le \frac{C}{1 + |x|^{1+\delta'}}.$$

4.2 Proof of Theorem 2

Under the Cramér condition (A3), we apply Nagaev' s inequality to obtain exponential tail bounds for S_n. Combining with the decomposition (2.1) and Lemma 5, we derive for $|x| \le n^{1/4}$:

$$\left|P\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \le x\right) - \Phi(x)\right| \le C(1 + x^2)\exp\left(-\frac{cx^2}{1 + |x|/\sqrt{n}}\right).$$

For $|x| > n^{1/4}$, the bound follows from Lemma 5 and standard Gaussian tail estimates:

$$1 - \Phi(x) \le \frac{e^{-x^2/2}}{\sqrt{2\pi}(1 + x)}.$$

4.3 Proof of Theorem 3

The confidence interval construction uses the quantile function $\Phi^{-1}(p)$. For $p \to 0$, we have the asymptotic expansion:

$$\Phi^{-1}(1-p) = -\sqrt{2\log(1/p)} + O(\log\log(1/p)).$$

Applying Theorem 1 with $\kappa_n \to 0$ slowly enough that $|\log \kappa_n| = o(\log n)$, we obtain:

$$P\left(\frac{\log Z_n - n\mu}{\sigma\sqrt{n}} \in \left[-\Phi^{-1}(1-\kappa_n), \Phi^{-1}(1-\kappa_n)\right]\right) = 1 - \kappa_n + o(1).$$

This yields the claimed confidence interval for μ.

References

1. Smith, W. L. and Wilkinson, W. E. (1969). On branching processes in random environments. *Ann. Math. Stat.* 40(3), 814–827.
2. Böinghoff, C. (2014). Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions. *Stochastic Process. Appl.* 124(11), 3553–3577.
3. Li, Y., Liu, Q., Gao, Z.Q. and Wang, H. (2014). Asymptotic properties of supercritical branching processes in random environments. *Front. Math. China* 9, 737–751.
4. Wang, Y. and Liu, Q. (2017). Limit theorems for a supercritical branching process with immigration in a random environment. *Sci. China Math.* 60(12), 2481–2502.
5. Fan, X., Hu, H. and Liu, Q. (2020). Uniform Cramér moderate deviations and Berry-Esseen bounds for a supercritical branching process in a random environment. *Front. Math. China* 15(5), 891–914.
6. Böinghoff, C and Kersting, C. (2010). Upper large deviations of branching processes in a random environment for offspring distributions with geometrically bounded tails. *Stochastic Process. Appl.* 120(10), 2064–2077.
7. Huang, C. and Liu, Q. (2012). Moments, moderate and large deviations for a branching process in a random environment. *Stochastic Process. Appl.* 122(2), 522–545.
8. Xu, H. (2021). Deviation inequalities for a supercritical branching process in a random environment. *J. Math. Research Appl.*, to appear. arXiv: 2109.03489.
9. Grama, I., Liu, Q. and Miqueu, E. (2017). Berry-Esseen bound and Cramér large deviation expansion for a supercritical branching process in a random environment. *Stochastic Process. Appl.* 127(4), 1255–1281.
10. Gao, Z.Q. (2021). Exact convergence rate in the central limit theorem for a branching process in a random environment. *Stat. Probab. Letters* 178: 109194.

11. Bikelis, A. (1966). On estimates of the remainder term in the central limit theorem. *Lith. Math. J.* 6(3), 323-346.
12. Chen, L.H.Y. and Shao, Q.M. (2001). A non-uniform Berry-Esseen bound via Stein' s method. *Probab. Theory Relat. Fields* 120(2): 236-254.
13. Tanny, D. (1988). A necessary and sufficient condition for a branching process in a random environment to grow like the product of its means. *Stochastic Process. Appl.* 28(1), 123-139.
14. Fan, X., Grama, I. and Liu. Q. (2017). Nonuniform Berry-Esseen bounds for martingales with applications to statistical estimation. *Statistics* 51(1), 105-122.
15. Grama, I., Liu, Q. and Miqueu, E. (2021). Asymptotic of the distribution and harmonic moments for a supercritical branching process in a random environment. (hal-03416307).
16. Fan, X., Grama, I. and Liu, Q. (2017). Deviation inequalities for martingales with applications. *J. Math. Anal. Appl.* 448(1), 538-566.
17. Liu, Q. (1999). Asymptotic properties of supercritical age-dependent branching processes and homogeneous branching random walks. *Stochastic Process. Appl.* 82(1), 61-87.
18. Liu, Q. (2001). Local dimensions of the branching measure on a Galton-Watson tree. *Ann. Inst. Henri Poincaré Probab. Stat.* 37(2), 195-222.

Note: Figure translations are in progress. See original paper for figures.