On the number of edges in some graphs (postprint)

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Abstract

Let f(n) denote the maximum possible number of edges in an n-vertex graph containing no cycles of equal length. The problem of determining f(n) was posed by Erdos in 1975. This paper provides a lower bound for f(n).

Full Text

Preamble

On the Number of Edges in Some Graphs

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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph with n vertices in which any two cycles have different lengths. The sequence $(c_1, c_2, \cdots, c_n)$ represents the cycle length distribution of a graph G with n vertices, where c_i is the number of cycles of length i in G. Let $f(a_1, a_2, \cdots, a_n)$ denote the maximum possible number of edges in a graph satisfying $c_i \leq a_i$, where a_i is a nonnegative integer. In 1991, Shi posed the problem of determining $f(a_1, a_2, \cdots, an)$, which extended Erdős' s problem. It is clear that $f(n) = f(1, 1, \cdots, 1)$. Let $g(n, m) = f(a_1, a_2, \cdots, a_n)$, where $a_i = 1$ if i/m is an integer, and $a_i = 0$ otherwise. It is clear that $f(n) = g(n, 1)$.

We prove that

$$\liminf_{n\to\infty} \frac{g(n,m) - n}{\sqrt{n}} \geq \sqrt{2 + \frac{7654}{99}} \approx 2.444$$

for all even integers m, which improves upon previous bounds (Lai, 2017). We show that

$$\liminf_{n\to\infty}\frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{7654}{19071}}$$

which is better than the previous bounds: $\sqrt{2}$ (Shi, 1988) and earlier results. We make the following conjecture:

$$\liminf_{n\to\infty}\frac{f(n)-n}{\sqrt{n}} = \sqrt{2+\frac{7654}{19071}}.$$

Key words: Graph, cycle, number of edges
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Introduction

Let $f(n)$ be the maximum number of edges in a graph with n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see Bondy and Murty [?], p. 247, Problem 11). Shi [?] proved the following lower bound:

Theorem 1 (Shi [?]). $f(n) \geq n + \lfloor (\sqrt{8n-23}+1)/2 \rfloor$ for $n \geq 3$.

Additional related results were obtained by Chen, Lehel, Jacobson and Shreve [?], Jia [?], Lai [?, ?, ?, ?], and Shi [?, ?]. Boros, Caro, Füredi and Yuster [?] proved the following upper bound:

Theorem 2 (Boros, Caro, Füredi and Yuster [?]). For n sufficiently large, $f(n) < n + 1.98\sqrt{n}$.

Lai [?] improved Shi' s lower bound as follows:

Theorem 3 (Lai [?]). Let $t = 1260r + 169$ $(r \geq 1)$, then

$$f(n) \geq n + \frac{t^2+87978t+15957}{4t+1}$$

for $n \geq \frac{2119}{4}t^2 + \frac{26399}{2}t + 6932215$.

Lai [?] proposed the following conjecture:

Conjecture 4 (Lai [?]).

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = \sqrt{2}.$$

It would be nice to prove that $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2}$. Survey papers on this problem can be found in Tian [?], Zhang [?], and Lai and Liu [?]. The progress on all 50 problems in [?] can be found in Locke [?].

The sequence $(c_1, c_2, \cdots, c_n)$ is the cycle length distribution of a graph G with n vertices, where c_i is the number of cycles of length i in G. Let $f(a_1, a_2, \cdots, a_n)$ denote the maximum possible number of edges in a graph which satisfies $c_i \le a_i$, where a_i is a nonnegative integer. Shi [?] posed the problem of determining $f(a_1, a_2, \cdots, a_n)$, which extended the problem due to Erdős. It is clear that $f(n) = f(1, 1, \cdots, 1)$. Let $g(n, m) = f(a_1, a_2, \cdots, a_n)$, where $a_i = 1$ if i/m is an integer, and $a_i = 0$ otherwise. It is clear that $f(n) = g(n, 1)$.

In this paper, we obtain the following results:

Theorem 5. Let m be even, $s_1 > s_2$, $s_1 + 3s_2 > k$, then

$$g(n, m) \ge n + (k + s_1 + 2s_2 + 1)t - 1$$

for $n \ge \left(\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{4}mk + \frac{1}{2}mks_2 + \frac{1}{2}ms_{1s}2 + \frac{9}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1\right)t + 1$.

Theorem 6. Let $t = 1260r + 169$ $(r \ge 1)$, then for $n \ge \frac{1309}{2}t^2 - \frac{1349159}{3}t + 6932215$,

$$f(n) \ge n + \frac{119}{3}t - 26399.$$

Proof of Theorem 5

Proof. Let

$$n_t = \left(\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{4}mk + \frac{1}{2}mks_2 + \frac{1}{2}ms_{1s}2 + \frac{9}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1\right)t+1,$$

where m is even, $s_1 > s_2$, $s_1 + 3s_2 > k$, and $n \ge n_t$. It suffices to show that there exists a graph G on n vertices with $n + (k + s_1 + 2s_2 + 1)t - 1$ edges such that all cycles in G have distinct lengths and all cycle lengths are multiples of m.

We construct the graph G consisting of several subgraphs B_i for $0 \le i \le s_{1t}$, $i = s_{1t} + j$ $(1 \le j \le s_{2t})$, and $i = s_{1t} + s_{2t} + j$ $(1 \le j \le t)$. These subgraphs all share a single common vertex x, and are otherwise vertex-disjoint.

For $1 \le i \le s_{2t}$, let the subgraph $B_{s_{1t}+i}$ consist of a cycle

$$C_{s_{1t}+i} = x a_1^i a_2^i \cdots a^i_{ms_{1t}+2ms_{2t}+mi-1} x$$

and a path

$$P_{s_{1t}+i} = a^i_{ms_{1t}+ms_{2t}+mi} \cdots a^i_{ms_{1t}+2ms_{2t}+mi-2}.$$

Based on this construction, $B_{s_{1t}+i}$ contains exactly three cycles of lengths $ms_{1t}+mi$, $ms_{1t}+ms_{2t}+mi$, and $ms_{1t}+2ms_{2t}+mi$.

For $1 \le i \le t$, let the subgraph $B_{s_{1t}+s_{2t}+i}$ consist of a cycle

$$C_{s_{1t}+s_{2t}+i} = xy^i_1 \cdots y^i_{ms_{1t}+3ms_{2t}+mk(k+1)t+mi-1}x$$

and k paths sharing the common vertex x, with their other endpoints on the cycle $C_{s_{1t}+s_{2t}+i}$:

$$P_{s_{1t}+s_{2t}+i,p} = y^i_{ms_{1t}+3ms_{2t}-mkt+m(p-1)t+mi} \cdots y^i_{ms_{1t}+3ms_{2t}+mk(2p-1)t+m(p-1)t+mi} \quad (p = 1, 2, \ldots, k).$$

As a cycle with k chords contains exactly $\binom{k+2}{2}$ distinct cycles, $B_{s_{1t}+s_{2t}+i}$ contains cycles of lengths $ms_{1t}+3ms_{2t}+mkht+(h+j-1)mt+mi$ for $j \ge 1$, $h \ge 0$, and $k+1 \ge j+h$.

B_0 is a path with one endpoint at x and length $n-n_t$. All other B_i are simply cycles of length mi.

Therefore, $g(n,m) \ge n+(k+s_1+2s_2+1)t-1$ for $n \ge n_t$. This completes the proof.

From Theorem 5, we have

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{n}} \ge \sqrt{\frac{k+s_1+2s_2+1}{\frac{3}{4}mk^2+\frac{1}{2}mks_1+\frac{3}{4}mk+\frac{1}{2}mks_2+\frac{1}{2}ms_{1s}2+\frac{9}{4}ms_2+\frac{1}{2}m}}$$

for all even integers m.

Let $s_1 = 28499066$, $s_2 = 4749839$, $k = 14249542$. Then

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{n}} \ge \sqrt{2.444}$$

for all even integers m.

Proof of Theorem 6

Proof. Let $t = 1260r+169$ with $r \ge 1$, and let

$$n_t = \frac{1309}{2}t^2 - \frac{1349159}{3}t + 6932215.$$

For $n \ge n_t$, it suffices to show that there exists a graph G on n vertices with $n+\frac{119}{3}t-26399$ edges such that all cycles in G have distinct lengths.

We construct the graph G consisting of several subgraphs B_i for $0 \le i \le 22t$, $i = 22t+j$ $(1 \le j \le \frac{5t-8}{3})$, $i = 23t+\frac{2t-2}{3}+j$ $(1 \le j \le \frac{5t-8}{3})$, and $i = 32t+j-60$

($58 \leq j \leq t-742$). These subgraphs all share a single common vertex x, and are otherwise vertex-disjoint.

For $1 \leq i \leq \frac{5t-8}{3}$, let the subgraph B_{22t+i} consist of a cycle

$$C_{22t+i} = xa_1^i a_2^i \cdots a^i_{28t+\frac{2t-2}{3}+2i-3}x$$

and a path

$$P_{22t+i} = a^i_{56t-2} \cdots a^i_{76t-4}.$$

Based on this construction, B_{22t+i} contains exactly three cycles of lengths $22t+i$, $25t+\frac{t-1}{3}+i-1$, and $28t+\frac{2t-2}{3}+2i-2$.

For $1 \leq i \leq \frac{5t-8}{3}$, let the subgraph $B_{23t+\frac{2t-2}{3}+i}$ consist of a cycle

$$C_{23t+\frac{2t-2}{3}+i} = xb_1^i b_2^i \cdots b^i_{28t+\frac{2t-2}{3}+2i-2}x$$

and a path

$$P_{23t+\frac{2t-2}{3}+i} = b^i_{11t-1} \cdots b^i_{76t-4}.$$

Based on this construction, $B_{23t+\frac{2t-2}{3}+i}$ contains exactly three cycles of lengths $23t+\frac{2t-2}{3}+i$, $27t+i-1$, and $28t+\frac{2t-2}{3}+2i-1$.

For $58 \leq i \leq t-742$, let the subgraph $B_{32t+i-60}$ consist of a cycle

$$C_{32t+i-60} = xy_1^i \cdots y^i_{137t+11i+890}x$$

and ten paths sharing the common vertex x, with their other endpoints on the cycle $C_{32t+i-60}$:

$$\begin{aligned}
P_{32t+i-60,1} &= y^i_{11t-2} \cdots y^i_{21t-59+i},\\
P_{32t+i-60,2} &= y^i_{12t-2} \cdots y^i_{31t-53+2i},\\
P_{32t+i-60,3} &= y^i_{12t-2} \cdots y^i_{41t+156+3i},\\
P_{32t+i-60,4} &= y^i_{13t-2} \cdots y^i_{51t+155+4i},\\
P_{32t+i-60,5} &= y^i_{13t-2} \cdots y^i_{61t+155+5i},\\
P_{32t+i-60,6} &= y^i_{14t-2} \cdots y^i_{71t+154+6i},\\
P_{32t+i-60,7} &= y^i_{14t-2} \cdots y^i_{81t+153+7i},\\
P_{32t+i-60,8} &= y^i_{15t-2} \cdots y^i_{91t+147+8i},\\
P_{32t+i-60,9} &= y^i_{15t-2} \cdots y^i_{101t+149+9i},\\
P_{32t+i-60,10} &= y^i_{16t-2} \cdots y^i_{111t+151+10i}.
\end{aligned}$$

As a cycle with d chords contains exactly $\binom{d+2}{2}$ distinct cycles, $B_{32t+i-60}$ contains cycles of lengths: $32t+i-60$, $33t+i+4$, $34t+i+207$, $35t+i-3$, $36t+i-2$, $37t+i-3$, $38t+i-3$, $39t+i-8$, $40t+i$, $41t+i$, $42t+i+739$, $43t+2i-54$, $43t+2i+213$, $45t+2i+206$, $45t+2i-3$, $47t+2i-3$, $47t+2i-4$, $49t+2i-9$,

$49t+2i-6$, $51t+2i+2$, $51t+2i+741$, $53t+3i+155$, $54t+3i+212$, $55t+3i+206$, $56t+3i-4$, $57t+3i-4$, $58t+3i-10$, $59t+3i-7$, $60t+3i-4$, $61t+3i+743$, $64t+4i+154$, $64t+4i+212$, $66t+4i+205$, $66t+4i-5$, $68t+4i-10$, $68t+4i-8$, $70t+4i-5$, $70t+4i+737$, $74t+5i+154$, $75t+5i+211$, $76t+5i+204$, $77t+5i-11$, $78t+5i-8$, $79t+5i-6$, $80t+5i+736$, $85t+6i+153$, $85t+6i+210$, $87t+6i+198$, $87t+6i-9$, $89t+6i-6$, $89t+6i+735$, $95t+7i+152$, $96t+7i+204$, $97t+7i+200$, $98t+7i-7$, $99t+7i+735$, $106t+8i+146$, $106t+8i+206$, $108t+8i+202$, $108t+8i+734$, $116t+9i+148$, $117t+9i+208$, $118t+9i+943$, $127t+10i+150$, $127t+10i+949$, $137t+11i+891$.

B_0 is a path with one endpoint at x and length $n-n_t$. All other B_i are simply cycles of length i.

Therefore, $f(n) \geq n + \frac{119}{3}t - 26399$ for $n \geq n_t$. This completes the proof.

From Theorem 6, we have

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{7654}{19071}},$$

which is better than the previous bounds: $\sqrt{2}$ (see [?]) and $\sqrt{2+\frac{7654}{19071}}$ (see [?]). Combining this with Boros, Caro, Füredi and Yuster' s upper bound (Theorem 2), we obtain

$$1.98 \geq \limsup_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{7654}{19071}}.$$

From the proof of Theorem 6, we have

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{n}} \geq \sqrt{2+\frac{7654}{19071}}$$

for all integers m.

If $m=1$ and $1 \leq i \leq t$, we could construct a subgraph similar to $B_{s_{1t}+s_{2t}+i}$ consisting of a cycle $C_{s_{1t}+s_{2t}+i}$ and k paths sharing a common vertex x, with their other endpoints on the cycle $C_{s_{1t}+s_{2t}+i}$, such that all cycles in $B_{s_{1t}+s_{2t}+i}$ have distinct lengths. This would yield $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2.444} > \sqrt{2+\frac{7654}{19071}}$. However, we have only constructed such a subgraph for $m=1$ and $58 \leq i \leq t-742$, using a cycle with ten paths sharing a common vertex x to obtain $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{7654}{19071}}$.

Since the liminf for $\frac{g(n,m)-n}{\sqrt{n}}$ for even m is $\sqrt{2.444}$, it is reasonable to suspect that such a lower bound also holds for $\frac{f(n)-n}{\sqrt{n}}$. We make the following conjecture:

Conjecture 7.

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = \sqrt{2.444}.$$

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Note: Figure translations are in progress. See original paper for figures.