Variable Volatility Elasticity Model for Commodity Markets

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Abstract

In this paper, we propose and study a novel continuous-time model, based on the well-known constant elasticity of variance (CEV) model, to describe the asset price process. The basic idea is that the volatility elasticity of the CEV model cannot be treated as a constant from the perspective of stochastic analysis. To address this issue, we deduce the price process of assets from the perspective of volatility elasticity, propose the constant volatility elasticity (CVE) model, and further derive a more general variable volatility elasticity (VVE) model. Moreover, our model can describe the positive correlation between volatility and asset prices existing in the commodity markets, while the CEV model can only describe the negative correlation. Through empirical research on the financial market, many assets, especially commodities, often exhibit this positive correlation phenomenon during certain time periods, which demonstrates that our model has strong practical application value. Finally, we provide the explicit pricing formula for European options based on our model. This formula has an elegant form that is convenient to calculate, which is similar to the renowned Black-Scholes formula and is of great significance to research in the derivatives market.

Full Text

The Variable Volatility Elasticity Model from Commodity Markets

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Abstract

In this paper, we propose and study a novel continuous-time model, based on the well-known constant elasticity of variance (CEV) model, to describe the asset price process. The fundamental insight is that the volatility elasticity of the CEV model cannot be treated as a constant from the perspective of stochastic analysis. To address this limitation, we derive the price process from the volatility elasticity viewpoint, propose the constant volatility elasticity (CVE) model, and further develop a more general variable volatility elasticity (VVE) model. Moreover, our model can capture the positive correlation between volatility and asset prices that exists in commodity markets, whereas the CEV model can only describe negative correlation. Empirical research on financial markets demonstrates that many assets, particularly commodities, frequently exhibit this positive correlation phenomenon during certain periods, indicating that our model has substantial practical value. Finally, we provide an explicit pricing formula for European options under our model. This formula has an elegant and computationally convenient form similar to the renowned Black-Scholes formula, which is of great significance for derivatives market research.

Keywords: volatility elasticity, commodity market, pricing.

1 Introduction

Continuous-time models are fundamental and powerful tools in financial market theory, where stochastic techniques play a prominent role. For instance, in option pricing, hedging, and portfolio research, stochastic analysis methods are typically employed to derive explicit solutions or closed-form expressions that provide strong guidance for practical applications.

As early as 1965, Samuelson proposed describing stock price evolution using geometric Brownian motion [**?**, **?**], establishing the classic continuous-time financial model. In 1973, Black and Scholes applied Itô's formula from stochastic analysis to derive an explicit European option pricing formula based on this model—the classical Black-Scholes formula [**?**, **?**]. This work represented a breakthrough in option pricing theory.

However, empirical data reveals that "volatility smile" or skew phenomena frequently appear in most stock markets [**?**, **?**]. Consequently, numerous studies have been devoted to proposing more sophisticated continuous-time financial models to correct volatility biases, such as the constant elasticity of variance (CEV) model [**?**, **?**], stochastic volatility models [**?**, **?**], the V.G. model [**?**, **?**], GARCH models [**?**, **?**], and others.

Recently, the classical CEV model has attracted considerable attention due to its wide applications in financial markets. In this model, volatility is negatively correlated with price, enabling it to describe derivatives and volatility of stocks exhibiting this inverse phenomenon [?, ?]. Additionally, many extensions based on the CEV model have been proposed, such as [?, ?, ?, ?, ?, ?]. However, we identify a fundamental issue with the constant elasticity assumption in the CEV model. Before discussing this problem in detail, we first introduce the concepts of volatility and elasticity.

1.1 Volatility and Elasticity

The volatility of financial assets is a key measure of return uncertainty, representing the risk level of financial assets. Mathematically, it is defined as the conditional standard deviation of the return rate, expressed as:

$$v_t = \sqrt{\mathrm{Var}(r_{t+1} \mid \mathcal{F}_t)}.$$

Here, r_t represents the rate of return at time t, and the filtration $(\mathcal{F}_t)$ encompasses all market information prior to time t. In this paper, we assume market efficiency, meaning that asset prices fully reflect all available market information. Volatility plays a crucial role in many financial market decisions, including risk management, derivatives pricing, and portfolio optimization. Moreover, derivatives based on volatility indices are becoming increasingly active in the market, providing investors with diverse investment and hedging tools that help mitigate risks and reduce irrational market volatility [?, ?]. In developed financial markets, various volatility indices and their derivatives are widely traded, such as futures, options, and ETFs based on the CBOE Volatility Index (VIX). Therefore, the study of volatility is of great significance in both theory and practice.

The concept of elasticity, first introduced by Marshall [?, ?], has been widely used in economics. It is an important measure of how sensitive one economic factor is to another—for example, how supply or demand changes in response to price changes, or how demand responds to income changes. Moreover, elasticity can help monitor risk exposure in stock markets. In this paper, we focus on the elasticity of an asset' s volatility with respect to its price, which we call volatility elasticity and denote by λ_t.

According to the definition of elasticity, λ_t measures the sensitivity of volatility to changes in asset price, expressed as:

$$\lambda_t = \frac{dv_t/v_t}{dS_t/S_t}.$$

In essence, asset prices and their volatility vary according to market conditions. More specifically, as time passes, various informational events continuously occur in the market, causing asset prices to fluctuate constantly. Therefore, the common driving force behind these two stochastic processes is the time scale t. As $dt \to 0$, the continuous formula for volatility elasticity becomes:

$$\lambda_t = \frac{dv_t}{dS_t} \cdot \frac{S_t}{v_t} \tag{1.1}$$

where dv_t and dS_t denote the stochastic differentials of v_t and S_t, respectively.

1.2 Existing Problems in the CEV Model

Let S_t denote the price of a risky asset. In the CEV model [?, ?], the price process satisfies the following stochastic differential equation (SDE):

$$dS_t = S_t \mu dt + \sigma S_t^{\beta/2-1} dB_t \tag{1.2}$$

where μ and σ are constants, $\beta \in (0, 2]$ is called the elasticity factor, and B_t represents Brownian motion. It has been shown in [?, ?] that the asset price volatility is $v_t = \sigma S_t^{\beta/2-1}$ and the volatility elasticity equals a constant $\beta/2 - 1$. We note that these results are derived in a deterministic sense, where v_t and S_t are assumed to be deterministic variables and the derivative of v_t with respect to S_t is computed straightforwardly. However, according to the economic definition of elasticity, v_t and S_t are intrinsically random processes in real markets. We should use stochastic differentials to treat volatility elasticity, which cannot be simply regarded as a constant. Consequently, the main objective of this paper is to develop a new model based on stochastic analysis that accurately describes real markets.

1.3 Contribution

In this paper, we design two SDE models to describe asset prices using stochastic differential tools. The first model assumes constant volatility elasticity, which we call the constant volatility elasticity (CVE) model. We then derive the second model, which we call the variable volatility elasticity (VVE) model, by assuming that elasticity is related to price. Analysis of actual market data demonstrates that our models describe the price processes of certain assets well, particularly commodities, providing valuable assistance for forecasting and guiding economic decision-making in these markets. Finally, we provide an explicit option pricing formula based on our proposed models, which is of great value for practical applications in derivatives markets.

1.4 Organization

The remainder of this paper is organized as follows. In Section 2, we derive the stochastic differential equation model for the asset price process under the assumption of constant volatility elasticity. This model is then extended to the general case of time-varying volatility elasticity in Section 3. The potential of our proposed models is verified through empirical research using actual market data in Section 4. Additionally, an explicit pricing formula for European options is presented in Section 5. Finally, we conclude the paper in the last section.

2 Constant Volatility Elasticity (CVE) Model

In this paper, we consider a market composed of two types of assets: risky and risk-free. Without loss of generality, we assume that asset price processes in the market are time-homogeneous Markov processes and that assets do not pay dividends. Suppose the price process of the risk-free asset satisfies:

$$d\beta_t = r\beta_t dt, \tag{2.1}$$

where r is the risk-free rate.

In financial markets, stock volatility is time-varying and often exhibits volatility clustering and fat tails in the distribution of logarithmic price returns [**?**, **?**]. According to the efficient market hypothesis and the assumption of time-homogeneous Markov price processes, volatility can be expressed as a function of price by definition. Let S_t denote the price of the risky asset. Then we assume S_t satisfies the following stochastic process:

$$dS_t = S_t[\mu dt + \theta(S_t)dB_t], \tag{2.2}$$

where $S_0 > 0$ and $\mu > 0$ is the instantaneous expected rate of return. We assume that $\theta(S_t)$ satisfies appropriate conditions to guarantee the existence of solutions. It can be readily verified that $\theta(S_t)$ is precisely the volatility function of S_t. Moreover, for convenience, the instantaneous rate of return of risky assets is simply assumed to be constant. In fact, our conclusions can easily be extended to more general cases, as discussed in Remark 1.

Now, based on the assumption that volatility elasticity is constant, we can derive that $\theta(S_t)$ admits a closed-form solution.

Theorem 2.1. Suppose the price of a risky asset satisfies SDE (2.2), $\theta(x)$ is twice continuously differentiable, and the volatility elasticity is constant. Then:

$$\theta(S_t) = CS_t^{\alpha} \tag{2.3}$$

where $\alpha \in \{0, 1\}$ is the value of volatility elasticity, and C is a positive constant.

Proof. Since the price satisfies SDE (2.2), we know that $\theta(S_t)$ is the volatility of S_t. Applying Itô' s rule to $\theta(S_t)$, we obtain:

$$\begin{aligned}
dv_t &= d\theta(S_t) \\
&= \theta'(S_t)dS_t + \frac{1}{2}\theta''(S_t)d\langle S\rangle_t \\
&= \theta'(S_t)S_t[\mu dt + \theta(S_t)dB_t] + \frac{1}{2}\theta''(S_t)[S_t\theta(S_t)]^2 dt \\
&= \theta(S_t)\left(\frac{\theta'(S_t)}{\theta(S_t)}S_t\mu + \frac{\theta''(S_t)S_t^2\theta(S_t)}{2}\right)dt + \theta'(S_t)S_t dB_t,
\end{aligned}$$

which yields:

$$dv_t = \left(\frac{\theta'(S_t)}{\theta(S_t)}S_t\mu + \frac{\theta''(S_t)S_t^2}{\theta(S_t)}\right)dt + \theta'(S_t)S_t dB_t. \tag{2.4}$$

We assume that volatility elasticity is a constant α. According to the definition of elasticity, it follows that:

$$\frac{dv_t}{v_t} = \alpha\frac{dS_t}{S_t},$$

which further implies:

$$\left(\frac{\theta'(S_t)}{\theta(S_t)}S_t\mu + \frac{\theta''(S_t)S_t^2}{\theta(S_t)}\right)dt + \theta'(S_t)S_t dB_t = \alpha[\mu dt + \theta(S_t)dB_t].$$

Therefore, we can deduce that:

$$\theta'(S_t)S_t = \alpha\theta(S_t), \tag{2.5}$$

$$\frac{\theta'(S_t)}{\theta(S_t)}S_t\mu + \frac{\theta''(S_t)S_t^2}{\theta(S_t)} = \alpha\mu. \tag{2.6}$$

Clearly, (2.5) is an ordinary differential equation (ODE), and its solution is:

$$\theta(S_t) = CS_t^\alpha, \tag{2.7}$$

where C is a positive constant. Substituting (2.7) into (2.6), we have:

$$\alpha(\alpha-1)C^2S_t^{2\alpha} = 0. \tag{2.8}$$

Since S_t is not identically equal to 0, we can conclude that:

$$\alpha = 0, \text{ or } \alpha = 1, \text{ or } C = 0. \tag{2.9}$$

If $C = 0$, SDE (2.2) becomes $dS_t = S_t \mu dt$. This means that S_t is the price of a risk-free asset, which contradicts our assumption. Therefore, we obtain:

$$\theta(S_t) = CS_t^{\alpha}, \tag{2.10}$$

where $\alpha \in \{0, 1\}$, and C is a positive constant.

Theorem 2.1 shows that when volatility elasticity is constant, its value can only be 0 or 1. The asset price can then be expressed as either:

$$dS_t = S_t[\mu dt + CdB_t], \tag{2.11}$$

or:

$$dS_t = S_t[\mu dt + CS_t dB_t]. \tag{2.12}$$

The first model (2.11) is simply the classical Black-Scholes model. In this case, asset volatility is constant σ and volatility elasticity is obviously 0. We will not discuss this trivial case further in this paper.

The second model (2.12) is similar in form to the CEV model, corresponding to the case where $\beta = 4$. However, it should be noted that β lies in $(0, 2]$ in the CEV model [**?**, **?**]. Therefore, our model is not a special case of the CEV model. Since volatility elasticity is assumed constant, (2.12) is called the Constant Volatility Elasticity (CVE) model.

Remark 1. In the proof of Theorem 2.1, the constant μ can be replaced by an adaptive process $\mu(t)$ without changing the conclusion.

3 Variable Volatility Elasticity (VVE) Model

In financial markets, return series of risky asset prices often exhibit excessive volatility and volatility clustering [**?**, **?**]. Numerous models attempt to capture these phenomena, such as stochastic volatility models [**?**, **?**], GARCH [**?**, **?**], and EGARCH [**?**, **?**]. In real markets, the characteristics of volatility series are often complex and elusive. Therefore, assuming that the elasticity of volatility with respect to price is constant is overly simplistic.

In this section, we discuss the stochastic differential equation model when volatility elasticity varies with market information. Recall that we assume market efficiency, meaning that the flow of market information is generated by the asset price. Therefore, we naturally assume that volatility elasticity depends on the asset price. The following theorem derives the resulting model.

Theorem 3.1. Suppose the price of a risky asset satisfies SDE (2.2), $\theta(x)$ is twice continuously differentiable, and the volatility elasticity depends on price. Then:

$$dS_t = S_t[\mu dt + (\sigma + C_1 S_t)dB_t]. \tag{3.1}$$

where σ and C_1 are positive constants.

Proof. According to the proof of Theorem 2.1, we know that the volatility of S_t is $\theta(S_t)$ and:

$$dv_t = \left(\frac{\theta'(S_t)}{\theta(S_t)} S_t \mu + \frac{\theta''(S_t)S_t^2}{\theta(S_t)}\right) dt + \theta'(S_t)S_t dB_t. \tag{3.2}$$

Suppose the volatility elasticity is $\alpha(S_t)$, namely:

$$\frac{dv_t}{v_t} = \alpha(S_t)\frac{dS_t}{S_t}, \tag{3.3}$$

which implies:

$$\left(\frac{\theta'(S_t)}{\theta(S_t)} S_t \mu + \frac{\theta''(S_t)S_t^2}{\theta(S_t)}\right) dt + \theta'(S_t)S_t dB_t = \alpha(S_t)[\mu dt + \theta(S_t)dB_t]. \tag{3.4}$$

Therefore, we can deduce that:

$$\theta'(S_t)S_t = \alpha(S_t)\theta(S_t), \tag{3.5}$$

$$\frac{\theta'(S_t)}{\theta(S_t)} S_t \mu + \frac{\theta''(S_t)S_t^2}{\theta(S_t)} = \alpha(S_t)\mu. \tag{3.6}$$

Solving ODE (3.5), we obtain:

$$\theta(S_t) = C_2 \exp\left(\int \frac{\alpha(S_t)}{S_t} dS_t\right), \tag{3.7}$$

where C_2 is a positive constant. It is straightforward to verify that:

$$\theta'(S_t) = C_2 \exp\left(\int \frac{\alpha(S_t)}{S_t} dS_t\right) \frac{\alpha(S_t)}{S_t},$$

$$\theta''(S_t) = C_2 \exp\left(\int \frac{\alpha(S_t)}{S_t} dS_t\right)\left(\frac{\alpha^2(S_t) + \alpha'(S_t)S_t - \alpha(S_t)}{S_t^2}\right).$$

Substituting these relationships into (3.6) yields:

$$\theta^2(S_t)[\alpha^2(S_t) + \alpha'(S_t)S_t - \alpha(S_t)] = 0. \tag{3.8}$$

Combining this with the fact that $\theta(S_t) > 0$ implies:

$$\alpha^2(S_t) + \alpha'(S_t)S_t - \alpha(S_t) = 0. \tag{3.9}$$

Solving this ODE, we obtain:

$$\alpha(S_t) = \frac{1}{1 + C_3 S_t}, \tag{3.10}$$

where C_3 is a positive constant. Therefore, we can further conclude that:

$$\theta(S_t) = C_2 \exp\left(\int \frac{1}{S_t(1 + C_3 S_t)} dS_t\right) = C_2(1 + C_3 S_t).$$

The proof is completed by setting $C_1 = C_2 C_3$ and $\sigma = C_2$.

Theorem 3.1 establishes the model for the case where volatility elasticity is not constant but depends on price. Therefore, (3.1) is called the Variable Volatility Elasticity (VVE) model. It is obvious that (2.12) is a special case of (3.1) when $\sigma = 0$. Consequently, we will focus on the VVE model hereafter.

3.1 Existence of Solutions

We now verify the existence of solutions to SDE (3.1) in the following lemma, which proves that the proposed model is well-defined.

Lemma 3.2. There exists a solution to SDE (3.1), which can be expressed as:

$$S_t = \frac{\sigma S_0 e^{\gamma t + \sigma B_t}}{C_1 S_0 e^{\gamma t + \sigma B_t} - \frac{\mu}{\gamma} C_1 S_0 e^{\sigma B_t} + \sigma + C_1 S_0}, \tag{3.11}$$

where $\gamma = \mu - \frac{\sigma^2}{2}$.

Proof. We use the nonlinear transformation method to solve SDE (3.1). Suppose $Y_t = F(S_t)$ where $F(x)$ is a twice continuously differentiable function. Then it can be readily verified that:

$$\begin{aligned}
dY_t &= F'(S_t)dS_t + \frac{1}{2}F''(S_t)d\langle S\rangle_t \\
&= F'(S_t)[\mu S_t dt + S_t(\sigma + C_1 S_t)dB_t] + \frac{1}{2}F''(S_t)S_t^2(\sigma + C_1 S_t)^2 dt \\
&= \left(\mu F'(S_t)S_t + \frac{1}{2}F''(S_t)S_t^2(\sigma + C_1 S_t)^2\right) dt + F'(S_t)S_t(\sigma + C_1 S_t)dB_t.
\end{aligned}$$

Upon taking:

$$F'(x)x(\sigma + C_1 x) = \sigma F(x), \tag{3.12}$$

we can obtain:

$$F'(S_t)S_t(\sigma + C_1 S_t)dB_t = \sigma F(S_t)dB_t = \sigma Y_t dB_t.$$

The solution to ODE (3.12) can be represented as:

$$F(x) = \frac{\sigma}{\sigma + C_1 x}, \tag{3.13}$$

where A is a constant. Without loss of generality, we set $A = 1$. Then we have:

$$Y_t = \frac{\sigma}{\sigma + C_1 S_t}. \tag{3.14}$$

It is straightforward to verify that:

$$F'(x) = \frac{\sigma}{x(\sigma + C_1 x)}F(x),$$

$$F''(x) = \frac{\sigma}{x(\sigma + C_1 x)}F'(x) - \frac{\sigma(1 + 2C_1 x/\sigma)}{x^2(\sigma + C_1 x)^2}F(x) = -\frac{2C_1\sigma}{x^2(\sigma + C_1 x)^2}F(x),$$

which further implies:

$$\mu F'(S_t)S_t = \mu\frac{\sigma}{\sigma + C_1 S_t}F(S_t) = \frac{\mu\sigma}{\sigma + C_1 S_t}Y_t,$$

$$\frac{1}{2}F''(S_t)S_t^2(\sigma + C_1 S_t)^2 = -F(S_t)C_1\sigma S_t = -\sigma C_1 S_t Y_t.$$

Thus, we can obtain:

$$dY_t = Y_t\left(\frac{\mu\sigma}{\sigma + C_1 S_t} - \sigma C_1 S_t\right)dt + \sigma dB_t = Y_t\left(\mu - \frac{(2\mu+\sigma^2)C_1 Y_t}{1 - C_1 Y_t} + \frac{\mu C_1^2 Y_t^2}{1 - C_1 Y_t}\right)dt + \sigma dB_t.$$

Next, we solve the following SDE:

$$dY_t = \hat{\mu}(Y_t)dt + \hat{\sigma}(Y_t)dB_t,$$

where $\hat{\sigma}(Y_t) = \sigma Y_t$ and:

$$\hat{\mu}(Y_t) = \mu - \frac{(2\mu+\sigma^2)C_1 Y_t}{1 - C_1 Y_t} + \frac{\mu C_1^2 Y_t^2}{1 - C_1 Y_t}.$$

We first solve the following ODE:

$$\frac{dy(\omega)}{d\omega} = \hat{\sigma}(y(\omega)) = \sigma y(\omega), \quad Y_0 = \xi. \tag{3.15}$$

The solution is $y(\omega) = \xi e^{\sigma\omega}$. Let $\Phi(\omega, \xi) = y(\omega) = \xi e^{\sigma\omega}$. Then we solve the following ODE with parameter ω:

$$\frac{d\xi_t}{dt} = \exp\left(\int_0^{B_t(\omega)} \frac{b(\Phi(B_t(\omega), \xi_t)) - \hat{\sigma}(\Phi(B_t(\omega), \xi_t)) \cdot \hat{\sigma}'(\Phi(B_t(\omega), \xi_t))}{\hat{\sigma}^2(\Phi(B_t(\omega), \xi_t))} dB_t(\omega)\right), \quad \xi_0 = Y_0. \tag{3.16}$$

In fact, the above ODE can be simplified as:

$$\frac{d\xi_t}{dt} = \frac{2r - \sigma^2}{2}\xi_t - \frac{(2r - \sigma^2)}{2rC_1 e^{\sigma B_t(\omega)}}\xi_t^2, \quad \xi_0 = Y_0.$$

It is straightforward to verify that the solution is:

$$\xi_t(\omega) = \frac{\sigma S_0 e^{\sigma B_t(\omega)}}{(\sigma + C_1 S_0)e^{\sigma B_t(\omega)} - \frac{\mu}{\gamma} C_1 S_0 (e^{\gamma t + \sigma B_t(\omega)} - e^{\sigma B_t(\omega)})},$$

where $\gamma = \mu - \frac{\sigma^2}{2}$. Therefore:

$$Y_t(\omega) = \Phi(B_t(\omega), \xi_t(\omega)) = \frac{\sigma S_0 e^{\gamma t + \sigma B_t(\omega)}}{C_1 S_0 e^{\gamma t + \sigma B_t(\omega)} - \frac{\mu}{\gamma} C_1 S_0 e^{\sigma B_t(\omega)} + \sigma + C_1 S_0} \tag{3.17}$$

is a solution to SDE (3.15). Finally, it follows from the relationship (3.14) that:

$$S_t = \frac{\sigma S_0 e^{\gamma t + \sigma B_t}}{C_1 S_0 e^{\gamma t + \sigma B_t} - \frac{\mu}{\gamma} C_1 S_0 e^{\sigma B_t} + \sigma + C_1 S_0}, \tag{3.18}$$

where $\gamma = \mu - \frac{\sigma^2}{2}$. This completes the proof.

In Lemma 3.2, we skillfully apply nonlinear reduction to SDE (3.1), directly solve the transformed SDE, and then transform the solution back to obtain the solution of the original model. Lemma 3.2 not only proves the existence of solutions but also provides a closed-form expression, which is highly valuable for pricing derivatives based on our model.

3.2 Qualitative Analysis

In this subsection, we conduct a qualitative analysis of our model from an economic perspective to demonstrate its effectiveness and significance.

Volatility. The volatility of our VVE model is $v_t = \sigma + C_1 S_t$, where σ and C_1 are positive constants. It can describe stock markets with positive correlation between volatility and stock price, which differs from the CEV model. This positive correlation has been investigated in the literature. For instance, Emanuel and MacBeth [**?**, **?**] verified its existence using real market data. They analyzed closing price data for Avon Products, Eastman Kodak, International Business Machines, and Xerox from 1976 to 1978. Through parameter estimation in the CEV model, they concluded that there was a positive correlation between volatility and price in 1978. Unlike financial securities, commodity markets possess intrinsic physical attributes, causing commodity price fluctuations to exhibit distinctive features.

More importantly, Geman [**?**, **?**] points out that for the vast majority of commodities, volatility is positively related to price, which will be described in detail in the next section. Therefore, our model will be highly useful for pricing and hedging commodity derivatives.

Pricing of contingent claims. Derivatives play a crucial role in finance. Measured by underlying assets, the scale of derivatives markets far exceeds that of stock markets. Currently, the total notional value of outstanding derivatives is many times larger than the world' s total economic output [**?**, **?**]. Derivative pricing theory is one of the most important research topics in the field of derivatives. Under our model, we can derive an explicit pricing formula for European options (presented in Section 5). It is worth noting that this formula has a form similar to the Black-Scholes pricing formula, making it easy to compute in practice.

4 Empirical Analysis of Commodity Markets

In financial investment markets, bulk commodities refer to homogeneous, tradable goods widely used as basic industrial raw materials, such as crude oil, non-ferrous metals, steel, agricultural products, iron ore, coal, etc. They include three categories: agricultural products, metals, and energy. Commodities constitute the only spot market in human history, which is closely related to national economies and people' s livelihoods. Since most commodities are fundamental to industrial production, changes in their futures and spot prices reflecting supply and demand directly affect the entire economic system. As an important component of financial markets, research on commodities and their derivatives is of great significance. Commodities possess both capital and goods attributes, leading to significant differences in price characteristics compared to securities.

In commodity markets, the theory of storage is often used to explain spot price volatility, as the supply-demand relationship caused by the physical properties of commodities is the primary factor affecting their price changes [**?**, **?**]. The most important result of storage theory is that there is a negative correlation between inventory levels and commodity volatility. Furthermore, according to supply-demand theory, commodity price is negatively correlated with inventory. Therefore, there is a positive correlation between commodity price and volatility, which distinguishes it from the negative correlation commonly observed in stock markets.

Production capacity and inventory levels are two key factors in predicting commodity prices. Fama and French [**?**, **?**] conducted statistical analysis on data from 21 commodities (including wood, livestock, metals, and agricultural products) and concluded that commodity price variance decreases as inventory levels increase. Geman and Nguyen [**?**, **?**] used soybean data from the United States, Brazil, and Argentina to reconstruct monthly, quarterly, and annual global soybean databases, showing that commodity price volatility is an increasing linear function of inverse inventory. Geman [**?**, **?**] notes that this property also holds in energy markets. When estimated oil reserves in the United States or other regions decline, oil price volatility increases sharply, and prices rise substantially. Deaton and Laroque [**?**, **?**] analyzed and simulated annual data for 13 commodities, finding that the conditional variance of price is a non-decreasing function of price. In summary, numerous studies demonstrate a positive correlation between price and volatility in commodity markets.

Figure 1: Close price and volatility of soybean meal and aluminum.

Next, we perform an empirical study on China' s commodity markets, which demonstrates that a linear positive correlation between volatility and commodity price exists during many time intervals. Specifically, we select market data for soybean meal and aluminum for analysis. For soybean meal, we use closing price data for 263 trading days (2020/01/02-2021/01/29). For aluminum, we use closing price data for 164 trading days (2017/08/01-2018/04/02). All data are collected from the Wind database.

In Figure 1, the daily closing price data and historical volatility (calculated using 30-day price data and recorded as HV30) for the two commodities are depicted.

Clearly, the trends of the two lines show a positive correlation. Furthermore, we analyze the correlation between closing price and volatility data and conduct linear fitting using R software. The results are shown in Table 1.

Table 1: Linear regression results of closing price and volatility data for soybean meal and aluminum.

Commodity	Slope	P-value of Slope	Intercept	P-value of Intercept	R-squared	Correlation Coefficient
Soybean Meal (SM)	1.53e-04	<2e-16	-0.0015	<2e-16	0.7471	0.8644
Aluminum (AL)	4.61e-05	<2e-16	0.0003	<2e-16	0.8207	0.9059

We have the following observations. First, the correlation coefficients for these two datasets are very close to 1. Second, the R-squared values for these linear fittings are 0.7471 and 0.8207, respectively. Third, the p-values for both slope and intercept are less than 0.01. Thus, during this time interval, the price and volatility of these two commodities exhibit a linear positive correlation, which is consistent with the VVE model. We can conclude that the VVE model can serve as an approximate continuous-time model for commodity prices, which is helpful for studying commodity and commodity derivatives markets.

5 Option Pricing

In this section, we consider option pricing under the VVE model (3.1). The following theorem provides an explicit pricing formula for European call options.

Theorem 5.1. Suppose the market is complete and there exists a unique equivalent martingale measure. Let $\xi = (S_T - K)^+$ be a replicable European contingent claim, where K is the strike price. Then its price process is $V_t = C(t, S_t)$, where:

$$C(t,x) = \sigma S_0 e^{-r(T-t)}\mathbb{E}[g(Z)\mathbf{1}_{(d,+\infty)}] - Ke^{-r(T-t)}(1 - N(d)).$$

Here:

$$\delta = \frac{r}{\sigma}\left(\frac{r-\sigma^2/2}{r-\sigma^2/2} - 1\right),$$

$$d = \frac{f_T^{-1}(K) - f_t^{-1}(x)}{\sqrt{T-t}},$$

$$g(z)=\frac{\sigma S_0 e^{-r(T-t)}}{(\delta-1-\delta e^{-r\delta T})C_1 S_0+(\sigma+C_1 S_0)e^{-\sigma(z+f_t^{-1}(x))}},$$

$$f_t^{-1}(x)=\frac{\ln\left(\frac{\sigma+C_1 x}{\sigma}\right)-(r-\sigma^2/2)t}{\sigma},$$

and Z follows a standard normal distribution, with $N(d)$ denoting the standard normal cumulative distribution function.

Proof. Since we assume the market is complete, the discounted price process $(e^{-rt}S_t)$ is a martingale under the risk-neutral probability measure $\mathbb{P}^*$. In fact, $\mathbb{P}^*$ can be defined as:

$$\left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_T}=\exp\left(-\int_0^T\frac{\mu-r}{\sigma+C_1 S_u}dB_u-\frac{1}{2}\int_0^T\left(\frac{\mu-r}{\sigma+C_1 S_u}\right)^2du\right).$$

According to Girsanov’ s theorem, we have that:

$$B_t^*=B_t+\int_0^t\frac{\mu-r}{\sigma+C_1 S_u}du$$

is a Brownian motion under $\mathbb{P}^*$, and SDE (3.1) becomes:

$$dS_t=S_t[rdt+(\sigma+C_1 S_t)dB_t^*]. \tag{5.1}$$

It follows from Lemma 3.2 that the solution of (5.1) is:

$$S_t=\frac{\sigma S_0 e^{(r-\sigma^2/2)t+\sigma B_t^*}}{C_1 S_0 e^{(r-\sigma^2/2)t+\sigma B_t^*}-\frac{r}{r-\sigma^2/2}C_1 S_0 e^{\sigma B_t^*}+\sigma+C_1 S_0}=:f_t(B_t^*). \tag{5.2}$$

According to the risk-neutral pricing formula, the price of the replicable European contingent claim ξ at time t is:

$$V_t=\mathbb{E}^*\left[e^{-r(T-t)}(S_T-K)^+\mid\mathcal{F}_t\right].$$

By the Markov property of the diffusion process (S_t), we have:

$$V_t=\mathbb{E}^*\left[e^{-r(T-t)}(S_T-K)^+\mid S_t\right].$$

Thus, we can denote $V_t=C(t,S_t)$. Then:

$$
\begin{aligned}
C(t,x) &= \mathbb{E}^*\left[e^{-r(T-t)}(S_T-K)^+ \mid S_t=x\right] \\
&= e^{-r(T-t)}\mathbb{E}^*\left[(f_T(B_T^*)-K)^+ \mid B_t^*=f_t^{-1}(x)\right] \\
&= e^{-r(T-t)}\int_{f_T^{-1}(K)}^{\infty}(f_T(y)-K)\cdot p_{B^*}(t,f_t^{-1}(x);T,y)dy \\
&= e^{-r(T-t)}\int_{f_T^{-1}(K)}^{\infty}\frac{\sigma S_0 e^{(r-\sigma^2/2)T+\sigma y}}{C_1S_0e^{(r-\sigma^2/2)T+\sigma y}-\frac{r}{r-\sigma^2/2}C_1S_0e^{\sigma y}+1}\cdot\frac{e^{-\frac{(y-f_t^{-1}(x))^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}dy \\
&\quad -Ke^{-r(T-t)}\int_{f_T^{-1}(K)}^{\infty}\frac{e^{-\frac{(y-f_t^{-1}(x))^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}dy \\
&= \sigma S_0e^{-r(T-t)}\int_{f_T^{-1}(K)}^{\infty}\frac{e^{\sigma y-\frac{(y-f_t^{-1}(x))^2}{2(T-t)}}}{(\delta-1)C_1S_0e^{r\delta T+\sigma y}-\delta C_1S_0e^{\sigma y}+1}\frac{dy}{\sqrt{2\pi(T-t)}} \\
&\quad -Ke^{-r(T-t)}(1-N(d)) \\
&= \sigma S_0e^{-r(T-t)}\int_{d}^{\infty}\frac{e^{-\frac{z^2}{2}}}{(\delta-1-\delta e^{-r\delta T})C_1S_0+(\sigma+C_1S_0)e^{-\sigma(z+f_t^{-1}(x))}}\frac{dz}{\sqrt{2\pi}} \\
&\quad -Ke^{-r(T-t)}(1-N(d)) \\
&= \sigma S_0e^{-r(T-t)}\mathbb{E}[g(Z)\mathbf{1}_{(d,+\infty)}]-Ke^{-r(T-t)}(1-N(d)),
\end{aligned}
$$

where:

$$
\delta=\frac{r}{\sigma}\left(\frac{r-\sigma^2/2}{r-\sigma^2/2}-1\right),
$$

$$
d=\frac{f_T^{-1}(K)-f_t^{-1}(x)}{\sqrt{T-t}},
$$

$$
g(z)=\frac{\sigma S_0e^{-r(T-t)}}{(\delta-1-\delta e^{-r\delta T})C_1S_0+(\sigma+C_1S_0)e^{-\sigma(z+f_t^{-1}(x))}},
$$

$$
f_t^{-1}(x)=\frac{\ln\left(\frac{\sigma+C_1x}{\sigma}\right)-(r-\sigma^2/2)t}{\sigma},
$$

and Z follows a standard normal distribution, with $N(d)$ denoting the standard normal cumulative distribution function.

6 Conclusion

In this paper, we demonstrate that the volatility elasticity of the CEV model cannot be simply treated as a constant. To address this issue, we employ stochastic analysis tools to derive a stochastic differential equation model when volatility elasticity is constant. Since volatility elasticity for most risky assets is not fixed, we then extend this model to the general case of time-varying volatility elasticity. Our model can capture the positive correlation between volatility and asset price that frequently occurs in commodity markets, in contrast to the CEV model which can only describe negative correlation. These theoretical findings are validated using actual market data. Furthermore, we derive an explicit pricing formula for European options based on our model, which has a form similar to the Black-Scholes formula and is computationally convenient. This formula has important guiding significance for the practical application of derivatives pricing in commodity markets.

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