The Silver Ratio in the Maximum Deng Entropy Triangle

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Date: 2022-03-18T22:56:37+00:00

Abstract

Pascal' s triangle is the triangular arrangement of binomial coefficients, from which one can obtain the Fibonacci sequence and the golden ratio (approximately 1.618). A question arises: Can the silver ratio (approximately 2.414) be obtained from Pascal' s triangle? This paper first establishes a Maximum Deng Entropy Triangle (MDET) based on the mass function distribution of maximum Deng entropy, which is a Pascal-like triangle. The general term formula of the MDET sequence is derived, and the limiting ratios in the MDET sequence are analyzed. The classical Pascal' s triangle is left-right symmetric, and through diagonal summation (Diagonal sum) one can only obtain a single sequence—the Fibonacci sequence, whose limiting ratio is the golden ratio. Unlike Pascal' s triangle, the MDET proposed in this paper has an asymmetric structure; through right-diagonal summation and left-diagonal summation, two sequences can be generated—the right MDET sequence and the left MDET sequence. This paper proves that the limiting ratio in the right MDET sequence converges to the silver ratio, while the limiting ratio in the left MDET sequence converges to the numerical value 2. This paper illustrates the properties of the proposed MDET and its sequences through numerical examples.

Full Text

Preamble

Silver Ratio in Maximum Deng Entropy Triangle

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Abstract

Pascal' s triangle is a mathematical triangle of combinatorial numbers, from which the Fibonacci number sequence and golden ratio can be obtained. Similarly, the silver ratio can be generated based on the Pell number sequence. Recently, the relations between Pascal' s triangle and maximum Deng entropy (MXDE) have been studied and presented. A straightforward question arises: if we design a triangle based on MXDE, what will the associated number sequence and the limiting ratio be like? Hence, this paper proposes a Pascal-like triangle based on MXDE, called the maximum Deng entropy triangle (MDET). Additionally, the number sequences based on MDET are investigated. Next, the general term for the MDET sequence is presented and the limiting ratio in the MDET sequence is analyzed. We prove that the limiting ratio in the right MDET sequence converges to the silver ratio $1 + \sqrt{2}$. Moreover, some examples are given to expound MDET and the MDET sequence.

Keywords: Pascal' s triangle, Deng entropy, Maximum Deng entropy triangle (MDET), Number sequence, Silver ratio

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Preprint submitted to Elsevier March 17, 2022

1. Introduction

Pascal' s triangle refers to a triangular arrangement of combinatorial numbers and has numerous properties [1]. One appealing property of Pascal' s triangle is that its diagonal sums generate the Fibonacci number sequence [2, 3, 4]. The Fibonacci numbers occur in many fields, from the pattern of gerbera flower heads [5] to the rhythm of architecture [6]. The limiting ratio of successive terms in the Fibonacci sequence converges to 1.618, which demonstrates the relationship between Pascal' s triangle and the golden ratio $(1 + \sqrt{5})/2$ [2, 7, 8]. Apart from the golden ratio, another important irrational mathematical constant is the silver ratio $1 + \sqrt{2}$. Similar to the way of generating the golden ratio from Fibonacci numbers, the silver ratio can be obtained based on the limiting ratio of successive numbers in the Pell number sequence [9]. Golden ratio and silver ratio have attracted much attention and have been widely used in modern sciences, such as set theory [10], architecture [6, 11], shape optimization [12], nonlinear systems [13], generalized golden ratio [14], silver structure [15], and continued fraction algorithm [16].

The relationship between Pascal' s triangle and entropy is a fascinating topic. Based on probability theory, Tsallis entropy is a generalization of Boltzmann-

Gibbs statistics [17], whose relationship to Pascal' s triangle is investigated in [18]. Recently, Deng proposed a new entropy, called Deng entropy [19], which is an extension of Shannon entropy based on Dempster-Shafer evidence theory [20, 21]. Then, Kang and Deng presented the maximum Deng entropy (MXDE) [19], where the analytical solution of MXDE and its associated distribution are analyzed. Since the form of MXDE contains combinatorial numbers, Deng entropy has many relations to Pascal' s triangle. In 2019, Gao and Deng pointed out that a pseudo-Pascal' s triangle can be generated from MXDE [22]. In 2021, Song and Deng explained the power set in evidence theory from the perspective of Pascal's triangle and entropy [23], in which the relation between Deng entropy and Pascal' s triangle is also discussed.

On the one hand, Pascal's triangle can construct the Fibonacci number sequence and further yield the golden ratio. On the other hand, Pascal's triangle has many relations to MXDE. A straightforward question arises: if we design a triangle based on MXDE, what will the generated number sequence and the associated limiting ratio be like? To address this issue, this paper proposes a Pascal-like triangle based on MXDE, called the maximum Deng entropy triangle (MDET). Additionally, the number sequence based on MDET is investigated. MDET can yield two types of sequences, i.e., the left and the right MDET sequence. Next, the general term for the MDET sequence is presented and proved. Based on the general term, the limiting ratio for successive terms in the MDET sequence is analyzed. We prove that, under the condition of $n \to \infty$, the limiting ratio in the right MDET sequence converges to the silver ratio $1 + \sqrt{2}$. Moreover, some examples are shown to expound MDET and the MDET sequence, where the MDET sequence is compared with several well-known number sequences, namely Fibonacci numbers, Pell numbers, and Jacobsthal numbers.

Section 2 reviews some preliminaries. In Section 3, the definition of MDET is proposed, and then the corresponding sequence and ratio of MDET are analyzed. Section 4 shows some examples for illustration. Section 5 makes a conclusion.

2. Preliminaries

2.1. Evidence Theory

Evidence theory [20, 21] is a generalization of probability theory and has various applications, such as reasoning [24] and knowledge representation [25, 26]. In evidence theory, the frame of discernment (FOD) is a finite sample space denoted by $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, which is the same as in probability theory. One of the biggest differences compared to probability theory is that the event space of evidence theory is the power set 2^Θ, which considers all possible subsets of Θ:

$$2^\Theta = \{\emptyset, \{\theta_1\}, \cdots, \{\theta_N\}, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_2, \theta_3\}, \cdots, \Theta\}$$

As an extension of probability distribution, a basic probability assignment (BPA) is a mapping function indicated by $m : 2^\Theta \to [0, 1]$ satisfying $\sum_{A \in 2^\Theta} m(A) = 1$

and $m(\emptyset) = 0$. Particularly, the element $A \in 2^{\Theta}$ with $m(A) > 0$ is called the focal element.

2.2. Deng Entropy

For handling the uncertainty of BPA, Deng entropy is proposed based on the framework of evidence theory and is an extension of Shannon entropy [19, 27]. Several properties of Deng entropy are analyzed in [19, 28]. Deng entropy has a variety of applications, such as time series analysis [29, 30], evidential reasoning [24], fractal-based eXtropy [31, 32], information volume [33, 34, 35], and target classification [36, 37]. Given a BPA defined on FOD Θ, Deng entropy is defined by [19]:

$$H_{DE}(m) = -\sum_{A \in 2^{\Theta}} m(A) \log \left(\frac{m(A)}{2^{|A|} - 1} \right)$$

where $|\cdot|$ represents the cardinality. The maximum entropy principle is important in statistics. The maximum value of Deng entropy and its BPA distribution are discussed in [19]. If and only if the BPA satisfies:

$$m(A) = \frac{2^{|A|} - 1}{\sum_{A \in 2^{\Theta}} (2^{|A|} - 1)}, \quad A \in 2^{\Theta}$$

the maximum Deng entropy (MXDE) appears, which is as follows:

$$H_{MXDE} = \log \sum_{A \in 2^{\Theta}} (2^{|A|} - 1)$$

2.3. Typical Number Sequences and Their Limiting Ratios

In this subsection, the recurrence relations of several well-known number sequences and their corresponding general terms are briefly reviewed. In addition, the associated limiting ratios of these sequences are presented, in which the limiting ratio of Fibonacci numbers is the golden ratio, and that of Pell numbers is the silver ratio.

- **Fibonacci number sequence and its limiting ratio (golden ratio)** [7, 38, 39, 40]:

  Recurrence: $F(n) = F(n-1) + F(n-2)$ with $F(0) = 0, F(1) = 1$

  General term: $F(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$

  Limiting ratio: $\lim_{n\to\infty} \frac{F(n)}{F(n-1)} = \frac{1+\sqrt{5}}{2}$

- **Pell number sequence and its limiting ratio (silver ratio)** [9, 41]:

  Recurrence: $P(n) = 2P(n-1) + P(n-2)$ with $P(0) = 0, P(1) = 1$

  General term: $P(n) = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right]$

  Limiting ratio: $\lim_{n\to\infty} \frac{P(n)}{P(n-1)} = 1 + \sqrt{2}$

- **Jacobsthal number sequence and its limiting ratio** [41, 42]:

  Recurrence: $J(n) = J(n-1) + 2J(n-2)$ with $J(0) = 0, J(1) = 1$

  General term: $J(n) = \frac{1}{3}\left[2^n - (-1)^n\right]$

  Limiting ratio: $\lim_{n\to\infty} \frac{J(n)}{J(n-1)} = 2$

3. Maximum Deng Entropy Triangle and Its Sequences

3.1. Maximum Deng Entropy Triangle

Pascal' s triangle is an arrangement of combinatorial numbers, illustrated in Figure 1(a). According to [19, 22], given a certain FOD Θ, the BPA distribution of MXDE is shown in Table 1, where the numbers of focal elements can actually construct a Pascal' s triangle (shown in red). Gao and Deng pointed out that there are inextricable connections between Pascal' s triangle and MXDE [22]. Inspired by this, a Pascal-like triangle is designed based on MXDE, called the maximum Deng entropy triangle (MDET). The definition of MDET is as follows:

Definition 3.1 (Maximum Deng entropy triangle). In the maximum Deng entropy triangle (MDET), the k-th element of the n-th row is defined by:

$$MDET(n,k) = C(n,k) \times (2^k - 1)$$

where $C(n,k) = \frac{n!}{k!(n-k)!}$, $n \geq 0$, and $0 \leq k \leq n$.

For better understanding, the illustration of MDET is shown in Figure 1(b).

3.2. MDET Sequence

According to [2], the Fibonacci number sequence can be derived from the diagonal sum of entries in Pascal' s triangle, as illustrated in Figure 2(a). Inspired by this idea, we present the diagonal sum of MDET to obtain number sequences based on MDET. It should be noted that, since MDET is asymmetric, there are two directions for the diagonal sum of MDET: the left diagonal sum and the right diagonal sum, shown in Figure 2(b) and (c).

Based on these two directions of diagonal sums, two types of number sequences are derived, called the left MDET sequence and the right MDET sequence. Specifically, as shown in Figure 2(b) and (c), the left (right) MDET sequence is generated based on the left (right) diagonal sum of MDET. These two types of sequences are mathematically defined as follows:

Definition 3.2 (MDET sequence). Based on the left and right diagonal sums of MDET, the left MDET sequence $D_L(n)$ and the right MDET sequence $D_R(n)$ are defined by:

$$D_L(n) = \sum_{k=0}^{\lfloor n/2 \rfloor + 1} MDET(n-k, k)$$

$$D_R(n) = \sum_{k=0}^{\lfloor n/2 \rfloor + 1} MDET(n-k, n-k)$$

where $\lfloor \cdot \rfloor$ is the floor function, which takes a real number as input and returns the greatest integer less than or equal to that number.

3.3. General Term of MDET Sequence

In this subsection, the general terms for the two types of MDET sequences are investigated. Substituting Eq. (15) into Eqs. (16) and (17) yields the recurrence relations of the left MDET sequence and the right MDET sequence:

Left MDET sequence recurrence:

$$D_L(n) = 2D_L(n-1) + 2D_L(n-2) - 3D_L(n-3) - 2D_L(n-4)$$

with initial values $D_L(0) = D_L(1) = 0, D_L(2) = 1, D_L(3) = 2$

Right MDET sequence recurrence:

$$D_R(n) = 3D_R(n-1) - 3D_R(n-3) - D_R(n-4)$$

with initial values $D_R(0) = 0, D_R(1) = 1, D_R(2) = 3, D_R(3) = 9$

Based on these recurrence relations, the general terms for the two MDET sequences can be derived:

Theorem 1. The general term of the left MDET sequence is:

$$D_L(n) = \frac{1}{3}\left[2^{n+1} - (-1)^{n+1}\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Theorem 2. The general term of the right MDET sequence is:

$$D_R(n) = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Proof 3.1 (Proof for Theorem 1). The characteristic equation corresponding to Eq. (18) is:

$$\phi^4 - 2\phi^3 - 2\phi^2 + 3\phi + 2 = 0$$

from which the characteristic roots can be solved: $\phi_1 = 2$, $\phi_2 = -1$, $\phi_3 = \frac{1+\sqrt{5}}{2}$, $\phi_4 = \frac{1-\sqrt{5}}{2}$.

Based on these roots, the general term of the left MDET sequence can be constructed:

$$D_L(n) = c_1 2^n + c_2(-1)^n + c_3\left(\frac{1+\sqrt{5}}{2}\right)^n + c_4\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Substituting the initial values $D_L(0) = D_L(1) = 0$, $D_L(2) = 1$, $D_L(3) = 2$ and solving for the constants yields: $c_1 = \frac{2}{3}$, $c_2 = -\frac{1}{3}$, $c_3 = -\frac{1}{\sqrt{5}}$, $c_4 = \frac{1}{\sqrt{5}}$.

Hence, the general term of the left MDET sequence is obtained:

$$D_L(n) = \frac{2}{3}\cdot 2^n - \frac{1}{3}(-1)^n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

which simplifies to the form shown in Theorem 1.

Proof 3.2 (Proof for Theorem 2). The characteristic equation associated with Eq. (19) is:

$$\delta^4 - 3\delta^3 + 3\delta - 1 = 0$$

from which the characteristic roots can be solved: $\delta_1 = 1+\sqrt{2}$, $\delta_2 = 1-\sqrt{2}$, $\delta_3 = \frac{1+\sqrt{5}}{2}$, $\delta_4 = \frac{1-\sqrt{5}}{2}$.

Based on these roots, the general term of the right MDET sequence can be constructed:

$$D_R(n) = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n + c_3\left(\frac{1+\sqrt{5}}{2}\right)^n + c_4\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Substituting the initial values $D_R(0) = 0$, $D_R(1) = 1$, $D_R(2) = 3$, $D_R(3) = 9$ yields the constants: $c_1 = \frac{1}{2\sqrt{2}}$, $c_2 = -\frac{1}{2\sqrt{2}}$, $c_3 = -\frac{1}{\sqrt{5}}$, $c_4 = \frac{1}{\sqrt{5}}$.

As a result, the general term of the right MDET sequence is obtained:

$$D_R(n) = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

which matches the form shown in Theorem 2.

Remark 1. Based on Theorem 1 and Eqs. (7)-(13), the left MDET sequence can be calculated using Jacobsthal numbers $J(n)$ and Fibonacci numbers $F(n)$:

$$D_L(n) = J(n+1) - F(n+1), \quad n \geq 0$$

Remark 2. According to Theorem 2 and Eqs. (7)-(10), the right MDET sequence can be calculated based on Pell numbers $P(n)$ and Fibonacci numbers $F(n)$:

$$D_R(n) = P(n+1) - F(n+1), \quad n \geq 0$$

3.4. Silver Ratio in MDET

In this subsection, we investigate the silver ratio in MDET. Based on Theorems 1 and 2, the limiting ratios for successive terms in the left and right MDET sequences satisfy the following theorems:

Theorem 3. When $n \to \infty$, the limiting ratio for successive terms in the left MDET sequence converges to:

$$\lim_{n\to\infty} \frac{D_L(n)}{D_L(n-1)} = 2$$

Theorem 4. When $n \to \infty$, the limiting ratio for successive terms in the right MDET sequence converges to:

$$\lim_{n\to\infty} \frac{D_R(n)}{D_R(n-1)} = 1 + \sqrt{2}$$

Proof 3.3 (Proof for Theorem 3). Based on Theorem 1, the limiting ratio can be calculated as:

$$\lim_{n\to\infty}\frac{D_L(n)}{D_L(n-1)}=\lim_{n\to\infty}\frac{\frac{1}{3}\left[2^{n+1}-(-1)^{n+1}\right]-\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]}{\frac{1}{3}\left[2^{n}-(-1)^{n}\right]-\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right]}$$

Since $\phi_1 = 2$ is the maximum characteristic root among the four roots, the limiting ratio can be written as:

$$\lim_{n\to\infty}\frac{D_L(n)}{D_L(n-1)}\to\lim_{n\to\infty}\frac{\frac{2}{3}\cdot 2^n}{\frac{1}{3}\cdot 2^n}=2$$

Proof 3.4 (Proof for Theorem 4). Based on Theorem 2, the limiting ratio can be calculated as:

$$\lim_{n\to\infty}\frac{D_R(n)}{D_R(n-1)}=\lim_{n\to\infty}\frac{\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\right]-\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]}{\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n}-(1-\sqrt{2})^{n}\right]-\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right]}$$

Since $\delta_1 = 1+\sqrt{2}$ is the maximum characteristic root among the four roots, the limiting ratio can be written as:

$$\lim_{n\to\infty}\frac{D_R(n)}{D_R(n-1)}\to\lim_{n\to\infty}\frac{(1+\sqrt{2})^{n+1}}{(1+\sqrt{2})^{n}}=1+\sqrt{2}$$

Remark 3. Based on Theorem 3, under the condition $n\to\infty$, the limiting ratio in the left MDET sequence is 2, which is the same as that in Jacobsthal numbers.

Remark 4. According to Theorem 4, when $n\to\infty$, the limiting ratio in the right MDET sequence is actually the silver ratio $1+\sqrt{2}$, which is the same as that in Pell numbers.

4. Numerical Examples

This section presents examples to illustrate the proposed maximum Deng entropy triangle (MDET) as well as the left and right MDET sequences.

Example 4.1. Assume $n = 0, 1, 2, 3, \ldots, 9$. The associated Pascal' s triangle and MDET are shown in Tables 2 and 3. It can be seen from the tables that Pascal' s triangle has only one type of diagonal sum since it is symmetric. By contrast, because MDET is asymmetric, there exist two types of diagonal sums for MDET. Based on these diagonal sums, Fibonacci numbers (shown in Table 2) and two MDET sequences (shown in Table 3) can be obtained. As shown in the tables, due to the two types of diagonal sums of MDET shown in Eqs. (16) and (17), there are two different kinds of MDET sequences, namely the left MDET sequence $D_L(n)$ and the right MDET sequence $D_R(n)$.

Example 4.2. Let n range from 0 to 10. The trend illustrations for Pell numbers $P(n)$, Fibonacci numbers $F(n)$, Jacobsthal numbers $J(n)$, left MDET sequence $D_L(n)$, and right MDET sequence $D_R(n)$ are shown in Figure 3. This example shows that $P(n)$ and $D_R(n)$ grow much faster than the other sequences because the maximum characteristic roots of both $P(n)$ and $D_R(n)$ are $1+\sqrt{2}$, which is much larger than that of the other sequences. Although $D_R(n)$ and $P(n)$ have the same maximum characteristic root value, $D_R(n)$ grows faster than $P(n)$ because the initial values of $D_R(n)$ (0, 1, 3, 9) are larger than those of $P(n)$ (0, 1, 2, 5). The growth rate of the Fibonacci number sequence is the lowest since its maximum characteristic root is $(1+\sqrt{5})/2$, which is much lower than that of the other sequences.

Example 4.3. Given n from 0 to 100, the values of the ratio for successive terms in Pell numbers $P(n)$, Fibonacci numbers $F(n)$, Jacobsthal numbers $J(n)$, left MDET sequence $D_L(n)$, and right MDET sequence $D_R(n)$ are shown in Figure 4. It can be seen from the figure that as n becomes larger, the ratio for each sequence finally converges to a certain value. The limiting ratio of $P(n)$ and $D_R(n)$ have the same value, which is the silver ratio. The limiting ratio of $J(n)$ equals that of $D_L(n)$, which is 2. The limiting ratio of $F(n)$ is the golden ratio.

5. Conclusion

Pascal' s triangle is a mathematical form of combinatorial numbers. Based on Pascal' s triangle, the Fibonacci number sequence and the golden ratio can be generated. Similar to the golden ratio, the silver ratio can be obtained based on the Pell number sequence. Recently, the relations between Pascal' s triangle and maximum Deng entropy (MXDE) have been studied and presented. A straightforward question arises: if we design a triangle based on MXDE, what will the associated number sequence and the limiting ratio be like? Hence, this paper proposes a novel triangle based on MXDE and analyzes its number sequence as well as limiting ratio.

The major contributions of this paper are as follows:

- A Pascal-like triangle is proposed based on MXDE, called the maximum Deng entropy triangle (MDET).
- The number sequences based on MDET are investigated. The general term for the MDET sequence is presented and the limiting ratio in the MDET sequence is analyzed. It is proved that the limiting ratio in the right MDET sequence converges to the silver ratio $1+\sqrt{2}$.
- Numerical examples are given to expound MDET, and the MDET sequence is compared with several well-known number sequences.

Acknowledgment

The work is partially supported by National Natural Science Foundation of China (Grant No. 61973332); JSPS Invitational Fellowships for Research in Japan (Short-term).

Declaration of Interest

All the authors certify that there is no conflict of interest with any individual or organization for this work.

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Note: Figure translations are in progress. See original paper for figures.