A Statistical Fields Theory underlying the Thermodynamics of Ricci Flow and Gravity

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Abstract

The paper proposes a statistical fields theory of quantum reference frame underlying the Perelman' s analogies between his formalism of the Ricci flow and the thermodynamics. The theory is based on a $d = 4 - \epsilon$ quantum non-linear sigma model (NLSM), interpreted as a quantum reference frame system which a to-be-studied quantum system is relative to.The statistic physics and thermodynamics of the quantum frame fields is studied by the density matrix of them obtained by the Gaussian approximation quantization. The induced Ricci flow of the frame fields and the Ricci-DeTurck flow of the frame fields associated with the density matrix is deduced. In this framework, the diffeomorphism anomaly of the theory has deep thermodynamic interpretation. The trace anomaly is related to a Shannon entropy in terms of the density matrix, which monotonically flows and achieves its maximal value at the flow limit,called the Gradient Shrinking Ricci Soliton (GSRS). A relative Shannon entropy w.r.t. the maximal entropy gives a statistical interpretation to Perelman' s partition function, which is also monotonic and giving an analogous H-theorem to the statistical frame fields system. We find that a temporal static 3-space of the GSRS spacetime is in a thermal equilibrium state, and Perelman' s analogies between his formalism and the thermodynamics of the frame fields in equilibrium can be explicitly given in the framework. As a possible underlying microscopic theory of the gravitational system, the theory is also applied to understand the thermodynamics of the Schwarzschild black hole. The cosmological constant in the effective theory of gravity at cosmic scale is also briefly given.

Full Text

Preamble

This paper proposes a statistical field theory of quantum reference frames underlying Perelman' s analogies between his Ricci flow formalism and thermodynamics. The theory is based on a $d = 4 - \epsilon$ quantum non-linear sigma model

(NLSM), interpreted as a quantum reference frame system relative to which a to-be-studied quantum system is defined. The statistical physics and thermodynamics of the quantum frame fields are studied using a density matrix obtained through Gaussian approximation quantization. The induced Ricci flow of the frame fields and the associated Ricci-DeTurck flow are derived from this density matrix. In this framework, the diffeomorphism anomaly of the theory possesses a deep thermodynamic interpretation. The trace anomaly is related to a Shannon entropy expressed in terms of the density matrix, which flows monotonically and achieves its maximal value at the flow limit, called the Gradient Shrinking Ricci Soliton (GSRS), corresponding to a thermal equilibrium state of spacetime. A relative Shannon entropy with respect to this maximal entropy provides a statistical interpretation of Perelman' s partition function, which is also monotonic and yields an H-theorem analogous to that for the statistical frame fields system. A temporal static 3-space of a GSRS 4-spacetime is also a GSRS in lower dimensions; we find that it is in a thermal equilibrium state, and Perelman' s analogies between his formalism and the thermodynamics of the frame fields in equilibrium can be explicitly realized within this framework. Extending the validity of the Equivalence Principle to the quantum level, the quantum reference frame theory at low energy yields an effective theory of gravity, recovering a scale-dependent Einstein-Hilbert action plus a cosmological constant. As a possible underlying microscopic theory of the gravitational system, the theory is also applied to understand the thermodynamics of the Schwarzschild black hole.

Introduction

Recent works [1, 2] have revealed possible relations between Perelman' s formalism of the Ricci flow and fundamental problems in quantum spacetime and quantum gravity, such as the trace anomaly and the cosmological constant problem. Perelman' s seminal work (Section 5 of [3]) and further developments by Li [4, 5] also suggest deep connections between the Ricci flow and thermodynamic systems, encompassing both irreversible non-equilibrium and thermal equilibrium thermodynamics of some underlying microscopic system. Perelman introduced a partition function and his functionals without specifying what the underlying microscopic ensemble actually is (in physics). Thus far, it remains unclear whether these beautiful thermodynamic analogies are physical or mere coincidences.

On the other hand, inspired by the surprising analogies between black holes and thermodynamic systems, it is widely believed that black holes possess temperature and entropy. Work along these lines has shown, in many respects, that gravitational systems are profoundly related to thermodynamic systems (see the recent review [6] and references therein). It is generally conjectured that there exists some underlying statistical theory for the microscopic quantum degrees of freedom of gravity, which has gradually become a touchstone for quantum gravity.

The motivations of this paper are, firstly, to propose an underlying statistical field theory for Perelman' s seminal thermodynamic analogies of his Ricci flow formalism, and secondly, to understand the possible microscopic origin of spacetime thermodynamics, particularly for the Schwarzschild black hole. We hope this work will advance understanding of the mysterious interplay between Perelman' s Ricci flow formalism and quantum spacetime and gravity. To our knowledge, several tentative works have been devoted to this goal (see, e.g., [7-10]), but frankly speaking, the physical picture underlying the Ricci flow remains unclear without a fundamental physical theory underlying the Ricci flow and a fundamental theory of quantum spacetime.

Building on our previous works [1, 2, 11-16] on quantum reference frames and their relation to Perelman' s Ricci flow formalism, we propose a statistical field theory of the quantum reference frame as a possible underlying theory for Perelman' s seminal analogies between his geometric functionals and thermodynamic functions. In Section II, we review the theory of quantum reference frames based on a $d = 4 - \epsilon$ quantum non-linear sigma model. At the Gaussian approximation quantization level, we obtain a density matrix of the frame fields system as the physical foundation for the statistical interpretation of the theory. The induced Ricci flow of the frame fields and the associated Ricci-DeTurck flow are derived. In Section III, we discuss the diffeomorphism and related trace anomaly of the quantum frame fields theory and its profound implications for the irreversible non-equilibrium thermodynamics of the frame fields, including the statistical entropy, an H-theorem for the frame fields, and the effective gravity theory at cosmic scale (particularly the emergence of the cosmological constant). In Section IV, the thermal equilibrium state of the frame fields as a flow limit configuration (the Gradient Shrinking Ricci Soliton) is discussed, where the density matrix recovers the thermal equilibrium canonical ensemble density. This section provides a physical foundation for Perelman' s seminal thermodynamic analogies. In Section V, the framework is applied to provide a possible microscopic understanding of the thermodynamics of the Schwarzschild black hole. Finally, we summarize the paper and present conclusions in Section VI.

II. Quantum Reference Frame

The reference frame is one of the most fundamental notions in physics. Any measurement in physics is performed or described with respect to a reference frame that is always explicitly or implicitly used. In classical physics, the reference frame is idealized through classical rulers and clocks that label spacetime coordinates, which are classical, external, and rigid without any fluctuations. Even in textbook quantum mechanics or quantum field theory, the spacetime coordinates remain classical. However, quantum principles tell us that all physical measuring instruments, including rulers and clocks, are inescapably subject to quantum fluctuations. Such idealized and classical treatment of reference frames works reasonably well in quantum mechanics and quantum field theory, largely because general coordinate transformations and gravitational effects are not seriously taken into account. As expected, when quantum principles are seriously applied to spacetime itself and gravitational phenomena, severe difficulties arise, such as information loss (non-unitarity), diffeomorphism anomaly, and the cosmological constant problem.

The quantum reference frame is a recurring theme in the literature (see, e.g., [17–24] and references therein) based on various physical motivations, ranging from quantum foundations to quantum information and quantum communication, to quantum gravity. For example, in Ref. [17], the author suggests a general relation between superselection rules and the lack of a reference frame. In Ref. [20], it is shown more practically that extra assumptions about superselection rules cannot be avoided from the viewpoint of quantum information and quantum communication theory if local observers do not share common information about their relative phase or Cartesian frames, etc. These extra assumptions about superselection rules may also be viewed as a weakness of textbook quantum mechanics, which can be overcome by introducing an appropriate quantum reference frame. Many models (e.g., [18, 22, 23]) of quantum reference frames and relational descriptions of the quantum system and the quantum reference frame as a whole have been suggested in quantum foundations. In recent works [24] and references therein, the authors review three approaches (relational Dirac observables, the Page-Wootters formalism, and quantum deparameterizations) to relational quantum dynamics and suggest their equivalence. Other authors focus on the possible role of quantum reference frames in decoherence in quantum gravity [21, 25]. Certainly, the list of works in this direction is far from complete, which is beyond the scope and ability of the author.

Fundamentally, our work shares a similar philosophical viewpoint regarding the role of quantum reference frames in quantum mechanics, such as the consideration that an appropriate materialized (but idealized) reference frame obeying the same laws of quantum mechanics must be taken into account, and that in the full quantum theory a relational description based on entanglement between a quantum system and the quantum reference frame as a whole must play a fundamental role. However, there are several important differences from past literature. First, we do not simply or merely treat the quantum clock as a quantum mechanical system ([23, 24]) (which is simpler and has fewer degrees of freedom to deal with, as discussed in most quantum reference frame literature; in fact, our early work [11, 12] also started from an operational treatment of quantum clocks to draw some general conclusions about vacuum energy and the cosmological constant problem). In this paper, we place both quantum space-rods and clock-time on an equal footing within the framework of quantum statistical fields, making the theory more appropriate for incorporating gravity under the assumption of a quantum version of the equivalence principle. In our understanding, the quantum clock can be viewed as a first-step model and is far from a complete theory. Second, based on the quantum spacetime reference frame model (i.e., the $d = 4 - \epsilon$ non-linear sigma model), our paper does not treat the genuine relational quantities from the very beginning (as most literature tends to announce), but rather we prepare the quantum frame fields of reference in a laboratory frame (the $d = 4 - \epsilon$ base spacetime of the non-linear sigma model) as the starting reference, and then quantum events are defined relative to the prepared quantum frame fields. In this sense, the framework effectively assumes the existence of an external, classical, and rigid (free from quantum fluctuations and with fixed volume) reference frame to serve as the laboratory frame, since the non-linear sigma model allows us to assign quantum states of spacetime reference (the target spacetime) to the base spacetime to arbitrary precision. However, it can be easily verified that the theory is independent of the laboratory frame (metric, signature, etc.) in the non-linear sigma model. The notion of an external and classical laboratory frame is merely for convenience, since a quantum statistical field theory is historically (and perhaps more appropriately) defined on an inertial frame (flat spacetime). Thus, the relational quantities describing the relation between the quantum system and the quantum spacetime reference system are essential in the framework. Third, also due to the base spacetime independence of the non-linear sigma model, whose Hamiltonian is trivial, the theory of spacetime reference frames is more properly quantized using path integral or functional methods rather than operator methods (e.g., relational Dirac observables quantization or relational Schrödinger picture in the Page-Wootters formalism). Fourth, there is a fundamentally non-unitary relation between two spacetime reference frames under a coordinate transformation due to an irreversible Ricci flow of the spacetime reference frame, unlike most approaches in which the coordinate transformation between different reference frames is assumed unitary. This is considered a key ingredient of quantum spacetime reference frames that is intrinsically ensemble statistical and thermal.

Generally speaking, our approach follows the general philosophy of quantum reference frames but is considered independent of the details of past literature. The framework associates with several elegant physical and mathematical structures not discussed in previous literature, such as the non-linear sigma model, Shannon entropy, the Ricci flow, and density Riemannian geometry, etc. Our previous works [1, 2, 11–16] have revealed very rich consequences of the framework (e.g., the accelerated expansion of the late universe, the cosmological constant, diffeomorphism anomaly, the inflationary early universe, local conformal stability and non-collapsibility, modified gravity, etc.), but frankly speaking, the possible consequences of the quantum reference frame are still far from fully discovered. The main motivation for a quantum treatment of a reference frame system is that it might form a foundation for constructing a theory of quantum spacetime and quantum gravity analogous to the way it is used to construct classical general relativity, and it is crucial for understanding the microscopic origin of spacetime thermodynamics.

A. Definition

In this section, we propose a quantum field theory of reference frames as a starting point for studying a quantum theory of spacetime and quantum gravity, based on an Equivalence Principle extended to reference frames described by quantum states (discussed through a paradox in Section V-B and in the conclusion of the paper). The generalization of the Equivalence Principle to the quantum level might form another foundation for quantum reference frames and quantum gravity. How the Equivalence Principle behaves at the quantum level has been discussed extensively in the literature (e.g., [26–30] and references therein, and [31, 32] for an extended thermal version). The Equivalence Principle is the physical foundation for measuring spacetime using physical material reference frames even at the quantum level, and it is the bridge between geometric curved spacetime and gravity; hence gravity is simply a relational phenomenon where the motion of a test particle in gravity manifests as relative motion with respect to the (quantum) material reference frame. Without the Equivalence Principle, we would lose the physical foundation of all these concepts. Therefore, the basic argument of the paper is that several pieces of evidence (e.g., the uniform quantum origin of the accelerating expansion of the universe proposed by the author in previous works [1, 2, 13], and a consistent incorporation of spacetime thermodynamics shown in this work) and the self-consistency of the framework all provide possible support for its validity for the quantum reference frame.

In this framework, a to-be-studied quantum system described by a state $|\psi\rangle$ and the spacetime reference system by $|X\rangle$ are both quantum. The states of the whole system are given by an entangled state

$$|\psi[X]\rangle = \sum_{\alpha ij} |\psi\rangle_i \otimes |X\rangle_j$$

in their direct product Hilbert space $\mathcal{H}_\psi \otimes \mathcal{H}_X$. The state (1) of the to-be-studied system and the reference frame system is an entangled state rather than a trivial direct product state for the purpose of calibration between them. Usually, a quantum measurement is performed as follows. In the preparation step of a quantum measurement, a one-to-one correlation between a quantum system $|\psi\rangle_i$ and a reference system $|X\rangle_j$ (a quantum instrument or ruler) is prepared, called calibration. This step in the usual sense is a comparison and adjustment of the measuring instrument $|X\rangle_j$ by a calibration standard $|\psi_{\text{standard}}\rangle_i$ which is physically similar to the to-be-studied system $|\psi\rangle_i = |\psi_{\text{standard}}\rangle_i$. A well-calibrated entangled state $\sum_{ij} \alpha_{ij} |\psi_{\text{standard}}\rangle_i \otimes |X\rangle_j$ can be used to measure the to-be-studied system $|\psi\rangle_i$ with reference to the quantum instrument $|X\rangle_j$. In essence, the measurement indirectly performs a comparison between $|\psi\rangle_i$ and the fiducial state $|\psi_{\text{standard}}\rangle_i$. Thus the entangled state $|\psi[X]\rangle$ is a superposition of all possible one-to-one correlations. According to the standard Copenhagen interpretation of a quantum state, the to-be-studied quantum system collapses into a state $|\psi\rangle_i$ together with the collapsing of the quantum reference system into the corresponding $|X\rangle_j$, happening with joint probability $|\alpha_{ij}|^2$, meaning that when the state of the quantum instrument is read out as being in state $|X\rangle_j$, then in this sense the to-be-studied system is inferred to be the corresponding $|\psi\rangle_i$. A simple and practical example is the Stern-Gerlach experiment (see [1]). The entangled state generalizes the textbook quantum description of the state $|\psi(x)\rangle$ with respect to an idealized parameter x of a classical reference system free from quantum fluctuations (in quantum mechanics x is Newtonian time, in quantum field theory x^a are Minkowskian spacetime coordinates).

The entangled state $|\psi[X]\rangle$ is inseparable, so the state can only be interpreted in a relational manner, i.e., the entangled state describes the "relation" between $|\psi\rangle$ and $|X\rangle$, but not each absolute state. The individual state $|\psi\rangle$ has physical meaning only when referenced to $|X\rangle$ entangled with it. When quantum mechanics is reformulated on the new foundation of the relational quantum state (the entangled state) describing the "relation" between the state of the under-studied quantum system and the state of the quantum reference system, a gravitational theory is automatically contained in the quantum framework without extra assumptions.

Since the state of reference $|X\rangle$ is also subject to quantum fluctuations, mathematically speaking, the state $|\psi[X]\rangle$ can be seen as the state $|\psi(x)\rangle$ with smeared spacetime coordinates, instead of the textbook state $|\psi(x)\rangle$ with definite and classical spacetime coordinates. The state $|\psi[X]\rangle$ could recover the textbook state $|\psi(x)\rangle$ only when the quantum fluctuations of the reference system are small enough to be ignored. More precisely, the second-order central moment (and even higher-order central moments) fluctuations of the spacetime coordinate $\langle\delta X^2\rangle$ (the variance) can be ignored compared with its first-order moment of quadratic distance $\langle\Delta X\rangle^2$ (squared mean), where $\langle\ldots\rangle$ represents the quantum expectation value by the state of the reference system $|X\rangle$. In this first-order approximation, this quantum framework recovers standard textbook quantum mechanics without gravity. When the quantum fluctuation $\langle\delta X^2\rangle$ as the second-order correction of the reference frame system is important and taken into account, gravity as a next-order effect emerges in the quantum framework, as if one introduces gravitation into standard textbook quantum mechanics, with details shown below and in previous works.

To find the state $|X\rangle \in \mathcal{H}_X$ of the quantum reference system, a quantum theory of the reference frame must be introduced. If the quantum spacetime reference frame $|X^\mu\rangle$ ($\mu = 0, 1, 2, \ldots, D-1$) itself is considered as the to-be-studied quantum system, with respect to the fiducial lab spacetime $|x^a\rangle$ as the reference system ($a = 0, 1, 2, \ldots, d-1$), the entangled state $|X(x)\rangle = \sum_{ij}\alpha_{ij}|X\rangle_i \otimes |x\rangle_j$ can be constructed by a mapping between the two states, i.e., $|x\rangle \to |X\rangle$. From a mathematical viewpoint, to define a D-dimensional manifold we need to construct a non-linear differentiable mapping $X(x)$ from a local coordinate patch $x \in \mathbb{R}^d$ to a D-manifold $X \in \mathcal{M}^D$. This mapping in physics is usually realized by a field theory for $X(x)$, the non-linear sigma model (NLSM) [33–40]

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$$S[X] = \sum_{\mu,\nu=0} \int d^{dx}\, g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu$$

where λ is a constant with dimension of energy density $[L^{-d}]$ taking the value of the critical density (68) of the universe.

In the action, x^a $(a = 0, 1, 2, \ldots, d-1)$, with dimension of length $[L]$, is called the base space in NLSM terminology, representing the coordinates of the local patch. They will be interpreted as the lab wall and clock frame as the starting reference, which is considered fiducial and classical with infinite precision. Since a quantum field theory must be formulated in a classical inertial frame, i.e., flat Minkowskian or Euclidean spacetime, the base space is considered flat. Without loss of generality, we consider the base space as Euclidean, i.e., $x \in \mathbb{R}^d$, which is better defined when one attempts to quantize the theory.

The differential mapping $X^\mu(x)$ $(\mu = 0, 1, 2, \ldots, D-1)$, with dimension of length $[L]$, gives the coordinates of a general Riemannian or Lorentzian manifold $\mathcal{M}^D$ (depending on the boundary condition) with curved metric $g_{\mu\nu}$, called the target space in NLSM terminology. We will work with real-valued coordinates for the target spacetime, and the Wick-rotated version is included in the general coordinate transformation of the time component. In the language of quantum field theory, $X^\mu(x)$ or $X_\mu(x) = \sum_{\nu=0}^{D-1} g_{\mu\nu}X^\nu(x)$ are the real scalar frame fields.

Here, unless specifically mentioned, we will use the Einstein summation convention to sum over index variables appearing twice (Latin indices for the lab frame from 0 to $d-1$ and Greek indices for the spacetime from 0 to $D-1$) and drop the summation notation Σ.

From a physical point of view, the reference frame fields can be interpreted as a physical coordinate system using particle/field signals, for instance, a multi-wire proportional chamber that measures coordinates of an event in a lab. To build a coordinate system, first we need to orient, align, and order the array of multi-wires with reference to the lab wall x^a $(a = 1, 2, 3)$. The electron fields (ignoring spin) in these arrays of multi-wires are considered as the scalar frame fields. With reference to the lab wall, to locate the position of an event, at least three electron signals X^1, X^2, X^3 must be received and read in three orthogonal directions. The location information can be measured from the wave function of the electron fields, e.g., from phase counting or particle number counting. Usually we could consider the electrons in the wires to be free, and the field intensity is not very large, so the intensity can be seen as a linear function of the coordinates of the lab' s wall, $X^\mu(x) = \sum_{a=1}^{3} e_a^\mu x^a$ $(\mu = 1, 2, 3)$, where e_a^μ is the intensity of the signals in each orthogonal direction. This means that when the direction μ is the lab' s wall direction a, the intensity of the electron beam is 1; otherwise the intensity is 0.

Similarly, one needs to read an extra electron signal X^0 to know when the event happens, with reference to the lab' s clock x^0. Thus, the fields of these $3+1$

electron signals can be given by

$$X^\mu(x) = \sum_{a=0}^{3} e_a^\mu \delta_a^\mu x^a \quad (\mu = 0, 1, 2, 3).$$

The intensity of the fields e_a^μ is in fact the vierbein, describing a mapping from the lab coordinate x^a to the frame fields X^μ. When the event happens at a long distance beyond the lab' s scale, for instance, at the scale of Earth or the solar system, we could imagine that extrapolating the multi-wire chamber to such long distance scales still seems acceptable, only replacing the electron beam in the wire with a light beam. However, if the scale is much larger than the solar system, for instance, reaching galactic or cosmic scales, when the signal travels along such long distances and is read by an observer, we could imagine that the broadening of the light beam fields or other particle fields gradually becomes non-negligible. More precisely, the second (or higher) order central moment fluctuations of the frame fields signals cannot be neglected; the distance of Riemannian/Lorentzian spacetime as a quadratic form must be modified by the second moment fluctuation or variance $\langle \delta X^2 \rangle$ of the coordinates:

$$\langle (\Delta X)^2 \rangle = \langle \Delta X \rangle^2 + \langle \delta X^2 \rangle.$$

A local distance element in spacetime is given by a local metric tensor at the point, so it is convenient to think of the location point X being fixed and interpret the variance of the coordinate as affecting only the metric tensor $g_{\mu\nu}$ at the location point. As a consequence, the expectation value of a metric tensor $g_{\mu\nu}$ is corrected by the second central moment quantum fluctuation of the frame fields:

$$\langle g_{\mu\nu} \rangle = \left\langle \frac{\partial X_\mu}{\partial x^a} \frac{\partial X_\nu}{\partial x_a} \right\rangle = \left\langle \frac{\partial X_\mu}{\partial x^a} \right\rangle \left\langle \frac{\partial X_\nu}{\partial x_a} \right\rangle + \left\langle \delta \left(\frac{\partial X_\mu}{\partial x^a} \right) \delta \left(\frac{\partial X_\nu}{\partial x_a} \right) \right\rangle$$

where $\langle \delta X^\mu \delta X^\nu \rangle = g_{\mu\nu}^{(1)}(X) - \delta g_{\mu\nu}^{(2)}(X)$,

$$g_{\mu\nu}^{(1)}(X) = \left\langle \frac{\partial X_\mu}{\partial x^a} \right\rangle \left\langle \frac{\partial X_\nu}{\partial x_a} \right\rangle = \langle e_a^\mu \rangle \langle e_a^\nu \rangle$$

is the first-order moment (mean value) contribution to the classical spacetime. For the contribution of the second-order central moment $\delta g_{\mu\nu}^{(2)}$ (variance), the expectation value of the metric generally tends to be curved and deformed; the longer the distance scale, the more important the broadening of the frame fields, making the spacetime geometry gradually deform and flow at long distance scales.

Since the classical solution of the frame fields (3) given by the vierbein satisfies the classical equation of motion of the NLSM, it provides a frame field interpretation of the NLSM in a lab: the base space of the NLSM is interpreted as a starting reference provided by the lab' s wall and clock, the frame fields $X(x)$ in the lab are the physical instruments measuring the spacetime coordinates. In this interpretation we consider $d = 4 - \epsilon$ $(0 < \epsilon \ll 1)$ in (2) and $D = 4$ as the minimal number of frame fields.

There are several reasons why d is not precisely but very close to 4 in the quantum frame fields interpretation of the NLSM. First, certainly at the scale of a lab it is our common sense; second, if we consider the entangled system $\mathcal{H}_\psi \otimes \mathcal{H}_X$ between the to-be-studied physical system and the reference frame fields system, without loss of generality we could take a scalar field ψ as the to-be-studied (matter) system, which shares the common base space with the frame fields. The total action of the two entangled systems is a direct sum of each system:

$$S[\psi, X] = \int d^{4x} \left[\frac{1}{2} \partial_a \psi \partial^a \psi - V(\psi) + \frac{\lambda}{2} g_{\mu\nu} \partial_a X^\mu \partial^a X^\nu \right]$$

where $V(\psi)$ is some potential of the ψ fields. It can be interpreted as an action of a quantum field ψ on general spacetime coordinates X. Since both the ψ field and the frame fields X share the same base space x, here they are described with respect to the lab spacetime x as in textbook quantum field theory defined on an inertial frame x. If we interpret the frame fields as the physical general spacetime coordinates, the coordinates of the ψ field must be transformed from the inertial frame x to general coordinates X. At the semi-classical level, or first moment approximation when the fluctuation of X can be ignored, it is simply a classical coordinate transformation:

$$S[\psi, X] \approx S[\psi(X)] = \int d^{4X} \sqrt{|\det g^{(1)}|} \left[\frac{1}{2} g^{(1)}_{\mu\nu} \frac{\delta \psi}{\delta X_\mu} \frac{\delta \psi}{\delta X_\nu} - V(\psi) \right]$$

where $\sqrt{|\det g^{(1)}|}$ is the Jacobian determinant of the coordinate transformation. Note that the determinant requires the coordinate transformation matrix to be square, so at the semi-classical level d must be very close to $D = 4$, which is not necessarily true beyond the semi-classical level when second-order moment quantum fluctuations are important.

For instance, since d is a parameter but an observable in the theory, it could even be non-integer but effectively fractal at the quantum level. That d is not precisely 4 is for quantum and topological reasons. To investigate this, we note that quantization depends on the homotopy group $\pi_d(\mathcal{M}^D)$ of the mapping $X(x) : \mathbb{R}^d \to \mathcal{M}^D$. If we consider the (Wick-rotated) spacetime $\mathcal{M}^D$ to be topologically S^D for simplicity, the homotopy group is trivial for all $d < D = 4$; in other words, when $d < 4$ the mapping $X(x)$ will be free from any unphysical singularities for topological reasons, and in this situation the target spacetime is always mathematically well-defined. However, the situation $d = 4$ is a little subtle, since $\pi_4(S^4) = \mathbb{Z}$ is non-trivial, the mapping might encounter intrinsic topological obstacles and become singular, i.e., a singular spacetime configuration. When the quantum principle is taken into account, this situation cannot be avoided, and through its RG flow the spacetime is possibly deformed into intrinsic singularities making the theory ill-defined at the quantum level and non-renormalizable (RG flow does not converge). So at the quantum level, $d = 4$ should not be precise; we must assume $d = 4 - \epsilon$ when the quantum principle applies, while at the classical or semi-classical level, considering $d = 4$ presents no serious problem. The above argument differs from the conventional simple power-counting argument, which claims the NLSM is perturbatively non-renormalizable when $d > 2$, but this is not necessarily the case. It is known that numerical calculations also support that $d = 3$ and $d = 4 - \epsilon$ are non-perturbatively renormalizable and well-defined at the quantum level.

B. Beyond the Semi-Classical Level: Gaussian Approximation

Going beyond the semi-classical or first-order moment approximation, we need to quantize the theory at least at the next leading order. If we consider the second-order central moment quantum fluctuations as the most important next-to-leading order contribution (compared with higher-order moments), we call it the Gaussian approximation or second-order central moment approximation, while higher-order moments are all called non-Gaussian fluctuations which might be important near local singularities of the spacetime when local phase transitions occur, although the intrinsic global singularity can be avoided by ensuring the global homotopy group is trivial.

At the Gaussian approximation, $\delta g_{\mu\nu}^{(2)}$ when it is relatively small compared with $g_{\mu\nu}^{(1)}$ can be given by a perturbative one-loop calculation [37, 38] of the NLSM:

$$\delta g_{\mu\nu}^{(2)}(X) = \frac{R_{\mu\nu}^{(1)}}{32\pi^2\lambda}\delta k^2$$

where $R_{\mu\nu}^{(1)}$ is the Ricci curvature given by the first-order metric $g_{\mu\nu}^{(1)}$, and k^2 is the cutoff energy scale of the Fourier component of the frame fields. The validity condition for the perturbative calculation $R^{(1)}\delta k^2 \ll \lambda$ is the validity condition for the Gaussian approximation, which can be seen as follows. It will be shown in a later section that λ is nothing but the critical density ρ_c of the universe, $\lambda \sim \mathcal{O}(H_0^2/G)$, where H_0 is Hubble' s constant and G is Newton' s constant. Thus for our concern of pure gravity in which matter is ignored, the condition $R^{(1)}\delta k^2 \ll \lambda$ is equivalent to $\delta k^2 \ll 1/G$, which is reliable except when some local singularities develop where the Gaussian approximation fails.

Equation (9) is nothing but an RG equation, known as the Ricci flow equation (see reviews in, e.g., [41–43]):

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu}$$

with flow parameter $\delta t = -\frac{1}{64\pi^2\lambda}\delta k^2$ having dimension of length squared $[L^2]$, which continuously deforms the spacetime metric driven by its Ricci curvature. Since the Ricci curvature is non-linear in the metric, the Ricci flow equation is a non-linear version of a heat equation for the metric, and flow along t introduces an averaging or coarse-graining process to the intrinsic non-linear gravitational system which is highly non-trivial [44–48]. In general, if the flow is free from local singularities there exists a long flow-time solution in $t \in (-\infty, 0)$, often called an ancient solution in mathematical literature. This range of the t-parameter corresponds to $k \in (0, \infty)$, that is from $t = -\infty$, i.e., the short-distance (high-energy) UV scale $k = \infty$, forward to $t = 0$, i.e., the long-distance (low-energy) IR scale $k = 0$. The metric at a certain scale t is given by averaging out the shorter-distance details, which produces an effective correction to the metric at that scale.

Thus along t, the manifold loses information at shorter distances, making the flow irreversible, i.e., generally having no backward solution, which is the underlying reason for the non-unitarity and existence of entropy of a spacetime.

As shown in (4) and (5), the second-order moment fluctuation modifies the local (quadratic) distance of the spacetime, so the flow is non-isometric. This is an important feature worth stressing, which is the underlying reason for the anomaly. The non-isometry is not important for topology, so along t, the flow preserves the topology of the spacetime but changes its local metric, shape, and size (volume). There also exists a very special solution of the Ricci flow called a Ricci soliton, which only changes the local volume while keeping its local shape unchanged. The Ricci soliton, and its generalized version, the Gradient Ricci Soliton, as flow limits, are generalizations of the notion of a fixed point in the sense of RG flow. The Ricci soliton is an important concept for understanding gravity at cosmic scale and studying the thermodynamics of the Ricci flow at equilibrium.

The Ricci flow was initially introduced in the 1980s by Friedan [34, 35] in $d = 2 + \epsilon$ NLSM and independently by Hamilton in mathematics [49, 50]. The main motivation for introducing it from the mathematical point of view is to classify manifolds; a specific goal is to prove the Poincaré conjecture. Hamilton used it as a useful tool to gradually deform manifolds into "simpler and better" manifolds whose topology can be readily recognized in simple cases. A general realization of this program was achieved by Perelman around 2003 [3, 51, 52], who introduced several monotonic functionals to successfully deal with local singularities that might develop in more general cases. The Ricci flow approach is powerful not only for compact geometry (as Hamilton' s and Perelman' s seminal works have shown) but also for non-compact [53–55] and Lorentzian geometry [15, 56–62].

C. The Wavefunction and Density Matrix at the Gaussian Approximation

So far we have not explicitly defined the quantum state of the reference frame $|X\rangle$ in (1). In fact, the previous second-order results, e.g., (5), (9), and hence the Ricci flow (10), can also equivalently be given by the expectation value $\langle O\rangle = \langle X|O|X\rangle$ via explicitly writing down the wavefunction $\Psi(X)$ of the frame fields at the Gaussian approximation.

Note that at the semi-classical level, the frame fields X are delta-distributed and peak at their mean value, and furthermore, the action of the NLSM resembles a collection of harmonic oscillators. Thus at the Gaussian approximation level, finite Gaussian width/second moment fluctuation of X must be introduced. When one performs canonical quantization of the NLSM at the Gaussian approximation level, the fundamental solution of the wave functional (as a functional of the frame fields X^μ) of the NLSM takes the Gaussian form, i.e., a coherent state:

$$\Psi[X^\mu(x)] = \frac{\lambda^{1/2}}{(2\pi)^{D/4}|\det\sigma_{\mu\nu}|^{1/4}|\det g_{\mu\nu}|^{1/4}} \exp\left[-\frac{1}{2}|X^\mu(x)\sigma_{\mu\nu}(x)X^\nu(x)|\right]$$

where the covariance matrix $\sigma_{\mu\nu}(x)$, playing the role of the Gaussian width, is the inverse of the second-order central moment fluctuations of the frame fields at point x:

$$\sigma_{\mu\nu}(x) = \frac{1}{\langle\delta X^\mu(x)\delta X^\nu(x)\rangle}$$

which is also given by perturbative one-loop calculation up to a diffeomorphism of X. The absolute value symbol $|X^\mu\sigma_{\mu\nu}X^\nu|$ in the exponential is used to guarantee that the quadratic form and hence the determinant of $\sigma_{\mu\nu}$ induced from the Gaussian integral over X remain positive even in Lorentzian signature.

We can also define a dimensionless density matrix corresponding to the fundamental solution of the wavefunction:

$$u[X^\mu(x)] = \Psi^*(X)\Psi(X) = \frac{\lambda}{(2\pi)^{D/2}\sqrt{|\det\sigma_{\mu\nu}|}\sqrt{|\det g_{\mu\nu}|}} \exp\left[-|X^\mu(x)\sigma_{\mu\nu}X^\nu(x)|\right]$$

where $\frac{\lambda}{(2\pi)^{D/2}\sqrt{|\det\sigma_{\mu\nu}|}\sqrt{|\det g_{\mu\nu}|}}$ is a normalization parameter, so that

$$\int d^{DX}\,\Psi^*(X)\Psi(X) = \lambda\int d^{DX}\,u(X) = 1,$$

in which we often attribute the flow of the volume form d_t^{DX} to the flow of the metric g_t, for the volume element $d_t^{DX} \equiv dV_t(X^\mu) \equiv \sqrt{|g_t|}dX^0dX^1dX^2dX^3$. Then the expectation values $\langle O\rangle$ can be understood as $\lambda\int d_t^{DX}\, uO$.

As the quantum frame fields X are q-numbers in the theory, precisely speaking, the integral over them should, in principle, be a functional integral. Here the formal c-number integral $\int d_t^{DX}$ ... is conventional in the Ricci flow literature, where X is a coarse-grained c-number coordinate of manifolds at scale t. The exact functional integral of X is considered when calculating the partition function and related anomalies of the theory in Section III.

Under a diffeomorphism of the metric, the transformation of $u(X)$ is given by a diffeomorphism of the covariance matrix (where h is some function):

$$\sigma_{\mu\nu} \to \hat{\sigma}_{\mu\nu} = \sigma_{\mu\nu} + \nabla_\mu\nabla_\nu h.$$

Thus there exists an arbitrariness in the density $u(X)$ for different choices of diffeomorphism/gauge.

According to the statistical interpretation of the wavefunction with the normalization condition (14), $u(X^0, X^1, X^2, X^3)$ describes the probability density of finding these frame particles in the volume $dV_t(X^\mu)$. As the spacetime X flows along t, the volume ΔV_t over which the density is averaged also flows, so the density at the corresponding scale is coarse-grained. If we consider the volume of the lab, i.e., the base space, to be rigid and fixed by $\lambda\int d^{4x} = 1$, noting (14), we have

$$u[X^\mu(x), t] = \lim_{\Delta V_t\to 0}\frac{1}{\Delta V_t}\int_{\Delta V_t} d^{4x}.$$

We can see that the density $u(X,t)$ can be interpreted as a coarse-grained density in the volume element $\Delta V_t \to 0$ with respect to a fine-grained unit density in the lab volume element d^{4x} at UV $t\to-\infty$.

In this sense, the coarse-grained density $u(X,t)$ is analogous to Boltzmann' s distribution function, so it should satisfy an analogous irreversible Boltzmann equation, giving rise to an analogous Boltzmann monotonic H-functional. In the following sections, we will deduce such an equation and the functional of $u(X,t)$. The coarse-grained density $u(X,t)$ has profound physical and geometric meaning; it also plays a central role in analyzing the statistical physics of the frame fields and generalizes manifolds to density manifolds.

D. Ricci-DeTurck Flow

In the previous subsection, from the viewpoint of frame field particles, $u(X^\mu, t)$ has a coarse-grained particle density interpretation. Equation (16) can also be interpreted as a manifold density [63] from the geometric point of view. For instance, $u(X,t)$ associates a manifold density or density bundle to each point X of a manifold, measuring the "fuzziness" of the "point" . It is worth stressing that the manifold density $u(X,t)$ is not simply a conformal scaling of a metric by a factor, since if that were the case, the integral measure of $D=4$-volume or 3-volume in the expectation $\langle O\rangle = \lambda \int d^{DX}\, uO$ would scale by different powers.

There are various useful generalizations of the Ricci curvature to density manifolds; a widely accepted version is the Bakry-Émery generalization [64], which is also used in Perelman' s seminal paper. The density-normalized Ricci curvature is bounded from below:

$$R_{\mu\nu} \to R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u, \quad R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u \geq \sigma_{\mu\nu},$$

if the density manifolds have finite volume.

As a consequence, replacing the Ricci curvature by the density-normalized one, we obtain the Ricci-DeTurck flow [65]:

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2\left(R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u\right),$$

which is equivalent to the standard Ricci flow equation (10) up to a diffeomorphism. Mathematically, the Ricci-DeTurck flow has the advantage that it turns out to be a gradient flow of some monotonic functionals introduced by Perelman, which have profound physical meanings as shown later.

Equations (14) and (16) also give a volume constraint to the fiducial spacetime (the lab): the coarse-grained density $u(X,t)$ cancels the flow of the volume element $\sqrt{|\det g_{\mu\nu}|}$, so

$$\lambda \int d^{4x} = \int d_t^{DX}\, u(X,t) = \int d^{DX}\, \sqrt{|\det g_{\mu\nu}|}\, u(X,t) = 1.$$

Together with the Ricci-DeTurck flow equation (19), we have the flow equation of the density:

$$\frac{\partial u}{\partial t} = (R - \Delta_X)u,$$

which is analogous to the irreversible Boltzmann equation for his distribution function. Δ_X is the Laplacian operator in terms of the manifold coordinates X. Note the minus sign in front of the Laplacian; it is a backwards heat-like equation. Naively speaking, the solution of the backwards heat flow will not exist. However, we could also note that if one lets the Ricci flow reach a certain IR scale t_*, and at t_* one might then choose an appropriate $u(t_*) = u_0$ arbitrarily (up to a diffeomorphism gauge) and flow it backwards in $\tau = t_* - t$ to obtain a solution $u(\tau)$ of the backwards equation. Now since the flow is considered free from global singularities due to the triviality of the homotopy group, we could simply choose $t_* = 0$, so we define

$$\tau = -t = \frac{1}{64\pi^2\lambda}k^2 \in (0, \infty).$$

In this case, the density satisfies the heat-like equation

$$\frac{\partial u}{\partial \tau} = (\Delta_X - R)u,$$

which does admit a solution along τ, often called the conjugate heat equation in mathematical literature.

Thus far, (23) together with (19) transforms the mathematical problem of the Ricci flow of a Riemannian/Lorentzian manifold into a coupled system:

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2\left(R_{\mu\nu} - \nabla_\mu \nabla_\nu \log u\right),$$
$$\frac{\partial u}{\partial \tau} = (\Delta_X - R)u$$

with $dt = -d\tau$, and the manifold $(\mathcal{M}^D, g)$ is generalized to a density manifold $(\mathcal{M}^D, g, u)$ [63, 66, 67] with the constraint (14).

III. The Anomaly and Its Implications

At the semi-classical approximation, as seen in eq. (8), when the quantum fluctuations of the frame fields or spacetime coordinates are ignored, the general coordinate transformation is just a classical coordinate transformation. We will show that when quantum fluctuations are taken into account in the general coordinate transformation beyond the semi-classical approximation, quantum anomalies emerge. As seen in the previous section, the quantum fluctuation and hence the coarse-graining process of the Ricci flow does not preserve the quadratic distance of a geometry, see (4) and (5). The non-isometry of the quantum fluctuation induces a breakdown of diffeomorphism or general coordinate transformation at the quantum level, namely the diffeomorphism anomaly.

In this section, we derive the diffeomorphism anomaly of the theory, show its relation to the Shannon entropy whose monotonicity gives an analogous H-theorem for the frame fields system and the Ricci flow. Furthermore, as the quantum frame fields theory describes a quantum spacetime, together with the generalized quantum Equivalence Principle, the anomaly-induced effective action in terms of the Shannon entropy can also be interpreted as a gravity theory, which at low-energy expansion is a scale-dependent Einstein-Hilbert action plus a cosmological constant. This part has some overlap with previous work [2]; for the self-containedness of the paper, we hope this section provides general background and lays the foundation for the subsequent thermodynamic and statistical interpretation of the theory.

A. Diffeomorphism at the Quantum Level

First we consider the functional quantization of the pure frame fields without explicitly incorporating matter sources. The partition function is

$$Z(\mathcal{M}^D) = \int [DX] \exp(-S[X]) = \int [DX] \exp\left(-\frac{\lambda}{2} \int d^{4x}\, g_{\mu\nu} \partial_a X^\mu \partial^a X^\nu\right)$$

where $\mathcal{M}^D$ is the target spacetime, and the base space can be either Euclidean or Minkowskian. Since considering the action or the volume element $d^{4x} \equiv d^{4x} \det e$ ($\det e$ is a Jacobian) does not pick up any imaginary i factor regardless of whether the base space is Minkowskian or Euclidean (if one takes $dx^{(E)}$ then $\det e^{(E)} \to -i \det e^{(M)}$), without loss of generality we use the Euclidean base spacetime in the following discussions, and remind that the result is the same for Minkowskian.

Note that a general coordinate transformation $X^\mu \to \hat{X}^\mu = \frac{\partial \hat{X}^\mu}{\partial X^\nu} X^\nu = e^\mu_\nu X^\nu$ does not change the action $S[X] = S[\hat{X}]$, but the measure of the functional integral changes:

$$D\hat{X} = \prod_x d\hat{X}^\mu(x) = \prod_x |\det e(x)| \prod_a dX^a(x) = \prod_x \sqrt{|\det g_{\mu\nu}|} \prod_a dX^a(x)$$

where $\sqrt{|\det g_{\mu\nu}|} = |\det e^\mu_a|$ is the Jacobian of the diffeomorphism. The Jacobian is nothing but a local relative (covariant basis) volume element $dV(\hat{X}^\mu)$ with respect to the fiducial volume $dV(X^a)$. Note that the normalization condition (14) also defines a fiducial volume element $ud^{4X} \equiv udV(\hat{X}^\mu)$, so the Jacobian is related to the frame fields density matrix:

$$u(\hat{X}^\mu) = \frac{dV(X^a)}{dV(\hat{X}^\mu)} = |\det e^\mu_a| = \sqrt{|\det g_{\mu\nu}|}.$$

Here the absolute value of the determinant is used because the density u and the volume element are kept positive-definite even in Lorentzian signature. Otherwise, for Lorentzian signature, one should introduce some extra imaginary factor i into (30) to maintain condition (14). The density so defined following (14) is an explicit generalization from the standard 3-space density to a 4-spacetime version. It is the definition of the volume form and the manifold density that ensures the formalism of the framework is formally identical to Perelman' s standard form even in Lorentzian signature. The manifold density encodes the most important information of a Riemannian or Lorentzian geometry, i.e., the local volume comparison.

In this case, if we parameterize a dimensionless solution u of the conjugate heat equation as

$$u(\hat{X}) = \frac{\lambda}{(4\pi\tau)^{D/2}} e^{-f(\hat{X})},$$

then the partition function $Z(\mathcal{M}^D)$ transforms to

$$Z(\hat{\mathcal{M}}^D) = \int [D\hat{X}] \exp(-S[\hat{X}]) = \int [DX] |\det e| \exp(-S[X]) = \exp\left(\lambda \int d^{DX}\, u \log u\right) \int [DX] \exp(-S[X]).$$

Note that $N(\hat{\mathcal{M}}^D)$ in the exponential of the change of the partition function $Z(\hat{\mathcal{M}}^D) = e^{\lambda N(\hat{\mathcal{M}}^D)} Z(\mathcal{M}^D)$ is nothing but a pure real Shannon entropy in terms of the density matrix u:

$$N(\hat{\mathcal{M}}^D) = \int d^{DX}\, u \log u.$$

The classical action $S[X]$ is invariant under general coordinate transformation or diffeomorphism, but the quantum partition function is no longer invariant under general coordinate transformation or diffeomorphism, which is called diffeomorphism anomaly, meaning a breakdown of diffeomorphism at the quantum level. The diffeomorphism anomaly is purely due to the quantum fluctuation and Ricci flow of the frame fields which do not preserve the functional integral measure and change the spacetime volume at the quantum level. The diffeomorphism anomaly has many profound consequences for the theory of quantum reference frames, e.g., non-unitarity, the trace anomaly, the notion of entropy, reversibility, and the cosmological constant.

The non-unitarity is indicated by the pure real anomaly term, which is also induced by the non-isometry or volume change, and consequently the non-invariance of the measure of the functional integral during the Ricci flow. Because of the real-valued volume form (29) for both Euclidean and Lorentzian signatures, the pure real contribution of the anomaly and hence the non-unitarity are valid not only for spacetime with Euclidean but also for Lorentzian signature; it is a rather general consequence of the Ricci flow of spacetime. Essentially speaking, the reason for the non-unitarity is that we have enlarged the Hilbert space of the reference frame from a rigid classical frame to a fluctuating quantum frame. The non-unitarity implies the breakdown of the fundamental Schrödinger equation, which is only valid on classical time of an inertial frame, the solution of which lies in $\mathcal{H}_\psi$. A fundamental equation playing the role of the Schrödinger equation, which can arbitrarily choose any (quantum) physical system as time or reference frame, must be replaced by a Wheeler-DeWitt-like equation in some sense [11], the solution of which lies instead in $\mathcal{H}_\psi \otimes \mathcal{H}_X$. In this fundamental equation, the quantum fluctuation of physical time and frame, more generally a general physical coordinate system, must break unitarity. We know that in quantum field theory on curved spacetime or accelerating frames, the vacuum states of quantum fields in diffeomorphism-equivalent coordinate systems are unitarily inequivalent. The Unruh effect is a well-known example: accelerating observers in the vacuum will measure a thermal bath of particles. The Unruh effect shows how a general coordinate transformation (e.g., from an inertial to an accelerating frame) leads to the non-unitary anomaly (particle creation and hence particle number non-conservation), and how the anomaly relates to a thermodynamic system (thermal bath). In fact, like the Unruh effect, the Hawking effect [68] and all non-unitary particle creation effects in curved spacetime or accelerating frames are related to the anomaly in a general covariant or gravitational system. All these imply that the diffeomorphism anomaly will have a deep thermodynamic interpretation, which is the central issue of this paper.

Without loss of generality, if we simply consider the under-transformed coordinates X^μ identifying with the coordinates of the fiducial lab x^a which can be treated as classical parameter coordinates, in this situation the classical action of the NLSM is just a topological invariant, i.e., half the dimension of the target spacetime:

$$\exp(-S_{\text{cl}}) = \exp\left(-\frac{\lambda}{2}\int d^{4x}\, g_{\mu\nu}\partial_a x^\mu \partial^a x^\nu\right) = \exp\left(-\frac{\lambda}{2}\int d^{4x}\, g_{\mu\nu} g^{\mu\nu}\right) = e^{-\frac{\lambda D}{2}\int d^{4x}}.$$

Thus the total partition function of the frame fields takes a simple form:

$$Z(\hat{\mathcal{M}}^D) = e^{\lambda N(\hat{\mathcal{M}}^D) - \frac{\lambda D}{2}\int d^{4x}}.$$

B. The Trace Anomaly

The partition function is now non-invariant (32) under diffeomorphism at the quantum level, so if one deduces the stress tensor by $\langle T_{\mu\nu}\rangle = -\frac{2}{\sqrt{g}}\frac{\delta \log Z}{\delta g^{\mu\nu}}$, its trace $\langle g^{\mu\nu}\rangle\langle T_{\mu\nu}\rangle = 0$ differs from $\langle g^{\mu\nu} T_{\mu\nu}\rangle = \langle T^\mu_\mu\rangle$, giving the trace anomaly:

$$\langle \Delta T^\mu_\mu\rangle = \langle g^{\mu\nu}\rangle\langle T_{\mu\nu}\rangle - \langle g^{\mu\nu} T_{\mu\nu}\rangle = \lambda N(\mathcal{M}^D)$$

known as the trace anomaly. Cardy conjectured [69] that in a $d = 4$ theory, quantities like $\langle T^\mu_\mu\rangle$ could be a higher-dimensional generalization of the monotonic Zamolodchikov' s c-function in $d = 2$ conformal theories, leading to a suggestion of the a-theorem [70] in $d = 4$ and other suggestions (e.g., [71, 72]). In the following subsections, we will show that the Shannon entropy N and generalized $\tilde{N}$ are indeed monotonic, which might have more advantages, e.g., being suitable for Lorentzian target spacetime and for general D.

Note that the Shannon entropy $N(\mathcal{M}^D)$ can be expanded at small τ:

$$\lambda N(\hat{\mathcal{M}}^D) = \lambda \sum_n B_n \tau^n = \lambda(B_0 + B_1\tau + B_2\tau^2 + \dots) \quad (\tau \to 0).$$

For $D = 4$ the first few coefficients are:

$$B_0 = \lim_{\tau\to 0} \int d^{4X} \sqrt{|g|} \left\langle 1 + \log\left(\sqrt{\frac{\lambda}{4\pi\tau}}\right)\right\rangle,$$

$$B_1 = \lim_{\tau\to 0} \int d^{4X} \sqrt{|g|} \left\langle R + |\nabla f|^2 \right\rangle,$$

$$B_2 = \lim_{\tau\to 0} \int d^{4X} \sqrt{|g|} \left\langle \frac{1}{2}|R_{\mu\nu} + \nabla_\mu\nabla_\nu f|^2 \right\rangle,$$

where B_0 can be renormalized away, and a renormalized B_1 will contribute to the effective Einstein-Hilbert action of gravity, see the following subsection D. And B_2, as a portion of the full anomaly, plays the role of the conformal/Weyl anomaly up to some total divergence terms, for instance, ΔR terms and the Gauss-Bonnet invariant. That is, a non-vanishing B_2 term measures the breakdown of conformal invariance of $\mathcal{M}^{D=4}$; otherwise, a vanishing B_2 means that the manifold is a gradient steady Ricci soliton as the fixed point of the Ricci-DeTurck flow, which preserves its shape (conformal invariant) during the flows.

We note that B_2 as the only dimensionless coefficient measures the anomalous conformal modes; in this sense, $N(\mathcal{M}^D)$ indeed relates to certain entropy. However, since the conformal transformation is just a special coordinate transformation, it is clear that the single B_2 coefficient does not measure the total (general coordinate transformation) anomalous modes. Obviously this theory at $2 < d = 4 - \epsilon$ is not conformally invariant, thus as the theory flows along t, the degrees of freedom are gradually coarse-grained and hence the mode-counting should also change with the flow and the scale. Consequently, all coefficients B_n in the series and hence the total partition function $e^{\lambda N(\mathcal{M}^D)}$ should measure the total anomalous modes at a certain scale τ, leading to the full entropy and anomaly.

Different from some classically conformally invariant theories, e.g., string theory, in which we only need to cancel a single scale-independent B_k coefficient to avoid conformal anomaly, as the theory in higher than 2 dimensions is not conformally invariant, the full scale-dependent anomaly $N(\mathcal{M}^D)$ is required to be canceled at a certain scale. Fortunately, it will be shown in a later subsection that a normalized full anomaly $\lambda\tilde{N}(\mathcal{M}^D)$ can converge at UV due to its monotonicity, thus giving rise to a finite counterterm of order $\mathcal{O}(\lambda)$ playing the role of a correct cosmological constant. The idea that the trace anomaly might have a relation to the cosmological constant is a recurring subject in the literature [73-77]; in this framework, the cosmological constant naturally emerges in this way as the counterterm of the trace anomaly (see subsection D or [2]).

C. Relative Shannon Entropy and an H-Theorem for Non-Equilibrium Frame Fields

In the Ricci flow limit, i.e., the Gradient Shrinking Ricci Soliton (GSRS) configuration, the Shannon entropy N takes its maximum value N_*, similar to a thermodynamic system being in a thermal equilibrium state where its entropy is also maximal. In mathematical literature on Ricci flow, it is common to define a series of relative quantities with respect to the extreme values taken by the flow limit GSRS or analogous thermal equilibrium state, denoted by a subscript $*$.

In GSRS, the covariance matrix $\sigma_{\mu\nu}$ as the second central moment of the frame fields with an IR cutoff k is simply proportional to the metric:

$$\langle \delta X^\mu \delta X^\nu \rangle = \int_{|p|=k} \frac{d^{4p}}{(2\pi)^4} \frac{1}{p^2} g^{\mu\nu} = \tau g^{\mu\nu},$$

and then

$$\sigma_*^{\mu\nu} = (\sigma^*_{\mu\nu})^{-1} = \frac{1}{\tau} g^{\mu\nu},$$

which means a uniform Gaussian broadening is achieved. In this gauge, only the longitudinal part of the fluctuation exists.

When the density-normalized Ricci curvature is completely given by the longitudinal fluctuation $\sigma_{\mu\nu}$, i.e., inequality (18) saturates, we obtain a Gradient Shrinking Ricci Soliton (GSRS) equation:

$$R_{\mu\nu} + \nabla_\mu \nabla_\nu f = \frac{1}{2\tau} g_{\mu\nu}.$$

This means that, on the one hand, for a general $f(X) = \frac{1}{4\tau}|\sigma_{\mu\nu}X^\mu X^\nu|$, $R_{\mu\nu}$ seems to vanish, so the standard Ricci flow equation (10) terminates; and on the other hand, the Ricci-DeTurck flow (19) only changes the longitudinal size or volume of the manifolds while keeping its shape unchanged, thus the GSRS can also be seen as stopping changing, up to a size or volume rescaling. Therefore, the GSRS is a flow limit and can be viewed as a generalized RG fixed point.

In the following, we consider relative quantities with respect to the GSRS configuration. Considering a general Gaussian density matrix, in the GSRS limit it becomes

$$u(X)=\frac{\lambda}{(2\pi)^{D/2}\sqrt{|\det\sigma_{\mu\nu}|}\sqrt{|\det g_{\mu\nu}|}}\exp\left(-\frac{1}{2}|X^\mu\sigma_{\mu\nu}X^\nu|\right),$$

$$u_*(X)=\frac{\lambda}{(4\pi\tau)^{D/2}}\exp\left(-\frac{|X|^2}{4\tau}\right).$$

Therefore, in GSRS, a relative density can be defined by the general Gaussian density $u(X)$ relative to the density $u_*(X)$ in GSRS:

$$\tilde{u}(X)=\frac{u(X)}{u_*(X)}.$$

Using the relative density, a relative Shannon entropy $\tilde{N}$ can be defined by

$$\tilde{N}(\mathcal{M}^D)=-\int d^{DX}\,\tilde{u}\log\tilde{u}=-\int d^{DX}\,u\log u+\int d^{DX}\,u_*\log u_*=N-N_*=-\log Z_P\leq 0,$$

where Z_P is nothing but Perelman' s partition function:

$$\log Z_P=\int d^{DX}\,u_*\left(\frac{D}{2}+\log\frac{\lambda}{4\pi\tau}\right),$$

and N_* is the maximum Shannon entropy:

$$N_*=-\int d^{DX}\,u_*\log u_*=\int d^{DX}\,u_*\left[1+\log\left(\frac{\lambda}{4\pi\tau}\right)\right].$$

Since the relative Shannon entropy and the anomaly term are purely real, the change of the partition function under diffeomorphism is non-unitary. For the coarse-graining nature of the density u, it is proved that the relative Shannon entropy is monotonic non-decreasing along the Ricci flow (along t):

$$\frac{d\tilde{N}(\hat{\mathcal{M}}^D)}{dt}=-\tilde{F}\geq 0,$$

where $\tilde{F}=F-F_*\leq 0$ is the GSRS-normalized F-functional of Perelman:

$$F(u)=\int d^{DX}\,u\,(R+|\nabla f|^2)$$

with the maximum value (at GSRS limit) $F_* \equiv F(u_*) = \frac{D}{2\tau}$.

Inequality (50) gives an analogous H-theorem for the non-equilibrium frame fields and the irreversible Ricci flow. The entropy is non-decreasing along the Ricci flow, making the flow irreversible in many aspects similar to processes of irreversible thermodynamics, meaning that as the observation scale of the spacetime flows from short to long distance scales, the process loses information and the Shannon entropy increases. The equality in (50) is achieved when the spacetime configuration has flowed to a limit known as a Gradient Shrinking Ricci Soliton (GSRS), when the Shannon entropy takes its maximum value. Similarly, at the flow limit the density matrix u_* (eq. 45) takes the analogous standard Maxwell-Boltzmann distribution.

D. Effective Gravity at Cosmic Scale and the Cosmological Constant

In terms of the relative Shannon entropy, the total partition function (35) of the frame fields is normalized by the GSRS extreme value:

$$Z(\mathcal{M}^D) = e^{\lambda N - \frac{\lambda D}{2}\int d^{4x}} = e^{\lambda \tilde{N} - \frac{\lambda D}{2}\int d^{4x}} = Z_P^{-\lambda} e^{-\frac{\lambda D}{2}\int d^{4x}} = \exp\left(\int d^{DX}\, u(f-D)\right).$$

The relative Shannon entropy $\tilde{N}$ as the anomaly vanishes at GSRS or IR scale, but it is non-zero at ordinary lab scale up to UV where the fiducial volume of the lab is considered fixed $\lambda \int d^{4x} = 1$. The cancellation of the anomaly at the lab scale up to UV is physically required, which leads to the counterterm $\nu(\mathcal{M}^D_{\tau=\infty})$ or cosmological constant. The monotonicity of $\tilde{N}$ (eq. 50) and the W-functional implies [3, 78]:

$$\nu(\mathcal{M}^D_{\tau=\infty}) = \lim_{\tau\to\infty} \lambda \tilde{N}(\mathcal{M}^D, u, \tau) = \lim_{\tau\to\infty} \lambda W(\mathcal{M}^D, u, \tau) = \inf_{\tau} \lambda W(\mathcal{M}^D, u, \tau) < 0,$$

where W, Perelman' s W-functional, is the Legendre transformation of $\tilde{N}$ with respect to τ^{-1}:

$$W \equiv \tau \frac{d\tilde{N}}{d\tau} + \tilde{N} = \tau \tilde{F} + \tilde{N} = \int d^{DX}\, u\left[\tau(R + |\nabla f|^2) + f - D\right].$$

In other words, the difference between the effective actions (relative Shannon entropies) at UV and IR is finite:

$$\nu = \lambda(\tilde{N}_{\mathrm{UV}} - \tilde{N}_{\mathrm{IR}}) < 0.$$

Perelman used his analogies: temperature $T \sim \tau$, (relative) internal energy $U \sim -\tau^2 \tilde{F}$, thermodynamic entropy $S \sim -W$, and free energy $\mathcal{F} \sim \tau \tilde{N}$, up to proportional factors balancing dimensions on both sides of $\sim$. Equation (55) is analogous to the thermodynamic equation $U - TS = \mathcal{F}$. So in this sense the W-functional is also called the W-entropy. Whether the thermodynamic analogies are real and physical, or just pure coincidences, is an important issue discussed in the next sections.

In fact, $e^{\nu} < 1$ (usually called the Gaussian density [79, 80]) is a relative volume or the reduced volume $\tilde{V}(\mathcal{M}^D_{\tau=\infty})$ of the backwards limit manifolds introduced by Perelman, or the inverse of the initial condition of the manifold density at $\tau = 0$. A finite value of it makes an initial spacetime with unit volume flow from UV and converge to a finite $u_{\tau=0}$, and hence the manifold finally converges to a finite relative volume/reduced volume instead of shrinking to a singular point at $\tau = 0$.

As an example, for a homogeneous and isotropic universe for which the sizes of space and time (with a "ball" radius $a(\tau)$) are on an equal footing, i.e., a late-epoch FRW-like metric $ds^2 = a^2(\tau)(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2)$, which is a Lorentzian shrinking soliton configuration. Note that the shrinking soliton equation $R_{\mu\nu} = \frac{1}{2\tau} g_{\mu\nu}$ (29) is independent of the signature, so it can be approximately given by a 4-ball value $\nu(B^4_\tau) \approx -0.8$ [1, 2].

Thus the partition function, which is anomaly-canceled at UV and has a fixed-volume fiducial lab, is

$$Z(\mathcal{M}^D) = e^{\lambda \tilde{N} - \frac{\lambda D}{2}\int d^{4x}} = e^{-\nu}.$$

Since $\lim_{\tau\to 0} \tilde{N}(\mathcal{M}^D) = 0$, at small τ, $\tilde{N}(\mathcal{M}^D)$ can be expanded in powers of τ:

$$\tilde{N}(\mathcal{M}^D) = \int d^{DX}\, u_{\tau\to 0} \left[f_{\tau\to 0} - \frac{D}{2} \right] = \tau \tilde{F} + \mathcal{O}(\tau^2) = \tau \int d^{DX}\, u_{\tau\to 0} \left(R_{\tau\to 0} + |\nabla f_{\tau\to 0}|^2 \right) + \mathcal{O}(\tau^2),$$

in which $\lambda \int d^{DX}\, u_{\tau\to 0} \tau |\nabla f_{\tau\to 0}|^2 = \frac{D}{2}$ (at GSRS) has been used.

For $D = 4$ and small τ, the effective action of $Z(\mathcal{M}^4)$ can be given by

$$-\log Z(\mathcal{M}^4) = S_{\text{eff}} \approx \int d^{4X}\, u_0 \left(2\lambda - \lambda R_0 \tau + \lambda\nu \right) \quad (\text{small } \tau).$$

Considering $u_0 d^{4X} = \sqrt{|g_t|} dV = \sqrt{|g_t|} dX^0 dX^1 dX^2 dX^3$ is the invariant volume element, and using (22) to replace t or τ by cutoff scale k, we have

$$S_{\text{eff}} = \int dV \sqrt{|g_k|} \left(2\lambda - \frac{64\pi^2}{k^2} + \lambda\nu \right) \quad (\text{small } k).$$

The effective action can be interpreted as a low-energy effective action of pure gravity. As the cutoff scale k ranges from the lab scale to the solar system scale ($k > 0$), the action must recover the well-tested Einstein-Hilbert (EH) action. But at cosmic scale ($k \to 0$), we know that the EH action deviates from observations and the cosmological constant becomes important. In this picture, as $k \to 0$, the action leaving $2\lambda + \lambda\nu$ should play the role of the standard EH action with a limiting constant background scalar curvature R_0 plus the cosmological constant, so

$$2\lambda + \lambda\nu = R_0 - 2\Lambda.$$

While at $k \to \infty$, $\lambda\tilde{N} \to \nu$, the action leaving only the fiducial Lagrangian $2\lambda = \frac{D}{2}\lambda$ should be interpreted as a constant EH action without the cosmological constant. Thus we have the cosmological term:

$$\Lambda = -\lambda\nu(B_\infty^4) \approx 0.8\rho_c,$$

where ρ_c is the critical density. The action can be rewritten as an effective EH action plus a cosmological term:

$$S_{\text{eff}} = \int dV \sqrt{|g_k|} \left(\frac{R_k}{16\pi G_k} - 2\Lambda \right) \quad (\text{small } k),$$

where

$$\frac{1}{16\pi G_k} = \frac{64\pi^2}{k^2}, \quad \Lambda = 2\lambda + \lambda\nu,$$

which is nothing but the flow equation of the scalar curvature [43]:

$$\frac{dR_k}{dk^2} = \frac{1}{4\pi G_k} \quad \text{or} \quad R_\tau = \frac{1}{\tau} + \frac{D}{2\tau} R_0 \tau.$$

Since at cosmic scale $k \to 0$, the effective scalar curvature is bounded by R_0 which can be measured by Hubble's constant H_0 at the cosmic scale, λ is nothing but the critical density of the 4-spacetime Universe:

$$R_0 = D(D-1)H_0^2 = 12H_0^2 = \rho_c,$$

so the cosmological constant is always of order of the critical density with a "dark energy" fraction $\Omega_\Lambda = -\nu \approx 0.8$, which is close to the observational value. Detailed discussions about the cosmological constant problem and observational effects in cosmology, especially the modification of the distance-redshift relation leading to the acceleration parameter $q_0 \approx -0.68$, can be found in [1, 2, 12, 13].

If matter is incorporated into the gravity theory, consider the entangled system in $\mathcal{H}_\psi \otimes \mathcal{H}_X$ between the to-be-studied quantum system (matter) and the quantum reference frame fields system (gravity). The 2λ term in eq. (8) is normalized by the Ricci flow. Using eq. (60) and eq. (65), matter-coupled gravity emerges from the Ricci flow:

$$S[\psi, X] = \int dV \sqrt{|g_k|} \left[\frac{1}{2} g^{\mu\nu} \frac{\delta\psi}{\delta X^\mu} \frac{\delta\psi}{\delta X^\nu} - V(\psi) + 2\lambda - \frac{64\pi^2}{k^2} + \lambda\nu \right].$$

IV. Thermal Equilibrium State

A Gradient Shrinking Ricci Soliton (GSRS) configuration as a Ricci flow limit extremizes the Shannon entropy N and the W-functional. Similarly, a thermal equilibrium state also extremizes Boltzmann' s H-functional and the thermodynamic entropy. Thus the process of a generic Ricci flow evolving into a GSRS limit is analogous to a non-equilibrium state evolving into a thermal equilibrium state; they are not merely similar but even equivalent when the thermal system is precisely the frame fields system. In this section, following the previous discussions on the non-equilibrium state of the frame fields in 4 dimensions, in a proper choice of time, we will discuss the thermal equilibrium state of the frame particle system as a GSRS configuration in lower 3 dimensions, in which the temperature and several thermodynamic functions of the system can be explicitly calculated and the manifold density can be interpreted as the thermal ensemble density of the frame fields particles, giving a statistical interpretation to Perelman' s thermodynamic analogies of the Ricci flow.

A. A Temporal Static Shrinking Ricci Soliton as a Thermal Equilibrium State

When the shrinking Ricci soliton $\mathcal{M}^4$ is static in the temporal direction, i.e., being a product manifold $\mathcal{M}^4 = \mathcal{M}^3 \times \mathbb{R}$ with $\delta X/\delta X^0 = 0$, where $X^0 \in \mathbb{R}$ is the physical time and $X = (X^1, X^2, X^3) \in \mathcal{M}^3$ is a 3-space gradient shrinking Ricci soliton of lower dimension, we can prove here that the temporal static spatial part $\mathcal{M}^3$ is in thermal equilibrium with the flow parameter τ proportional to its temperature, and the manifold density u of $\mathcal{M}^3$ can be interpreted as the thermal equilibrium ensemble density.

According to Matsubara' s formalism of thermal field theory, the thermal equilibrium of the spatial frame fields can be defined by the periodicity $X(x, 0) = X(x, \beta)$ in their Euclidean time of the lab (recall that we start from the Euclidean base space for the frame fields theory), where $\beta = 1/T$ is the inverse temperature. Now the frame fields are a mapping $\mathbb{R}^3 \times S^1 \to \mathcal{M}^3 \times \mathbb{R}$. Then in such a configuration, the τ parameter of the 3-space shrinking soliton $\mathcal{M}^3$ becomes

$$\tau = \int \frac{d^{3p}\, d\omega_n}{(2\pi)^4} \frac{1}{p^2 + \omega_n^2} = \int \frac{d^{3p}}{(2\pi)^3} \frac{1}{p^2 + (2\pi nT)^2},$$

where $\omega_n = 2\pi nT$, and $\sum_n \int d\omega_n$ have been used. The calculation is a periodic-Euclidean-time version of the general eq. (41). Since the density matrix eq. (45) of the frame fields X^μ is Gaussian or a coherent state, in which the oscillators are almost condensed in the central peak, thus $\omega_0 = 0$ dominates the Matsubara sum:

$$\tau = \frac{1}{2\pi T} \int \frac{d^{3p}}{(2\pi)^3} \frac{1}{p^2} = \frac{T}{12\pi^2 \lambda_3},$$

where the 3-space energy density is $\lambda_3 = \lambda / \int dx^0$. Note that this differs from the naive notion of "temporal static" at the classical level, which means $\langle \delta X / \delta X^0 \rangle = 0$ with respect to the physical clock X^0 of the quantum reference frame. However, the notion of "temporal static" is a little subtle at the quantum level. Because there is no "absolute static" at the quantum or microscopic level, since at such microscopic scales the modes are always in motion or vibrating with respect to the infinitely precise lab time x^0, i.e., $\partial X(x)/\partial x^0 \neq 0$. Actually $\partial X/\partial x^0$ is generally non-zero even though its oscillation degrees of freedom are almost frozen (Matsubara frequency ω_n is zero for the Gaussian wavefunction), while the center of the Gaussian wave packet of X is in translational motion so $p \neq 0$, so its expectation value is in general finite, for instance, $\langle \partial X(x)/\partial x^0 \rangle \sim 3T < \infty$ claimed by the equipartition energy of the translational motion in 3-space.

In general, whether or not the modes of the spatial frame fields are temporal static depends on the scale at which one evaluates the average of the physical clock $\langle X^0 \rangle$. The notion of "thermal static" in the sense of statistical physics is approximate at a macroscopic scale rather than a microscopic scale, at which scale the molecules are always in motion (as is the physical clock X^0). The macroscopic scale of the thermal static system is at such a long physical time scale $\delta\langle X^0 \rangle \gg \delta x^0$ that the averaged physical clock is almost frozen $\partial\langle X^0 \rangle/\partial x^0 \to 0$ with respect to the infinitely precise lab time x^0, so that the thermal static condition $\langle \delta X/\delta X^0 \rangle = 0$ can be achieved. More precisely, when we mention that the 3-space is macroscopically "temporal static", an IR cutoff, for example, H_0 as a macroscopic Hubble scale should be taken into account. The fluctuation modes on the 3-space outside the Hubble scale $0 < |p| < H_0$ are frozen and temporal static, while those modes $|p| > H_0$ inside the Hubble horizon are dynamic. So with this cutoff scale we have

$$\frac{\partial\langle X^0 \rangle}{\partial x^0} \to 0,$$

and

$$\langle \delta X/\delta X^0 \rangle = \langle \partial X/\partial x^0 \rangle \cdot \partial x^0/\partial \langle X^0 \rangle \sim \frac{3}{12\pi^2\lambda_3} \int_{|p|=H_0} d^{3p} = \frac{T}{H_0} = 1,$$

where the 3-space energy density is $\lambda_3 = \lambda / \int dx^0$. Note that if we consider the temporal integral is also cutoff at about a long physical time scale, e.g., the age of the universe $\mathcal{O}(1/H_0)$, let the temporal direction be normalized as $\int_0^{12\pi^2/H_0} dx^0 H_0 = 1$, then the condition $\int d^{4x}\lambda \equiv 1$ gives its 3-space version $\int d^{3x}\lambda_3 = 1$, which is the definition of λ_3 on the 3-space slice generalizing the critical density λ in a 4-spacetime covariant theory.

It is worth stressing that since the spatial slice depends on the definition of time, the value of λ_3 is not universal (not necessarily equal to the above $12\pi^2\lambda$ in other frames or cutoffs, unlike the universal 4-spacetime critical density λ) but frame-dependent. If a specific gauge of time or frame is chosen, λ_3 could be considered fixed and used as a proportionality factor to correlate the τ parameter with the temperature of the temporal static frame fields configuration in that specific gauge of time. The 3-space energy density λ_3 is very useful when we consider a temporal static GSRS spacetime or the corresponding thermal equilibrium frame fields ensemble in later discussions.

In summary, an important observation is that when $\mathcal{M}^3$ is a shrinking Ricci soliton in a temporal static product shrinking soliton $\mathcal{M}^3 \times \mathbb{R}$, the global τ parameter of $\mathcal{M}^3$ can be interpreted as a thermal equilibrium temperature defined by the Euclidean time periodicity of the frame fields, up to a proportionality factor being a 3-space energy density λ_3 (satisfying eq. 74) balancing the dimensions between τ and T. Since temperature T is frame-dependent, so is the proportionality factor λ_3. This observation gives us a reason why in Perelman' s paper τ could be analogous to temperature T. The same results can also be obtained if one uses the Lorentzian signature for the lab or base spacetime of the frame fields theory (2). In this case the thermal equilibrium of the spatial frame fields is instead subject to periodicity in the imaginary Minkowskian time $X(x, 0) = X(x, i\beta)$, but even though the base spacetime is Wick-rotated, the path integral does not pick up any imaginary i factor in front of the action in (25) as the starting point, so the main results of the discussions remain independent of the signature of the base spacetime.

B. Thermodynamic Functions

For the thermodynamic interpretation of the quantum reference frame and gravity theory, in this subsection we derive other thermodynamic functions of the system besides the temperature in the previous subsection, which are similar to those of an ideal gas. Thus the frame fields system in the Gaussian approximation can be seen as a system of frame fields gas, which manifests an underlying statistical picture of Perelman' s thermodynamic analogies of his functionals.

As convention, we take the temperature $T = \lambda_3\tau$ (eq. 73), $D = 3$, and replace

λ by λ_3; this is equivalent to choosing a specific gauge of time for the thermal equilibrium frame fields configuration.

When the spatial shrinking soliton $\mathcal{M}^3$ is temporal static ($dX^0 = 0$) and in thermal equilibrium, the partition function of the thermal ensemble of the frame fields X can be given by the trace/integration of the density matrix:

$$Z_*(\tau) = \lambda_3 \int d^{3X}\, u(X) = \lambda_3 \int d^{3X}\, e^{-X^2/4\tau} = \lambda_3 (4\pi\tau)^{3/2},$$

and the normalized density u can be given by the 3-dimensional version of eq. (45):

$$u_*(X) = u(X) = \frac{\lambda_3}{(4\pi\tau)^{3/2}} e^{-X^2/4\tau}.$$

The partition function can also be consistently given by (35) with $D = 3$ in thermal equilibrium:

$$Z_*(\tau) = e^{\lambda_3 N_*(\mathcal{M}^3) - 3/2} = \exp\left(\int d^{3X}\, u_* \log u_* - \frac{3}{2}\right) = \lambda_3 (4\pi\tau)^{3/2} = V_3 \left(\frac{4\pi\lambda_3^{1/3}}{V_3^{2/3}}\tau\right)^{3/2} = Z_*(\beta),$$

where $V_3 = \int d^{3x}$ is the 3-volume with the constraint $\lambda_3 V_3 = 1$. The partition function is identified with the partition function of the canonical ensemble of an ideal gas (i.e., non-interacting frame fields gas in the lab) of temperature $1/\beta$ and gas particle mass $\lambda_3^{1/3}$. The interactions are effectively absorbed into the broadening of the density matrix and normalized mass of the frame fields gas particles.

The physical picture of frame fields gas in thermal equilibrium lays a statistical and physical foundation for Perelman' s analogies between his functionals and thermodynamic equations, as detailed below.

The internal energy of the frame fields gas can be given similarly to the standard internal energy of an ideal gas:

$$E_* = -\frac{\partial \log Z_*}{\partial \beta} = \frac{3\tau}{2}\lambda_3 F_* = \frac{3}{2}\lambda_3 \tau = \frac{3}{2} T,$$

given by the equipartition energy of translational motion in 3-space, where (52) with $D = 3$ and $\lambda \to \lambda_3$ have been used.

The fluctuation of the internal energy is given by

$$\langle E^2\rangle - \langle E_*\rangle^2 = \frac{\partial^2 \log Z_*}{\partial\beta^2} = \frac{3\tau^2}{2} = \frac{3}{2}T^2.$$

The Fourier transform of the density $u_*(X)$ is given by

$$u_*(K) = \int d^{3X}\, u_*(X)e^{-iK\cdot X} = e^{-\tau K^2},$$

since u satisfies the conjugate heat equation (23), so K^2 is the eigenvalue of the Laplacian $-4\Delta_X + R$ of the 3-space, taking the value of the F-functional:

$$K^2 = \lambda_3 \int d^{3X}\,(R|\Psi|^2 + 4|\nabla\Psi|^2) = \lambda_3 F, \quad u_*(K^2) = e^{-\lambda_3\tau F}.$$

For a state taking energy $\lambda_3^2\tau^2 F = E$, the probability density of the state can be rewritten as

$$u_*(E) = e^{-E/\lambda_3\tau} = e^{-E/T},$$

which is the standard Boltzmann probability distribution of the state. Thus we can see that the (Fourier-transformed) manifold density can be interpreted as the thermal equilibrium canonical ensemble density of the frame fields.

The free energy is given by

$$\mathcal{F}_* = -\frac{1}{\beta}\log Z_* = -\lambda_3\tau\log Z_* = -\frac{3}{2}\lambda_3\tau\log(4\pi\tau),$$

similar to the standard free energy of an ideal gas $-\frac{3}{2}T\log T$ up to a constant.

The minus H-functional of Boltzmann at an equilibrium limit and the thermal entropy of the frame fields gas can be given by the Shannon entropy:

$$\lambda_3 N_* = S_* = -\lambda_3\int d^{3X}\, u_*\log u_* = \frac{3}{2}[1+\log(4\pi\tau)],$$

similar to the thermal entropy of a fixed-volume ideal gas $\frac{3}{2}\log T + \frac{3}{2}$ up to a constant. The thermal entropy can also be consistently given by the standard formula:

$$S_* = \log Z_* - \beta\frac{\partial\log Z_*}{\partial\beta} = \frac{3}{2}[1+\log(4\pi\tau)],$$

which is analogous to the fact that the W-functional is the Legendre transformation of the relative Shannon entropy with respect to τ^{-1}. For this reason, the

W-functional is also an entropy function related to the (minus) thermodynamic entropy.

In summary, we have seen that, under general frame fields (coordinates) transformation, the Shannon entropy anomaly N appearing in the partition function (32) (or relative Shannon entropy $\tilde{N}$ with respect to N_*) has profound thermodynamic interpretations. The Ricci flow of frame fields leads to non-equilibrium and equilibrium thermodynamics of quantum spacetime. We summarize the comparisons between them in Tables I and II.

Table I: Frame fields in general Ricci flow at non-flow-limit and non-equilibrium thermodynamics.

Frame fields at non-Ricci-flow-limit	Non-equilibrium thermodynamics
Relative Shannon entropy: $\tilde{N} = -\int d^{3X}\, \tilde{u}(X,t) \log \tilde{u}(X,t)$	Boltzmann H-function: $H(t) = \int d^{3v}\, \rho(v,t) \log \rho(v,t)$
Ricci flow parameter: t	Newtonian time: t
Monotonicity: $d\tilde{N}/dt = -\tilde{F} \geq 0$	H-theorem: $dH/dt \leq 0$
Conjugate heat equation: $\partial u/\partial t = (-\Delta + R)u$	Boltzmann equation of ideal gas: $\partial\rho/\partial t = -v \cdot \nabla\rho$

Table II: Frame fields in Gradient Shrinking Ricci Soliton (GSRS) configuration and equilibrium thermodynamics of ideal gas.

Frame fields at Ricci-flow-limit (GSRS)	Equilibrium thermodynamics of ideal gas
Partition function: $Z_*(\tau) = \lambda_3(4\pi\tau)^{3/2}$	Partition function: $Z(T) = V_3(2\pi mT)^{3/2}$
GSRS flow parameter: $\lambda_3\tau$	Temperature: $T = \beta^{-1}$
Internal energy: $E_* = \frac{3\tau}{2}\lambda_3 F_* = \frac{3}{2}\lambda_3\tau$	Internal energy: $E = \frac{3}{2}T$
Manifold density: $u_*(K) = e^{-\tau K^2} = e^{-\lambda_3\tau F}$	Canonical ensemble density: $\rho = e^{-E/T}$
Free energy: $\mathcal{F}_* = -\lambda_3\tau \log Z_* = -\frac{3}{2}\lambda_3\tau \log(4\pi\tau)$	Free energy: $\mathcal{F} = -T \log Z(T) = -\frac{3}{2}T \log T$
Shannon entropy: $\lambda_3 N_* = \frac{3}{2}[1 + \log(4\pi\tau)]$	Thermodynamic entropy: $S_* = \frac{3}{2}(1 + \log T)$
W-functional: $W = \tau d\tilde{N}/d\tau + \tilde{N}$	First law: $E - TS = \mathcal{F}$
Monotonicity: $d\tilde{N}/dt \geq 0$	Second law: $\delta S \geq 0$

V. Application to the Schwarzschild Black Hole

In this section, we apply the general statistical and thermodynamic interpretation of the quantum frame fields to a physical gravitational system, as a touchstone of quantum gravity, i.e., to understand the statistical origin of the thermodynamics of the Schwarzschild black hole.

A. The Temperature of a Schwarzschild Black Hole

The region in the vicinity of the origin of a Schwarzschild black hole is an example of a classical static shrinking Ricci soliton. A rest observer distant from it sees an approximate metric $\mathcal{M}^3 \times \mathbb{R}$, where the region in the vicinity of the origin of the spatial part $\mathcal{M}^3$ is a shrinking Ricci soliton. The reason is as follows: because the black hole satisfies Einstein' s equation where the stress tensor is a point-distributed matter at rest with mass m at the origin $x = 0$ (as seen from the distant rest observer):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \quad T_{00} = m\delta^{(3)}(x), \quad T_{ij} = 0 \quad (i, j = 1, 2, 3),$$

where Latin indices i, j denote spatial indices in the following. Thus we have

$$R(x) = -8\pi G T^{\mu}_{\mu} = 8\pi G m \delta^{(3)}(x).$$

From Einstein' s equation we have the Ricci curvature of $\mathcal{M}^3$ proportional to the metric of $\mathcal{M}^3$:

$$R_{ij}(x) = 8\pi G T_{ij} + \frac{1}{2} g_{ij} R = 8\pi G m \delta^{(3)}(x) g_{ij} \quad (i, j = 1, 2, 3).$$

This equation is nothing but a normalized shrinking Ricci soliton equation (43) for $\mathcal{M}^3$:

$$R_{ij}(x) = \frac{g_{ij}(x)}{2\tau} \delta^{(3)}(x),$$

where $\delta^{(3)}(x)$ plays the role of the 3-space energy density λ_3 in the vicinity of the origin, satisfying $\int d^{3x}\, \delta^{(3)}(x) = 1$ as in eq. (74). Using the relation between τ and temperature T (73), we can directly read from the equation that the temperature seen by the lab' s infinitely distant rest observer is

$$T = \delta^{(3)}(x)\tau = \frac{1}{8\pi G m},$$

which is the standard Hawking temperature of the Schwarzschild black hole seen by a distant rest observer.

Is the vacuum region outside the origin of the black hole also a shrinking Ricci soliton? One may naively think that the answer is no, since at the classical level it seems $R_{ij} = 0$ (not a shrinking soliton eq. (91)), as outside the origin is just vacuum. But as discussed in the next subsection, we argue that this is not true at the quantum level: if the vacuum and the vicinity region of the origin are in thermal equilibrium, they must be shrinking Ricci solitons as a whole, i.e., $\langle R_{ij}\rangle = \frac{1}{2\tau}g_{ij} \neq 0$, eq. (101) in the "vacuum" . The above result can be extended to the "vacuum" region outside the origin; the price to pay is that the "vacuum" is full of internal energy corresponding to the Hawking temperature. If the whole spacetime has not yet reached thermal equilibrium, the configuration must irreversibly continue flowing to a common thermal equilibrium fixed point (a global shrinking Ricci soliton), leading to globally maximized entropy, as the H-theorem asserts.

B. The Energy of a Schwarzschild Black Hole

In classical general relativity, the mass m is often mentioned as the ADM energy of the black hole:

$$E_{\mathrm{ADM}} = \int d^{3x}\, T_{00} = \int d^{3x}\, m\delta^{(3)}(x),$$

seen by the distant rest observer (with respect to the lab time x^0). Here at the quantum level, the coordinates or frame fields and spacetime are quantum fluctuating, which gives rise to internal energy related to the periodicity of the (Euclidean) lab time x^0 (i.e., $\beta = 1/T$). So, mathematically speaking, the anomaly of the trace of the stress tensor will modify the total ADM mass at the quantum level, see (36). Since the anomaly of the action of the frame fields $\lambda_3 N_*$ representing the spacetime part is always real, the internal energy of the frame fields is given by (78):

$$E_* = -\frac{\partial \log Z_*}{\partial \beta} = \frac{3}{2}T = \frac{3}{16\pi G m},$$

in which we have considered the 3-space volume V_3 outside the origin is in thermal equilibrium with the Hawking temperature at the origin (eq. 93), sharing the same equilibrium temperature T in the 3-volume V_3.

We can see that the internal energy E_* is an extra contribution to the total energy of the (black hole + "vacuum") system seen by the distant rest observer. Essentially this term can be seen as a quantum correction or part of the trace anomaly contribution to the stress tensor. Thus the total energy of the black hole including the classical ADM energy and the quantum fluctuating internal energy of the metric is

$$m_{\mathrm{BH}} = \int d^{3x}\, \langle T_{00}\rangle = m + E_* = m + \frac{3}{16\pi G m},$$

where the classical stress tensor T_{00} is formally replaced by its quantum expectation value $\langle T_{00}\rangle = m\delta^{(3)}(x) + \frac{3}{16\pi G m}\lambda_3$.

A quantum Equivalence Principle should assert that the total energy rather than only the classical ADM mass contributes to gravitation.

For a macroscopic classical black hole, $m \gg \sqrt{G}$, the first term (ADM energy) dominates eq. (96):

$$m_{\rm BH} \approx m.$$

The second internal energy term becomes gradually non-negligible for a microscopic quantum black hole. An important effect of the existence of the second term in (96) is that for a microscopic quantum black hole, it makes the total energy bounded from below; the minimal energy is of order of the Planck mass $m_{\rm BH} \geq \sqrt{3/16\pi G} \sim \mathcal{O}(m_P)$, which seems to prevent the black hole from evaporating into nothing.

Furthermore, the internal energy $\frac{3}{2}T$ term contributing to the total energy $m_{\rm BH}$ and gravitation also demands that not only the vicinity of the origin of the black hole is a shrinking soliton (as the previous subsection claims), but at the quantum level the whole 3-space is also the same shrinking soliton (i.e., satisfying eq. (91) with the identical τ globally and hence the same temperature T everywhere for the whole 3-space), just replacing the $\delta^{(3)}$-density in eq. (93) by the λ_3-density, which extends the $\delta^{(3)}$-density at the origin to the outside region (the "vacuum"):

$$T = \lambda_3 \tau = \frac{\langle T_{00}\rangle}{3}, \quad \text{with} \quad \int d^{3x}\, \lambda_3 = \int d^{3x}\, \langle T_{00}\rangle$$

for the whole thermal equilibrium 3-space, although at the classical level the vacuum $R_{ij}(x \neq 0) = 0$ seems not to be a shrinking soliton outside the origin. The physical reason is transparent: the internal energy' s contribution $\frac{3}{2}T\langle T_{00}\rangle$ also plays the role of an additional source of gravity outside the origin. For the whole 3-space with $\langle T_{00}\rangle \neq 0$ and $\langle T_{ij}\rangle = 0$, Einstein' s equation for the whole 3-space is nothing but the Shrinking Ricci Soliton equation (91):

$$\langle R_{ij}\rangle = \frac{\langle R\rangle}{2} g_{ij} = 8\pi G\langle T_{ij}\rangle + \frac{1}{2} g_{ij}\langle R\rangle \approx \frac{\langle T_{00}\rangle}{3} g_{ij} = \lambda_3 g_{ij} = \frac{g_{ij}}{2\tau} \neq 0,$$

in which $\langle R\rangle = -8\pi G\langle T^\mu_\mu\rangle = 8\pi G\langle T_{00}\rangle \neq 0$ is used in the "vacuum" outside the origin. This equation is in fact the spatial components of the Gradient Shrinking Ricci Soliton equation (43) where $\langle R_{ij}\rangle = R_{ij} + \nabla_i\nabla_j f$; the Gaussian/thermal broadening of the density matrix u contributes to the classical curvature. The vicinity region of the origin plus the "vacuum" outside the origin of the black hole as a whole is nothing but globally a shrinking Ricci soliton.

The "vacuum" is not completely nothing at the quantum level but is full of thermal particles $\langle T_{00}(x \neq 0)\rangle \neq 0$. The Hawking temperature is essentially an Unruh effect; in a certain sense, the Gradient Shrinking Ricci soliton equation, eq. (101), might play a more fundamental role than Unruh' s formula, determining how local acceleration or gravitation gives rise to temperature.

The internal energy of the spacetime frame fields is an additional and necessary source of gravity. Although macroscopically it is too small to contribute, at the quantum level its contribution is crucial for the 3-space in thermal equilibrium to be exactly a global shrinking Ricci soliton. The thermal internal energy coming from the quantum fluctuation of the 3-space gravitates normally as the quantum Equivalence Principle asserts. Otherwise, we would face a paradox as follows. If we consider a frame x having $T_{\mu\nu}(x) = 0$ everywhere, then according to the classical Einstein equation we have $R_{\mu\nu}(x) = 0$ everywhere. If we transform to another accelerating frame x', one expects $T_{\mu\nu}(x) \to T'_{\mu\nu}(x') = 0$ everywhere. However, according to the Equivalence Principle, in the accelerating frame x' one should feel equivalent gravity $R'_{\mu\nu}(x') \neq 0$. Clearly something is missing; a new dimension of the Equivalence Principle must be considered. To solve this paradox and retain the Equivalence Principle, a quantum effect (actually the effect from the diffeomorphism anomaly such as the trace anomaly or the Unruh effect) must be introduced so that the accelerating frame must create particles from the "vacuum"and be thermal, which plays the role of an equivalent gravitational source making $R'_{\mu\nu}(x') \neq 0$. The Hawking temperature in the internal energy term of eq. (96) is essentially the Unruh temperature playing such a role. In this sense, the validity of the Equivalence Principle should be extended to reference frames described by quantum states.

C. The Entropy of a Schwarzschild Black Hole

In the general framework, the entropy of the black hole comes from the uncertainty or quantum fluctuation moment of the frame fields given by the manifold density u; more precisely, the thermalized black hole entropy is measured by the maximized Shannon entropy in terms of the probability distribution u of the frame fields in the background of the black hole. So in this subsection, we calculate the u density distributed around the Schwarzschild black hole and then evaluate the corresponding entropy as a measure of the black hole entropy. After a proper definition of a zero-point of the Shannon entropy, it yields the standard Bekenstein-Hawking entropy.

For an observer in the distant lab rest frame, the contributions to the temporal static u density around the black hole are twofold. Besides the thermal distribution u_* in the "vacuum" or bulk outside the black hole horizon, which gives rise to the ideal gas entropy (85) as the background entropy, there is an additional $\tilde{u}$ density distributed mostly in an exterior thin shell near the horizon and sparsely in the bulk outside the horizon, which we will focus on. The reason is as follows. Because $\tilde{u}$ density satisfies the conjugate heat equation (23) on the classical background of the black hole, and since the classical scalar curvature $R = 0$ outside the horizon, and the temperature (equivalently the parameter τ and the mass) can be seen as unchanged for the thermalized black hole, i.e., $\partial\tilde{u}/\partial\tau = 0$, the conjugate heat equation for $\tilde{u}$ is approximately given by the 4-Laplacian equation on the Schwarzschild black hole:

$$\Delta_X \tilde{u}(X) = 0, \quad (|X| \geq r_H).$$

Now the temporal static density $\tilde{u}(X)$ plays a similar role to a solution of the Klein-Gordon equation on the static background of the black hole. The approximation of the conjugate heat equation is equivalent to interpreting the Klein-Gordon modes as a "first-quantization" probability density (not second-quantization fields). As is well known, there are modes falling into the black hole horizon and hence disappearing from the outside observer' s view, just like negative Klein-Gordon modes falling into negative energy states below the ground state. In a flat background, the amplitudes of modes falling into and going out of the horizon are identical. So in second quantization, the negative mode falling into the horizon can be reinterpreted as a single anti-particle with positive energy modes going out of the horizon with identical amplitude. However, in a curved background, for instance, the spacetime near the black hole horizon, this statement is no longer true. The two amplitudes differ from each other by a non-unitary equivalence factor. Thus the negative mode falling into the black hole horizon can no longer be reinterpreted as a single anti-particle mode going out, but rather as a multi-particle thermo-ensemble. In this situation, the density $\tilde{u}$ describes the ensemble density of modes going exterior to the horizon $|X| \geq r_H$ which can be seen by an outside observer.

By a routine calculation of the solution near the exterior black hole horizon resembling a Rindler metric as a starting point, we denote the solution $\tilde{u}_k(\rho)$, where k represents the Fourier component/momentum in directions orthogonal to the radial direction with $\rho = \log(r - r_H)$, r the radius, $r_H = 2Gm$ the horizon radius. The equation becomes

$$\frac{\partial^2 \tilde{u}_k}{\partial \rho^2} + k^2 e^{2\rho} \tilde{u}_k = \omega^2 \tilde{u}_k,$$

where ω is the eigen-energy of the modes. Using a natural boundary condition that $\tilde{u}$ vanishes at infinity, we can see that each transverse Fourier mode $\tilde{u}_k$ can be considered as a free $1+1$ dimensional quantum field confined in a box, one wall of the box at the reflecting boundary $\rho_0 = \log \epsilon_0$ where $\epsilon_0 \approx 0$, and the other wall provided by the potential

$$V(\rho) = k^2 e^{2\rho},$$

which becomes large $V(\rho) \gg 1$ at $\rho > -\log k$. So we can approximate the potential by the second wall at $\rho_w = -\log k$.

Thus the length of the box is given by

$$\Delta\rho = \rho_w - \rho_0 = -\log(\epsilon_0 k).$$

The thickness of the horizon is about $\Delta r \sim e^{\Delta\rho} \sim \epsilon_0 k$.

The density $\tilde{u}_k(\rho)$ is located in the box $\rho \in (\rho_0, \rho_w)$. In other words, the solution of $\tilde{u}$ density is located mainly in a thin shell near the horizon $r \in (r_H, r_H + \epsilon_0 k)$. Furthermore, the modes k are assumed normally distributed (with a tiny width described by the parameter τ). In this picture, without solving the equation, we can approximately write down the natural solution as

$$\tilde{u}_k(r) \approx \delta(|k|)\delta(r - r_H),$$

while for finite and small τ, we have a nearly Gaussian form:

$$\tilde{u}_k(r) \approx \delta(|k|) \cdot \frac{e^{-(r-r_H)^2/(4\pi|k|^2\tau)}}{(4\pi|k|^2\tau)^{1/2}} = \delta(|k|) \cdot \frac{e^{-(r-r_H)^2/(4\tau)}}{(4\pi\tau)^{1/2}} \quad (r > r_H).$$

The exterior horizon solution can be considered as a standing wave solution as the superposition of modes falling into and coming out of the black hole horizon. Then we have (up to a constant)

$$\log \tilde{u}_k(r)\big|_{r \sim r_H} \approx -\log(|k|^2\tau).$$

A routine calculation of the relative Shannon entropy or W-functional gives the entropy of each k-mode in the limit where the width τ is very small:

$$\lambda_3 \tilde{N}(\tilde{u}_k) = -\lambda_3 \int d^{3X}\, \tilde{u}_k \log \tilde{u}_k = \delta(|k|) \int_{r_H}^{\infty} 4\pi r^2 dr \frac{e^{-(r-r_H)^2/(4\tau)}}{(4\pi\tau)^{1/2}} \log(|k|^2\tau) = \delta(|k|) A \log(|k|^2\tau),$$

where $A = 4\pi r_H^2$ is the area of the horizon.

It is natural to assume the momentum k in the horizon shell is homogeneous, $|k| = |k_r| = |k_\perp|$, where k_r is the momentum in the radial direction and $k_\perp$ in the transverse directions on the horizon. When we integrate over all k-modes, we have the total relative Shannon entropy weakly depending on τ:

$$\lambda_3 \tilde{N}(\tilde{u}) = \lambda_3 \int d^{3k}\, \tilde{N}(\tilde{u}_k) = \int \frac{dk_r \delta(k_r)}{(2\pi)^2} \int 2\pi k_\perp dk_\perp \log(|k_\perp|^2\tau) = \frac{A}{16\pi\epsilon^2} \left[1 - \log\left(\frac{|k_\perp|^2\tau}{\epsilon^2}\right)\right]\Bigg|_{1/\epsilon}^{\Lambda},$$

where the transverse momentum is effectively cut off at an inverse of a fundamental UV length scale ϵ^2.

The relative Shannon entropy gives an area law for the black hole entropy. To determine the UV length cutoff ϵ^2, we need to consider the scale at which the relative entropy is defined to be zero (not only is the black hole locally in thermal equilibrium, but also the asymptotic background spacetime is globally in thermal equilibrium). Thus we need to consider the flow of the asymptotic background spacetime. A natural choice of a thermal equilibrium Ricci flow limit of the background spacetime (in which the black hole is embedded) is an asymptotic homogeneous and isotropic Hubble universe with scalar curvature $R_0 = D(D-1)H_0^2$ at scale t_{UV}, where we could consider and normalize the relative entropy to be zero (leaving only the background ideal gas entropy), since there is no information about local shape distortions in such a GSRS background due to the vanishing of its Weyl curvature, while the global curvature is non-zero which encodes information about its global volume shrinking. Under this definition, taking the normalized Shrinking Ricci soliton equation (43) and (22), we have

$$0 = 12H_0^2 - \frac{64\pi^2}{\epsilon^2},$$

which, using the critical density (68), gives a natural cutoff corresponding to the scale t_{UV}:

$$\tau_{\mathrm{UV}} = -t_{\mathrm{UV}} = \frac{1}{64\pi^2\lambda_{\mathrm{UV}}}, \quad \epsilon^2 = k_{\mathrm{UV}}^{-2} = \frac{1}{12H_0^2}.$$

This is exactly the Planck scale, which is a natural cutoff scale induced from the Hubble scale H_0 and λ of the framework. However, it is worth stressing that the Planck scale is not the absolute fundamental scale of the theory; it only has meaning with respect to the asymptotic Hubble scale. The only fundamental scale of the theory is the critical density λ, which is given by a combination of both the Planck scale and Hubble scale, but each individual Planck or Hubble scale does not have absolute meaning. The UV (Planck) cutoff scale could tend to infinity while the complementary (Hubble) scale correspondingly tends to zero (asymptotic flat background), keeping λ finite and fixed.

At this point, if we define a zero-relative-entropy for an asymptotic Hubble universe of scalar curvature R_0, then the black hole in this asymptotic background has a non-zero thermodynamic entropy:

$$S = -\lambda_3 \tilde{N}(\tilde{u}) = \frac{A}{4G},$$

up to the bulk background entropy $\lambda_3 N_* = S_* \ll S$, eq. (86). Combining the relative Shannon entropy $\tilde{N}$ and the bulk thermal background entropy N_*, and using the total partition function eq. (35), $Z(\mathcal{M}^3) = e^{\lambda_3 N - 3/2} = e^{\lambda_3(\tilde{N}+N_*) - 3/2}$, we can also reproduce the total energy of the black hole in (96):

$$m_{\rm BH} = -\frac{\partial \log Z}{\partial \beta} = m + \frac{A}{4G}T,$$

in which eq. (47) and $A = 4\pi r_H^2 = 16\pi G^2 m^2 = \beta^2/(4\pi)$ have been used.

Different from the holographic idea that information or entropy is coded in the (infinitely thin and 2-dimensional) horizon or boundary of a gravitational system, in this framework where the coordinates of the spacetime geometry are smeared by quantum fluctuations, there is no mathematically precise notion of an infinitely thin boundary in a "density manifold" in general; it is just a semi-classical concept. Note that the manifold density u is mainly distributed at the horizon with a finite thickness (although very small), which contributes most of the anomaly and entropy to the black hole. Thus although the entropy (113) is proportional to the area, the geometric gravitational entropy given by the framework essentially comes from the 3-volume (note the 3D integral in eq. (108) and eq. (110)) rather than the 2-surface boundary. Or in equivalent words, here the area of the horizon is fluctuating (due to its finite thickness) rather than fixed, while the total energy and hence the temperature are fixed. In this sense, it is a canonical ensemble rather than an area ensemble as some ideas might suggest.

VI. Conclusions

In this paper, we have proposed a statistical field theory underlying Perelman's seminal analogies between his geometric functionals and thermodynamic functions. The theory is based on a $d = 4 - \epsilon$ quantum non-linear sigma model, interpreted as a quantum reference frame. When we quantize the theory at the Gaussian approximation, the wavefunction $\Psi(X)$ and hence the density matrix $u(X) = \Psi^*(X)\Psi(X)$ (eq. 13) can be written down explicitly.

Based on the density matrix, the Ricci flow of the frame fields (10) and the generalized Ricci-DeTurck flow (19) of the frame fields endowed with the density matrix are discussed. Furthermore, we find that the density matrix has profound statistical and geometric meanings: using it, the spacetime $(\mathcal{M}^D, g)$ as the target space of the NLSM is generalized to a density spacetime $(\mathcal{M}^D, g, u)$. The density matrix $u(X,\tau)$, satisfying a conjugate heat equation (23), not only describes a (coarse-grained) probability density of finding frame fields in a local volume, but also describes a volume comparison between a local volume and a fiducial one.

Due to the non-isometric nature of the Ricci or Ricci-DeTurck flow, classical diffeomorphism is broken down at the quantum level. Through functional integral quantization, the change of the measure of the functional integral can be expressed using a Shannon entropy N in terms of the density matrix $u(X,\tau)$. The induced trace anomaly and its relation to anomalies in conventional gravity theories are also discussed. As the Shannon entropy flows monotonically to its maximal value N_* in a limit called Gradient Shrinking Ricci Soliton (GSRS), a relative density $\tilde{u}$ and relative Shannon entropy $\tilde{N} = N - N_*$ can be defined with respect to the flow limit. The relative Shannon entropy provides a statistical interpretation underlying Perelman' s partition function (47). The monotonicity of $\tilde{N}$ along the Ricci flow yields an analogous H-theorem (50) for the frame fields system. As a side effect, the gravitational meaning of the theory is also discussed, in which a cosmological constant $-\lambda\nu(B^4_\infty) \approx 0.8\rho_c$ as a UV counterterm of the anomaly must be introduced.

We find that a temporal static GSRS, $\mathcal{M}^3$, as a 3-space slice of the 4-spacetime GSRS, $\mathcal{M}^4 = \mathcal{M}^3 \times \mathbb{R}$, is in a thermal equilibrium state, in which the temperature is proportional to the global τ parameter of $\mathcal{M}^3$ (73) up to a 3-space energy density λ_3 with normalization $\int d^{3x}\lambda_3 = 1$. The temperature and λ_3 both depend on the choice of time. In the sense that $\mathcal{M}^3$ is thermal, its Ricci soliton equation (91) or quantum (indistinguishable from thermal) fluctuation (41) can be considered as a generalization of Unruh' s formula, relating temperature to local acceleration or gravitation. Based on the statistical interpretation of the density matrix $u(X, \tau)$, we find that the thermodynamic partition function (75) at the Gaussian approximation is just a partition function of an ideal gas of frame fields. In this physical picture of a canonical ensemble of frame fields gas, several thermodynamic functions, including internal energy (78), free energy (84), thermodynamic entropy (85), and ensemble density (83), can be calculated explicitly and agree with Perelman' s formulae, providing an underlying statistical foundation for Perelman' s analogous functionals.

We find that the statistical field theory of quantum reference frames can be used to provide a possible microscopic origin of spacetime thermodynamics. The standard results of the thermodynamics of the Schwarzschild black hole, including the Hawking temperature, energy, and Bekenstein-Hawking entropy, can be successfully reproduced in this framework. We find that when the fluctuation internal energy of the metric is taken into account in the total energy, the energy of the black hole has a lower bound of order of the Planck energy, which prevents the quantum black hole from evaporating into nothing. The internal energy or related temperature of the spacetime frame fields is an additional source of gravity; although macroscopically it is very small, at the quantum level its contribution is necessary for a thermal equilibrium 3-space to be exactly a GSRS; otherwise, the Equivalence Principle would break down. In this paper, the extended quantum Equivalence Principle plays a fundamental role as a bridge from the quantum reference frame theory (as a statistical field or quantum field theory on the base/lab spacetime) to quantum gravity.

To summarize, this paper can be seen as an attempt to discuss the deep relations between three fundamental themes—the diffeomorphism anomaly, gravity, and spacetime thermodynamics—based on the statistical field theory of quantum spacetime reference frames and the quantum Equivalence Principle.

In the spirit of classical general relativity, if we trust the Equivalence Principle, one cannot in principle determine whether one is in an absolute accelerating frame or in an absolute gravitational background, which leads to a general covariance principle or diffeomorphism invariance of the gravitational theory. However, at the quantum level the issue is more subtle. If an observer in an accelerating frame sees the Unruh effect, i.e., thermal particles are created in the "vacuum" , this seems to lead to unitary inequivalence between the vacuums of, for instance, an inertial frame and an accelerating frame, and hence diffeomorphism invariance appears to break down, as discussed as the anomaly in this paper. Our treatment of the anomaly is that it is only canceled in an observer' s lab up to UV scale, where the frame can be considered classical, rigid, and cold, while at general scales the anomaly is not completely canceled.

Could one determine that he/she is in an absolute accelerating frame by detecting the anomaly (Shannon $\tilde{N}$ term) at general scale (e.g., by thermodynamic experiments detecting vacuum thermal particle creation and hence finding the non-unitarity)? We argue that if the answer is still "NO!" in the spirit of general relativity, then the anomaly term coming from a quantum general coordinate transformation must also be equivalently interpreted as the effects of spacetime thermodynamics and gravity. Because the second-order moment fluctuation of the quantum coordinates or a non-trivial manifold density u, which gives rise to the diffeomorphism anomaly, also contributes to other second-order quantities (series coefficients at second spacetime derivative) such as (i) acceleration (second time derivative of coordinates, e.g., leading to uniform accelerated expansion or other acceleration discrepancies in the universe [1]), (ii) gravity or curvature (second spacetime derivative of metric, e.g., see (9) and (18)), and (iii) thermal broadening (second spatial derivative of the manifold density or ensemble density, e.g., see (41) and (73)) at the same (second) order. In this sense, the validity of the classical Equivalence Principle would be generalized to the quantum level to incorporate the effects of quantum fluctuation of spacetime coordinates or frame fields, so that one still cannot determine and distinguish whether he/she is in an accelerating frame, in a gravitational field, or in a thermal spacetime (as a new dimension of the Equivalence Principle); these three things have no absolute physical meaning and are indistinguishable in this framework. The classical Equivalence Principle asserts the equivalence of the first two things at first order (mean level); the quantum Equivalence Principle asserts the equivalence of the three things even at second order (variance level), and even higher orders.

Data availability statement

All data that support the findings of this study are included within the article.

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Note: Figure translations are in progress. See original paper for figures.