Exact Decomposition of Multifrequency Discrete Real and Complex Signals

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Abstract

'The spectral leakage (SL) from windowing and the picket fence effect (PEF) from discretization' have been among the standard contents in textbooks for many decades. The SL and PEF would cause the distortions in amplitude, frequency, and phase of signals, which have always been of concern, and attempts have been made to solve them. This paper proposes two novel decomposition theorems that can totally eliminate the SL and PEF, they could broaden the knowledge of signal processing. First, two generalized eigenvalue equations are constructed for multifrequency discrete real signals and complex signals. The two decomposition theorems are then proved. On these bases, exact decomposition methods for real and complex signals are proposed. For a noise-free multifrequency real signal with m sinusoidal components, the frequency, amplitude, and phase of each component can be exactly calculated by using just 4m–1 discrete values and its second-order derivatives. For a multifrequency complex signal, only 2m–1 discrete values and its first-order derivatives are needed. The numerical experiments show that the proposed methods have very high resolution, and the sampling rate does not necessarily obey the Nyquist sampling theorem. With noisy signals, the proposed methods have extraordinary accuracy.

Full Text

Preamble

Exact Decomposition of Multifrequency Discrete Real and Complex Signals

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Abstract

For many decades, spectral leakage (SL) from windowing and the picket fence effect (PEF) from discretization have been standard topics in textbooks. These phenomena cause distortions in the amplitude, frequency, and phase of signals, which have long been a concern, and numerous attempts have been made to address them. This paper proposes two novel decomposition theorems that can completely eliminate SL and PEF, thereby broadening the knowledge base of signal processing. First, two generalized eigenvalue equations are constructed for multifrequency discrete real signals and complex signals, and the two decomposition theorems are then proved. On this basis, exact decomposition methods for real and complex signals are proposed. For a noise-free multifrequency real signal with m sinusoidal components, the frequency, amplitude, and phase of each component can be calculated exactly using just $4m-1$ discrete values and their second-order derivatives. For a multifrequency complex signal, only $2m-1$ discrete values and their first-order derivatives are needed. Numerical experiments demonstrate that the proposed methods possess very high resolution, and the sampling rate does not necessarily need to obey the Nyquist sampling theorem. For noisy signals, the proposed methods exhibit extraordinary accuracy.

Keywords: Exact decomposition; Multifrequency sinusoidal signal; Discrete real signal; Discrete complex signal; Decomposition theorem; Generalized eigenvalue equation

1. Introduction

Since Cooley and Tukey proposed the fast Fourier transform (FFT) algorithm in 1965 [?, ?], the technique has become indispensable in electronics, communications, signal analysis, digital image and audio processing, and many other fields [?, ?]. However, when performing the FFT algorithm, distortions in amplitude, frequency, and phase—caused by spectral leakage (SL) due to signal truncation and the picket fence effect (PEF) due to frequency discretization—are inevitable. These issues have long been a concern, and numerous attempts have been made to solve them.

In 1970, Rife and Vincent studied the correction of frequency and level measurements of tones using the discrete Fourier transform (DFT) [?], and many scholars have since investigated this issue [?]. Interpolation techniques are among the most studied and widely applied methods in many engineering fields \cite{4-10}. These techniques refer to interpolated DFT (IpDFT) or interpolated FFT (IpFFT), which reduce SL and PEF through windowing and interpolation, respectively. Generally, windows with maximum side lobe decay efficiently reduce SL; thus, the Hanning window is commonly used [?, ?, ?]. By weighting the discrete signals before performing DFT and FFT according to the distribution of spectral lines in the windows, the PEF is minimized, and accurate amplitudes, frequencies, and phases of the signal components can be computed \cite{4-10}. IpDFT and IpFFT can be implemented quickly and easily; however, their results are not accurate when sinusoids are not well separated in frequency \cite{5-8}. Weighted phase averaging (WPA) is another popular correction technique for FFT \cite{11-15}. Relying on weighted linear regression of the phases, WPA averages the weighted phase estimators obtained by FFT within different nonoverlapping segments of the signals. WPA has the advantage of accurately calculating frequencies and phases, but amplitudes remain dependent on the windows. Many other correction techniques exist, such as the phase difference correction method [?] and the sliding-window DFT method [?, ?, ?]. All these correction techniques have their respective advantages and disadvantages.

Distinct from techniques based on DFT and FFT, matrix-based singular value decomposition (SVD) and singular spectral decomposition (SSD) aim to represent signals as a linear superposition of elementary variable modes without requiring harmonic components \cite{19-22}. These techniques do not provide spectral estimators but serve as powerful denoising filters capable of separating autocoherent features, such as anharmonic oscillations and quasiperiodic phenomena, from random features. They are non-parametric techniques.

Other commonly used techniques for extracting signal features include the wavelet transform (WT) [?, ?], Hilbert-Huang transform (HHT) [?], estimation of signal parameters via rotational invariance technique (ESPRIT) [?, ?], and multiple signal classification (MUSIC) [?, ?]. However, it is impossible to exactly compute the frequencies, amplitudes, and phases of multifrequency discrete signals using these techniques.

This paper constructs generalized eigenvalue equations and proves decomposition theorems for multifrequency discrete real signals and complex signals. Based on these two decomposition theorems, decomposition methods for real and complex signals are proposed. For a noise-free real signal with m sinusoidal components, the frequency, amplitude, and phase of each component can be calculated exactly using just $4m-1$ discrete values and their second-order derivatives. For a complex signal, only $2m-1$ discrete values and their first-order derivatives are needed. The decomposition results are exact in theory. Numerical experiments show that the proposed methods have very high resolution, and the sampling rate does not necessarily need to obey the Nyquist sampling theorem. For noisy signals, the proposed methods exhibit extraordinary accuracy.

2. Decomposition Theorem of Multifrequency Discrete Real Signal

Multifrequency sinusoidal signals are among the most common signals in engineering. A multifrequency real signal can be expressed as follows:

$$x(t) = \sum_{i=1}^{m} a_i \sin(2\pi f_i t + \varphi_{i0})$$

where a_i, f_i, and φ_{i0} are the amplitude, frequency, and phase of the ith component of signal $x(t)$, and m is the number of component signals.

Discretizing $x(t)$, we construct the following generalized eigenvalue equation for the multifrequency discrete real sinusoidal signal:

$$-\mathbf{D}_x\mathbf{v} = \lambda\mathbf{X}\mathbf{v} \tag{2}$$

where λ is the generalized eigenvalue, $\mathbf{v}$ is the generalized eigenvector, $\mathbf{X}$ is the square Hankel matrix of $x(t)$, and $\mathbf{D}_x$ is the square Hankel matrix of the second-order derivative $\ddot{x}(t)$. These matrices are defined as:

$$\mathbf{X} = \begin{bmatrix} x_2 & \cdots & x_n & \\ x_3 & \cdots & x_{n+1} & \\ \vdots & \ddots & \vdots & \\ x_n & x_{n+1} & \cdots & x_{2n-1} \end{bmatrix}; \quad \mathbf{D}_x = \begin{bmatrix} \ddot{x}_2 & \cdots & \ddot{x}_n & \\ \ddot{x}_3 & \cdots & \ddot{x}_{n+1} & \\ \vdots & \ddots & \vdots & \\ \ddot{x}_n & \ddot{x}_{n+1} & \cdots & \ddot{x}_{2n-1} \end{bmatrix} \tag{3}$$

where $x_k, \ddot{x}_k$ $(k = 1, \cdots, 2n-1)$ are the discrete series of $x(t)$ and $\ddot{x}(t)$, respectively.

Theorem 1: When $n \geq 2m$, the generalized eigenvalue equation of the multi-frequency real sinusoidal signal has $2m$ nonzero generalized eigenvalues, which are given by:

$$\lambda_{2i-1} = \lambda_{2i} = 4\pi^2 f_i^2 \tag{4}$$

2.1 Proof

The square Hankel matrix $\mathbf{X}$ of signal $x(t)$ can be rewritten as:

$$\mathbf{X} = \sum_{i=1}^{m} a_i \mathbf{X}_i \tag{5}$$

where $a_i\mathbf{X}_i$ is the square Hankel matrix of the ith sinusoidal component. The element in the kth row and lth column of matrix $\mathbf{X}_i$ is given by:

$$\mathbf{X}_i(k,l) = \sin\left(2\pi f_i(k+l-2)/f_s + \varphi_{i0}\right), \quad (k,l = 1, \cdots, n) \tag{6}$$

where f_s is the sampling rate. The square Hankel matrix $\mathbf{D}_x$ is given by:

$$\mathbf{D}_x = \sum_{i=1}^{m} \mathbf{D}_{x_i} = \sum_{i=1}^{m} -4\pi^2 f_i^2 a_i \mathbf{X}_i \tag{7}$$

The ranks of matrices $\mathbf{X}$ and $\mathbf{D}_x$ are less than or equal to $2m$, because the ranks of the square Hankel matrices $\mathbf{X}_i$ and $\mathbf{D}_{x_i}$ are both two [?].

Suppose $\lambda = 4\pi^2 f_i^2$, and considering the relation $\mathbf{D}_{x_i} = -4\pi^2 f_i^2 a_i \mathbf{X}_i$, equation (2) is modified to obtain:

$$\left\{ \sum_{\substack{k=1\\k\neq i}}^{m} \mathbf{D}_{x_k} + 4\pi^2 f_i^2 \sum_{\substack{k=1\\k\neq i}}^{m} a_k \mathbf{X}_k \right\} \mathbf{v} = \mathbf{0} \tag{8}$$

Equation (8) has at least two nonzero solutions, which are $\mathbf{v}_{2i}$ and $\mathbf{v}_{2i-1}$, because it is a homogeneous equation in vector $\mathbf{v}$, and the rank of the coefficient matrix is less than or equal to $2m-2$.

In equation (8), replacing $\mathbf{v}$ with $\mathbf{v}_{2i}$ and left-multiplying by $\mathbf{v}_{2i}^T$ yields:

$$4\pi^2 f_i^2 = -\frac{\mathbf{v}_{2i}^T \left(\sum_{\substack{k=1\\k\neq i}}^{m} \mathbf{D}_{x_k} \right) \mathbf{v}_{2i}}{\mathbf{v}_{2i}^T \left(\sum_{\substack{k=1\\k\neq i}}^{m} a_k \mathbf{X}_k \right) \mathbf{v}_{2i}} \tag{9}$$

where $\mathbf{v}_{2i}^T$ is the transpose of $\mathbf{v}_{2i}$. Considering the following identity:

$$4\pi^2 f_i^2 = -\frac{\mathbf{v}_{2i}^T \mathbf{D}_{x_i} \mathbf{v}_{2i}}{\mathbf{v}_{2i}^T a_i \mathbf{X}_i \mathbf{v}_{2i}} \tag{10}$$

the following equation is derived from (9) and (10):

$$4\pi^2 f_i^2 = -\frac{\mathbf{v}_{2i}^T \mathbf{D}_x \mathbf{v}_{2i}}{\mathbf{v}_{2i}^T \mathbf{X} \mathbf{v}_{2i}} \tag{11}$$

Equation (11) reveals that $4\pi^2 f_i^2$ is a Rayleigh quotient of matrices $-\mathbf{D}_x$ and $\mathbf{X}$; that is, $\lambda = 4\pi^2 f_i^2$ is a generalized eigenvalue of equation (2), and $\mathbf{v}_{2i}$ is the corresponding generalized eigenvector [?]. Replacing $\mathbf{v}_{2i}$ with $\mathbf{v}_{2i-1}$, equation (11) also holds. Therefore, $\lambda = 4\pi^2 f_i^2$ is a double eigenvalue of equation (2).

Thus, Decomposition Theorem 1 for discrete real signals is proven.

2.2 Component Amplitudes and Phases of Multifrequency Discrete Real Signal

After computing the ith double eigenvalue and its two corresponding eigenvectors, and considering equation (6), $\mathbf{v}_{2i}^T a_i \mathbf{X}_i \mathbf{v}_{2i-1}$ can be expanded as:

$$\mathbf{v}_{2i}^T a_i \mathbf{X}_i \mathbf{v}_{2i-1} = \mathbf{v}_{2i}^T \mathbf{H}_{i1} \mathbf{v}_{2i} a_i \cos\varphi_{i0} + \mathbf{v}_{2i-1}^T \mathbf{H}_{i1} \mathbf{v}_{2i-1} a_i \cos\varphi_{i0} + \mathbf{v}_{2i}^T \mathbf{H}_{i2} \mathbf{v}_{2i} a_i \sin\varphi_{i0} + \mathbf{v}_{2i-1}^T \mathbf{H}_{i2} \mathbf{v}_{2i-1} a_i \sin\varphi_{i0} \tag{12}$$

where $\mathbf{H}_{i1}$ and $\mathbf{H}_{i2}$ are square matrices of dimension n. Their elements in the kth row and lth column are given by:

$$\mathbf{H}_{i1}(k,l) = \sin\left(2\pi f_i (k+l-2)/f_s\right), \quad \mathbf{H}_{i2}(k,l) = \cos\left(2\pi f_i (k+l-2)/f_s\right), \quad (k,l = 1, \cdots, n) \tag{13}$$

For the ith generalized eigenvector, $\mathbf{v}_{2i}^T a_i \mathbf{X}_i \mathbf{v}_{2i-1}$ can be replaced by $\mathbf{v}_{2i}^T \mathbf{X} \mathbf{v}_{2i-1}$ because the generalized eigenvectors are weighted orthogonal, that is, $\mathbf{v}_k^T \mathbf{X} \mathbf{v}_l = 0$ $(k \neq l)$. Thus, $\mathbf{v}_{2i}^T \mathbf{X} \mathbf{v}_{2i}$ and $\mathbf{v}_{2i-1}^T \mathbf{X} \mathbf{v}_{2i-1}$ can replace $\mathbf{v}_{2i}^T a_i \mathbf{X}_i \mathbf{v}_{2i}$ and $\mathbf{v}_{2i-1}^T a_i \mathbf{X}_i \mathbf{v}_{2i-1}$, respectively. Equation (12) is therefore modified to create:

$$\begin{bmatrix} \mathbf{v}_{2i}^T \mathbf{H}_{i1} \mathbf{v}_{2i} & \mathbf{v}_{2i}^T \mathbf{H}_{i2} \mathbf{v}_{2i} \\ \mathbf{v}_{2i-1}^T \mathbf{H}_{i1} \mathbf{v}_{2i-1} & \mathbf{v}_{2i-1}^T \mathbf{H}_{i2} \mathbf{v}_{2i-1} \end{bmatrix} \begin{bmatrix} a_i \cos\varphi_{i0} \\ a_i \sin\varphi_{i0} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{2i}^T \mathbf{X} \mathbf{v}_{2i} \\ \mathbf{v}_{2i-1}^T \mathbf{X} \mathbf{v}_{2i-1} \end{bmatrix} \tag{14}$$

The amplitude a_i and phase φ_{i0} can be calculated after solving for the unknowns $a_i \cos\varphi_{i0}$ and $a_i \sin\varphi_{i0}$ from equation (14). The frequencies, amplitudes, and phases of all sinusoidal component signals can be computed in this manner.

3. Decomposition Theorem of Multifrequency Discrete Complex Signal

The multifrequency complex signal can be expressed as:

$$y(t) = \sum_{i=1}^{m} a_i e^{j(2\pi f_i t + \varphi_{i0})} \tag{15}$$

where a_i, f_i, and φ_{i0} are the amplitude, frequency, and phase of the ith component of the complex signal $y(t)$, and m is the number of component signals.

Discretizing $y(t)$, we construct the generalized eigenvalue equation for the multifrequency discrete complex signal as:

$$-j\mathbf{D}_y \mathbf{v} = \lambda \mathbf{Y} \mathbf{v} \tag{16}$$

where λ is the generalized eigenvalue, $\mathbf{v}$ is the generalized eigenvector, $\mathbf{Y}$ is the square Hankel matrix of $y(t)$, and $\mathbf{D}_y$ is the square Hankel matrix of the first-order derivative $\dot{y}(t)$. These matrices are defined as:

$$\mathbf{Y} = \begin{bmatrix} y_2 & \cdots & y_n & \\ y_3 & \cdots & y_{n+1} & \\ \vdots & \ddots & \vdots & \\ y_n & y_{n+1} & \cdots & y_{2n-1} \end{bmatrix}; \quad \mathbf{D}_y = \begin{bmatrix} \dot{y}_2 & \cdots & \dot{y}_n & \\ \dot{y}_3 & \cdots & \dot{y}_{n+1} & \\ \vdots & \ddots & \vdots & \\ \dot{y}_n & \dot{y}_{n+1} & \cdots & \dot{y}_{2n-1} \end{bmatrix} \tag{17}$$

where $y_k, \dot{y}_k$ $(k = 1, \cdots, 2n{-}1)$ are the discrete series of $y(t)$ and $\dot{y}(t)$, respectively.

Theorem 2: When $n \geq m$, the generalized eigenvalue equation of the multifrequency complex signal has m nonzero generalized eigenvalues, which are given by:

$$\lambda_i = 2\pi f_i, \quad (i = 1, \cdots, m) \tag{18}$$

3.1 Proof

The square Hankel matrix $\mathbf{Y}$ of the complex signal $y(t)$ can be rewritten as:

$$\mathbf{Y} = \sum_{i=1}^{m} a_i \mathbf{Y}_i \tag{19}$$

where $a_i \mathbf{Y}_i$ is the square Hankel matrix of the ith component complex signal. The element in the kth row and lth column of matrix $\mathbf{Y}_i$ is given by:

$$\mathbf{Y}_i(k, l) = e^{j(2\pi f_i (k+l-2)/f_s + \varphi_{i0})}, \quad (k, l = 1, \cdots, n) \tag{20}$$

where f_s is the sampling rate. The square Hankel matrix $\mathbf{D}_y$ is given by:

$$\mathbf{D}_y = \sum_{i=1}^{m} \mathbf{D}_{y_i} = \sum_{i=1}^{m} j2\pi f_i a_i \mathbf{Y}_i \tag{21}$$

The ranks of matrices $\mathbf{Y}$ and $\mathbf{D}_y$ are less than or equal to m, because the ranks of the square Hankel matrices $\mathbf{Y}_i$ and $\mathbf{D}_{y_i}$ are both one [**?**].

Suppose $\lambda = 2\pi f_i$, and considering the relation $\mathbf{D}_{y_i} = j2\pi f_i a_i \mathbf{Y}_i$, equation (16) is modified to obtain:

$$\left\{ j \sum_{\substack{k=1 \\ k \neq i}}^{m} \mathbf{D}_{y_k} + 2\pi f_i \sum_{\substack{k=1 \\ k \neq i}}^{m} a_k \mathbf{Y}_k \right\} \mathbf{v} = \mathbf{0} \tag{22}$$

Equation (22) has at least one nonzero solution, which is $\mathbf{v}_i$, because it is a homogeneous equation in vector $\mathbf{v}$, and the rank of the coefficient matrix is less than or equal to $m - 1$.

In equation (22), replacing $\mathbf{v}$ with $\mathbf{v}_i$ and left-multiplying by $\mathbf{v}_i^T$ yields:

$$2\pi f_i = -\frac{\mathbf{v}_i^T \left(j \sum_{\substack{k=1 \\ k\neq i}}^{m} \mathbf{D}_{y_k} \right) \mathbf{v}_i}{\mathbf{v}_i^T \left(\sum_{\substack{k=1 \\ k\neq i}}^{m} a_k \mathbf{Y}_k \right) \mathbf{v}_i} \tag{23}$$

where $\mathbf{v}_i^T$ is the transpose of $\mathbf{v}_i$. Considering the following identity:

$$2\pi f_i = -\frac{\mathbf{v}_i^T \mathbf{D}_{y_i} \mathbf{v}_i}{\mathbf{v}_i^T a_i \mathbf{Y}_i \mathbf{v}_i} \tag{24}$$

the following equation is derived from (23) and (24):

$$2\pi f_i = \frac{\mathbf{v}_i^T j \mathbf{D}_y \mathbf{v}_i}{\mathbf{v}_i^T \mathbf{Y} \mathbf{v}_i} \tag{25}$$

Equation (25) reveals that $2\pi f_i$ is a Rayleigh quotient of matrices $-j\mathbf{D}_y$ and $\mathbf{Y}$; that is, $\lambda = 2\pi f_i$ is a generalized eigenvalue of equation (16), and $\mathbf{v}_i$ is the corresponding generalized eigenvector [**?**].

Therefore, $\lambda = 2\pi f_i$ is an eigenvalue of equation (16).

Thus, Decomposition Theorem 2 for discrete complex signals is proven.

3.2 Component Amplitudes and Phases of Multifrequency Discrete Complex Signal

After computing the ith eigenvalue and its corresponding eigenvector, and considering equation (20), $\mathbf{v}_i^T a_i \mathbf{Y}_i \mathbf{v}_i$ can be expressed as:

$$\mathbf{v}_i^T a_i \mathbf{Y}_i \mathbf{v}_i = a_i e^{j\varphi_{i0}} \mathbf{v}_i^T \mathbf{G}_i \mathbf{v}_i \tag{26}$$

where $\mathbf{G}_i$ is a square matrix of dimension n. Its element in the kth row and lth column is given by:

$$\mathbf{G}_i(k,l) = e^{j(2\pi f_i (k+l-2)/f_s)}, \quad (k, l = 1, \cdots, n) \tag{27}$$

For the ith generalized eigenvector, $\mathbf{v}_i^T a_i \mathbf{Y}_i \mathbf{v}_i$ can be replaced by $\mathbf{v}_i^T \mathbf{Y} \mathbf{v}_i$ because the generalized eigenvectors are weighted orthogonal, that is, $\mathbf{v}_i^T \mathbf{Y} \mathbf{v}_k = 0$ $(k \neq i)$. Thus, equation (26) becomes:

$$a_i e^{j\varphi_{i0}} = \frac{\mathbf{v}_i^T \mathbf{Y} \mathbf{v}_i}{\mathbf{v}_i^T \mathbf{G}_i \mathbf{v}_i}, \quad (i = 1, \cdots, m) \tag{28}$$

Equation (28) is a complex equation, so the amplitude a_i and phase φ_{i0} ($i = 1, \cdots, m$) can be calculated.

4.1 Decomposition of Noise-Free Multifrequency Real and Complex Signals

Based on equations (1) and (15) as noise-free multifrequency real and complex signals, respectively, the known frequencies, amplitudes, and phases of all component signals for $m = 10$ are listed in Table 1. Since the highest frequency of the component signal is 290 Hz, we can set the sampling rate $f_s = 299$ Hz, which is slightly higher than the highest frequency.

Table 1. Parameters of the known signal

f_j (Hz)	φ_{j0} (°)
…	…

The minimum frequency difference between two neighboring components is only 0.5 Hz, the relative difference is about 0.5%, and the corresponding amplitude difference is up to 20 times.

For the real signal with $n = 4m - 1 = 39$, we solve the generalized eigenvalue equation (2), computing ten double eigenvalues and twenty corresponding eigenvectors. Ten component frequencies were calculated using equation (4), and the component amplitudes and phases were computed using equation (14).

Figure 1 shows the known multifrequency real signal and its ten decomposed component signals. The 39 discrete points on the original multifrequency signal are marked with the symbol ‘o’. Although the minimum frequency difference between two neighboring components is only 0.5 Hz, the relative difference is approximately 0.5%, and the corresponding amplitude difference is up to 20 times, the two neighboring component signals can be decomposed accurately. The sampling rate is less than the Nyquist sampling rate, which would be 580 Hz—twice the highest component frequency.

Fig. 1. Multifrequency real signal and its ten decomposed component signals. The 39 discrete points on the original multifrequency signal are marked with symbol ‘o’. The maximum absolute errors in frequency, amplitude, and phase are 1.99×10^{-9}, 7.79×10^{-10}, and -4.39×10^{-8}, respectively.

For the complex signal with $n = 2m - 1 = 19$, we solve the generalized eigenvalue equation (16), computing ten eigenvalues and ten corresponding eigenvectors. Ten component frequencies were calculated using equation (18), and the component amplitudes and phases were computed using equation (28).

The decomposition results are similar to those for the real signal. However, with the multifrequency complex signal, the decomposed results contain both real and imaginary component signals, and the discrete points on the original signal are much fewer than for the real signal.

The absolute computational errors in the component frequencies, amplitudes, and phases are listed in Table 2. For the real signal, the maximum absolute errors in frequency, amplitude, and phase are 1.99×10^{-9}, 7.79×10^{-10}, and -4.39×10^{-8}, respectively. For the complex signal, these errors are 6.41×10^{-7}, -2.51×10^{-7}, and -6.72×10^{-6}, respectively. It should be noted that all maximum errors resulted from the two neighboring components with the minimum frequency difference and maximum amplitude difference, while the other errors were much smaller. Therefore, these errors are considered to be caused by computer precision rather than the method itself.

Thus, the proposed decomposition theorems are exact in theory, and the corresponding decomposition methods are highly accurate and possess high resolution, with no requirement for the sampling rate to obey the sampling theorem. In fact, it is impossible to extract accurate frequencies, amplitudes, and phases from multifrequency signals using other methods with so little discrete data.

Table 2. The absolute computation errors in the component frequencies, amplitudes, and phases

Errors of real signal	Errors of complex signal
Δf_j (Hz)	$\Delta \varphi_{j0}$ (°)
-5.76×10^{-12}	2.07×10^{-12}
...	...

The bold numbers represent the maximum absolute errors of the component parameters. All maximum errors resulted from the two neighboring components with minimum frequency difference and maximum amplitude difference, while the other errors are substantially smaller.

Generally, the first-order and second-order derivatives of signals are unknown. In such cases, they can be easily obtained through differential circuits or numerically calculated using the discrete values of the original signal.

4.2 Simulation of FMCW Radar Measurement

Frequency-modulated continuous-wave (FMCW) radar is widely used to measure target distance. The transmitting antenna of the radar system transmits an FMCW radio signal, the receiving antenna receives the reflected signal from the target, and mixing the transmission and reflection signals yields [**?**]:

$$z(t) = Ae^{j(2\pi f_b t + \varphi_b)} + w(t) \tag{29}$$

where A is the received signal power and $w(t)$ is the system noise. The frequency f_b and phase φ_b are given by:

$$f_b = \frac{4\pi f_c R}{cT_c}, \quad \varphi_b = \frac{4\pi B R^2}{cT_c} \tag{30}$$

where f_c is the chirp start frequency, B is the chirp bandwidth, T_c is the chirp duration, R is the distance to the target, and c is the speed of light.

According to equations (23) and (24), the distance to the target can be calculated by measuring the frequency f_b or the phase φ_b. In general, the distance measurement accuracy of the phase method is higher. However, due to phase periodic ambiguity, it is difficult to use the phase method when the measured distance is long [**?**]. Here, we simulate distance measurement of the target using the frequency method and investigate the influence of noise on measurement accuracy using the proposed method.

The radar parameters are set as follows: chirp start frequency $f_c = 24$ GHz, chirp bandwidth $B = 100$ MHz, chirp duration $T_c = 512$ s, and speed of light $c = 3 \times 10^8$ m/s. Suppose the target distance $R = 12.3456789$ m, and the noise $w(t)$ is zero-mean Gaussian white noise. With a specified signal-to-noise ratio (SNR), 1000 simulation measurements were conducted, and the results are shown in Figures 2 to 4.

Fig. 2. The absolute errors and relative errors of the average distance from 1000 simulation measurements vs. SNR. As seen in Figure 2, even when the SNR is very low (as low as -10 dB), the distance errors are less than 4 mm, and the relative error is approximately 0.032%. When SNR $\geq$ 10 dB, the absolute and relative errors are close to zero. The reason for the high distance accuracy is that the simulated measurements of frequency, amplitude, and phase of signal $z(t)$ are highly accurate. Figure 3 shows the comparison between the mean squared errors (MSEs) of the simulated frequency, amplitude, and phase measurements and their Cramér-Rao bounds (CRBs) [**?**].

Fig. 3. The MSEs and CRBs of frequency, amplitude, and phase of the signal vs. SNR. The MSEs of frequency, amplitude, and phase are less than their CRBs, indicating that the method has achieved extraordinary accuracy.

In general theory, it is impossible for the MSEs of frequency, amplitude, and phase to be less than their CRBs. However, the simulation results show that the MSEs are indeed less than the CRBs. This is very difficult to explain at present, but it indicates that the method has achieved extraordinary accuracy.

To investigate the robustness of the method, the MSE of the simulated distance measurements is calculated using:

$$\sigma_s^2 = \frac{1}{N}\sum_{i=1}^{N}(R_i - R)^2 \tag{31}$$

where $N = 1000$ is the number of simulations, R_i is the measured distance at a certain SNR, and R is the true target distance.

Figure 4 shows the curve of σ_s^2 vs. SNR, demonstrating that σ_s^2 decreases rapidly with increasing SNR. Thus, the method has good robustness.

Fig. 4. The MSE of distance vs. SNR

5. Conclusions

The key contribution of this paper is the construction of generalized eigenvalue equations for real and complex signals. For multifrequency real signals, the generalized eigenvalue equation is constructed using the square Hankel matrices of discrete values and their second-order derivatives. For multifrequency complex signals, the generalized eigenvalue equation is constructed using the square Hankel matrices of discrete values and their first-order derivatives. The aforementioned Hankel matrix can be replaced by a Toeplitz matrix or other matrices as long as the rank equals that of the corresponding square Hankel matrix. The first-order and second-order derivatives of signals can be obtained through differential circuits or numerically calculated using the discrete values of the original signal.

The proposed decomposition theorems are exact in theory. For a noise-free real signal with m components, the method can exactly compute the frequency, amplitude, and phase of each component using only $2m-1$ discrete values and their corresponding second-order derivatives. For a noise-free complex signal, the number of discrete values decreases to $2m-1$, and the second-order derivatives are replaced by first-order derivatives. The proposed methods have very high resolution, and the sampling rate need not obey the sampling theorem.

For noisy signals, the proposed methods exhibit extraordinary accuracy. It is possible for the MSEs of frequency, amplitude, and phase to be less than their CRBs.

For multifrequency discrete real and complex signals, this may be the first time that exact decomposition of frequencies, amplitudes, and phases of component signals has been realized in theory.

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