Classical Circuit Simulation of the Non-Reciprocal Aubry-Andr'e Model

Authors: 成恩宏 (Enhong Cheng), Li-Jun Lang, Li-Jun Lang

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Abstract

The introduction of non-Hermiticity extends concepts in traditional Hermitian quantum systems and induces many novel physical phenomena, such as the non-Hermitian skin effect unique to non-Hermitian systems, which has made the simulation of non-Hermitian quantum models a focus of attention. Compared with quantum platforms, classical systems possess advantages such as low cost, mature technology, and room-temperature operation, among which classical circuit systems are even more flexible. In principle, they can simulate quantum tight-binding models of arbitrary dimension, arbitrary inter-site hopping, and arbitrary boundary conditions, and have become a powerful platform for simulating quantum states of matter. In this work, using classical circuits, we successfully simulate the steady-state properties of an important non-Hermitian quantum model—the non-reciprocal Aubry-Andr'e model—via SPICE. This model features both non-reciprocal site hopping and quasi-periodic on-site potentials. Using this as an example, we elaborate in detail on how to establish the mapping between the Laplacian form of classical circuits and the Hamiltonian matrix of quantum tight-binding models under different boundary conditions, particularly how to construct the non-reciprocity of the model using current-type negative impedance converters. Then, based on the Green' s function of the circuit, by driving with AC current and measuring the voltage response, we use SPICE to simulate the complex energy spectrum and corresponding spectral winding number under periodic boundary conditions, as well as the competition between skin and localized modes under open boundary conditions. Furthermore, to prevent the circuit response from diverging, we also analytically provide the design principles for auxiliary components. The results show that the SPICE simulations agree well with theoretical calculations, providing detailed guidance for further experimental implementation. Due to the universality of the circuit design and measurement scheme in this work, it can in principle be directly applied to the circuit simulation of other non-Hermitian quantum models.

Full Text

Electrical Circuit Simulation of Nonreciprocal Aubry-André Models

Enhong Cheng and **Li-Jun Lang**1,2,*

1Guangdong Provincial Key Laboratory of Quantum Engineering and Quantum Materials, School of Physics and Telecommunication Engineering, South China Normal University, Guangzhou 510006, China
2Guangdong Provincial Key Laboratory of Nuclear Science, Institute of Quantum Matter, South China Normal University, Guangzhou 510006, China

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Abstract

The introduction of non-Hermiticity extends the concepts of traditional Hermitian quantum systems and induces many novel physical phenomena, such as the non-Hermitian skin effect that is unique to non-Hermitian systems. This has made the simulation of non-Hermitian quantum models a focus of attention. Compared to quantum platforms, classical systems offer advantages such as low cost, mature technology, and room-temperature operation. Among these, classical electrical circuits are particularly flexible, as they can in principle simulate quantum tight-binding models of arbitrary dimension, with arbitrary-range hopping and arbitrary boundary conditions, making them a powerful platform for quantum state simulation.

In this work, we successfully simulate the steady-state properties of an important non-Hermitian quantum model—the nonreciprocal Aubry-André (AA) model—using classical electrical circuits via SPICE. This model features both nonreciprocal hopping between sites and quasiperiodic on-site potentials. As a concrete example, we detail how to establish the mapping between the Laplacian formalism of classical circuits and the Hamiltonian matrix of quantum tight-binding models under different boundary conditions, with particular emphasis on constructing nonreciprocity using current-inversion negative impedance converters (INICs). Based on the circuit' s Green' s function, we then simulate the complex energy spectrum and corresponding spectral winding number under periodic boundary conditions, as well as the competition between skin and localization modes under open boundary conditions, by driving the circuit with AC currents and measuring voltage responses. To prevent divergent circuit responses, we also derive analytical principles for setting auxiliary components.

The results demonstrate excellent agreement between SPICE simulations and theoretical calculations, providing detailed guidance for future experimental implementations. Due to the universality of our circuit design and measurement scheme, it can in principle be directly applied to the circuit simulation of other non-Hermitian quantum models.

I. Introduction

In recent years, non-Hermitian physics [?] has attracted widespread attention across various fields of physics. Beyond its traditional role in describing gain and loss phenomena in classical systems, it has also been used to characterize the properties of open quantum systems [?]. Unlike the Hermitian Hamiltonians of closed systems, the introduction of non-Hermiticity extends the paradigm of conventional quantum mechanics [?], leading to concepts such as complex energy spectra and biorthogonal bases, which induce many novel phenomena: parity-time (PT) symmetry breaking [?], non-Hermitian degeneracies [?, ?], mode switching [?], and others. Non-Hermiticity has similarly expanded our understanding of topological states. A notable anomaly is that the bulk-boundary correspondence, a fundamental principle in Hermitian topological systems, does not always hold in non-Hermitian systems [?]. The non-Hermitian skin effect [?, ?], which exists exclusively in non-Hermitian systems, is considered a key reason for this breakdown. The discovery of the non-Hermitian skin effect has triggered extensive research into its competition with traditional Hermitian quantum effects, such as its interplay with Anderson localization [?] and Hubbard interactions [?].

Given the importance of non-Hermitian physics, experimental simulation of non-Hermitian models and their unique phenomena is particularly crucial. Compared to traditional quantum platforms (such as cold atom systems [?, ?, ?]), classical systems offer natural advantages for simulating non-Hermitian models. In addition to features like low cost and mature technology, they can directly implement non-Hermiticity through gain and loss, making them powerful platforms for simulating non-Hermitian systems, as demonstrated in optical systems [?, ?] and mechanical systems [?, ?]. Among these, classical electrical circuits, with their unrestricted network configurations and high degree of tunability, can in principle simulate quantum tight-binding models of arbitrary dimension, with arbitrary-range hopping and arbitrary boundary conditions, making them strong competitors for quantum system simulation. Numerous non-Hermitian quantum models and phenomena have already been successfully implemented in circuit systems, including PT symmetry breaking [?, ?], nonreciprocal Su-Schrieffer-Heeger (SSH) models [?, ?], two-dimensional nonreciprocal Chern insulators [?, ?], nonreciprocal higher-dimensional models [?, ?], nonreciprocal Dirac models [?], and non-Hermitian exceptional lines [?].

Regarding the competition between the non-Hermitian skin effect and quasi-disorder, Ref.~[?] proposed the nonreciprocal Aubry-André (AA) model:

$$\hat{H} = \kappa \sum_{n=1}^{N-1} (e^{\alpha}|n+1\rangle\langle n| + e^{-\alpha}|n\rangle\langle n+1| + v_n|n\rangle\langle n|) + (e^{\alpha}e^{-i\Phi}|1\rangle\langle N| + e^{-\alpha}e^{i\Phi}|N\rangle\langle 1|)$$

where $\kappa e^{\pm\alpha}$ describes the nearest-neighbor hopping strength. If $\alpha \neq 0$, the forward and backward hopping strengths are unequal, indicating nonreciprocity. The term $v_n = 2\lambda\cos(2\pi\beta n)$ represents the on-site quasiperiodic/disorder potential, with λ denoting the disorder strength and β generally taken as an irrational number. For finite-size systems, one can choose the number of sites $N = F_s$ and $\beta = F_{s-1}/F_s$ to ensure the quasiperiodic potential is incommensurate with the lattice periodicity within the system size, where F_s denotes the s-th Fibonacci number. To clearly represent the boundary conditions in finite-size systems, the hopping term between the first and last sites is explicitly written in Eq.~(1) (within the second parentheses) and can represent a Hamiltonian with magnetic flux Φ inserted into the one-dimensional ring (after gauge fixing). For convenience in subsequent circuit simulations, we write the Hamiltonian $\hat{H}$ in matrix form using the site basis $\{|n\rangle\}$:

$$H = \kappa \begin{pmatrix} v_1 & e^{-\alpha} & 0 & \cdots & e^{\alpha}e^{-i\Phi} \\ e^{\alpha} & v_2 & e^{-\alpha} & \cdots & 0 \\ 0 & e^{\alpha} & v_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & e^{-\alpha} \\ e^{-\alpha}e^{i\Phi} & 0 & 0 & e^{\alpha} & v_N \end{pmatrix}$$

The non-Hermitian topological properties of this model and the competition between skin and localization effects have been theoretically discussed in detail in Ref.~[**?**]. We briefly review the main conclusions here: Under periodic boundary conditions, when the quasi-disorder strength is weak ($\lambda < \max\{e^{\alpha}, e^{-\alpha}\} \equiv \lambda_c$), the system' s eigenstates are extended, and its eigenenergy spectrum forms a loop winding around the origin in the complex plane, indicating a non-Hermitian topological phase characterized by an energy winding number $\nu = \pm 1$. As the quasi-disorder strength increases beyond $\lambda > \lambda_c$, the eigenstates undergo a transition from extended to localized states, and simultaneously the eigenenergy spectrum collapses into a line on the real axis, indicating a topologically trivial phase with corresponding energy winding number $\nu = 0$. Interestingly, the topological phase transition coincides exactly with the localization transition point because localization changes the distribution of the eigenenergy spectrum in the complex plane, thereby affecting the energy winding number. Correspondingly, under open boundary conditions, because the localization transition makes the system insensitive to boundaries, the system exhibits the same phase transition point. The difference is that in the topological phase region, the eigenstates become skin modes localized at one edge (determined by the sign of α) and the eigenenergy spectrum becomes real; in the localization phase region, the decay lengths of eigenstates at both edges become different.

The main purpose of this paper is to provide a detailed introduction to simulating quantum tight-binding models using the circuit Laplacian formalism, using the nonreciprocal AA quantum model as an example, to facilitate interested readers in applying similar methods to simulate other quantum models and to provide detailed guidance for experimental implementation. The remainder of

the paper is organized as follows: Section II details how to construct the mapping between the Laplacian formalism of classical circuits and tight-binding models. Section III presents the specific circuit design for implementing the nonreciprocal AA model under different boundary conditions. Sections IV and V use SPICE to simulate the energy spectrum and winding number under periodic boundary conditions and the competition between skin and localization modes under open boundary conditions, respectively. The final section provides a summary.

II. Correspondence Between Circuit Laplacian Formalism and Tight-Binding Models

Any classical circuit network can be represented by a graph, with its nodes and edges corresponding to circuit connection points and components [**?**, **?**, **?**]. For circuits composed of passive elements such as resistors, inductors, and capacitors (RLC), the constitutive equations for each element are $V = RI$, $V = L\frac{dI}{dt}$, and $I = C\frac{dV}{dt}$, where V and I represent the voltage difference across and current through the element, respectively, and (R, L, C) are the resistance, inductance, and capacitance. Using Kirchhoff' s current law and the above constitutive equations, we obtain a differential equation for each node n in the graph:

$$I_n(t) = C_{ng}\frac{d_n^{2V}(t)}{dt^2} + R_{ng}^{-1}V_n(t) + L_{ng}^{-1}V_n(t) + \sum_{m\neq n}\Big[C_{nm}\frac{d^2}{dt^2}[V_n(t) - V_m(t)] + R_{nm}^{-1}[V_n(t) - V_m(t)] + L_{nm}^{-1}[V_n(t) - V$$

where $I_n(t)$ and $V_n(t)$ represent the external input current and ground-referenced voltage at circuit node n, respectively. We use (R_{nm}, L_{nm}, C_{nm}) to denote the effective resistance, inductance, and capacitance from node n to node m (subscript g represents ground), allowing description of more general nonreciprocal elements. Conventional passive RLC elements are reciprocal, i.e., $R_{nm} = R_{mn} \equiv R$, $L_{nm} = L_{mn} \equiv L$, $C_{nm} = C_{mn} \equiv C$. The above equation can be written in a more compact matrix form:

$$\mathbf{I}(t) = \mathbf{C}\frac{d^2\mathbf{V}(t)}{dt^2} + \mathbf{R}\frac{d\mathbf{V}(t)}{dt} + \mathbf{L}\mathbf{V}(t)$$

where $\mathbf{I}(t)$ and $\mathbf{V}(t)$ are column vectors of node input currents and ground-referenced voltages, respectively, and $(\mathbf{R}, \mathbf{L}, \mathbf{C})$ are coefficient matrices of effective resistances, inductances, and capacitances with matrix elements $C_{nn} = C_{ng} + \sum_{m\neq n} C_{nm}$, $R_{nn} = R_{ng}^{-1} + \sum_{m\neq n} R_{nm}^{-1}$, $L_{nn} = L_{ng}^{-1} + \sum_{m\neq n} L_{nm}^{-1}$, and for $n \neq m$: $C_{nm} = -C_{nm}$, $R_{nm} = -R_{nm}^{-1}$, $L_{nm} = -L_{nm}^{-1}$.

For the differential equation (5), we consider AC current sources $\mathbf{I}(t)$ and their voltage responses $\mathbf{V}(t)$ at a fixed frequency ω, with the form $\mathbf{I}(t) = \mathbf{I}e^{i\omega t}$ and

$\mathbf{V}(t) = \mathbf{V}e^{i\omega t}$. Substituting these into Eq.~(5) yields a time-independent matrix equation:

$$\mathbf{I} = \left(i\omega\mathbf{C} + \mathbf{R} + \frac{\mathbf{L}}{i\omega}\right)\mathbf{V} \equiv \mathbf{J}(\omega)\mathbf{V}$$

The defined $\mathbf{J}(\omega)$ is called the circuit Laplacian matrix or Kirchhoff matrix [**?**], which has the dimension of admittance and generally has complex matrix elements that depend on the driving frequency ω. Without external current input, i.e., $\mathbf{J}(\omega)\mathbf{V} = 0$, the condition $\det \mathbf{J}(\omega_c) = 0$ determines the circuit' s eigenfrequency spectrum $\{\omega_c\}$. Alternatively, we can invert Eq.~(8) to obtain $\mathbf{V} = \mathbf{J}^{-1}(\omega)\mathbf{I} \equiv \mathbf{G}(\omega)\mathbf{I}$, where $\mathbf{G}(\omega) \equiv \mathbf{J}^{-1}(\omega)$ is called the circuit Green' s function, which has the dimension of impedance.

In fact, for steady-state solutions of the form (7), any circuit network can be expressed in Laplacian form, including active circuits with amplifiers [**?**, **?**, **?**] and nonlinear circuits with nonlinear elements [**?**, **?**].

To simulate quantum tight-binding models using classical circuits, we can directly relate the circuit Laplacian matrix $\mathbf{J}(\omega)$ to the real-space Hamiltonian matrix $\mathbf{H}$ of the tight-binding model [**?**]. The eigenvalue equations of the Laplacian

$$\mathbf{J}(\omega)\psi_n^{(r)}(\omega) = j_n(\omega)\psi_n^{(r)}(\omega), \quad \mathbf{J}^\dagger(\omega)\psi_n^{(l)}(\omega) = j_n^*(\omega)\psi_n^{(l)}(\omega)$$

directly correspond to the stationary Schrödinger equation of the Hamiltonian matrix. Here, $j_n(\omega)$ is the n-th eigenvalue of the Laplacian matrix, forming the eigen-admittance spectrum that can completely simulate the energy spectrum of $\mathbf{H}$. In particular, from $\det \mathbf{J}(\omega_c) = 0$, we know that at eigenfrequencies ω_c, at least one eigen-admittance vanishes, i.e., $j_n(\omega_c) = 0$.

Since $\mathbf{J}(\omega)$ is generally non-Hermitian, i.e., $\mathbf{J}^\dagger(\omega) \neq \mathbf{J}(\omega)$, the corresponding eigenvectors typically include both right and left eigenvectors $\psi^{(r,l)}(\omega)$. Through AC analysis of circuits, we can obtain the right eigenmodes $\psi_n^{(r)}(\omega)$ of $\mathbf{J}(\omega)$, thereby simulating the right eigenstates of $\mathbf{H}$; the left eigenstates can be simulated using $\mathbf{J}^\dagger(\omega)$. Through design and adjustment of components and driving frequencies, $\mathbf{J}(\omega)$ is highly controllable and can in principle simulate a very broad range of quantum models, including arbitrary dimensions, boundary conditions, nonlinearities, and non-Hermitian properties.

Next, we apply the above circuit Laplacian formalism to construct the correspondence between $\mathbf{J}(\omega)$ and the Hamiltonian matrix $\mathbf{H}$ of the nonreciprocal AA model, thereby simulating its steady-state properties, including the energy spectrum and winding number under periodic boundary conditions, as well as the competition between skin and localization modes under open boundary conditions.

III. Circuit Laplacian for the Nonreciprocal AA Model

Under AC driving, passive elements typically exhibit reciprocity, as determined by the fundamental Kirchhoff current law. For example, the admittances of capacitors and inductors, $J_C(\omega) = i\omega C$ and $J_L(\omega) = 1/i\omega L$, are independent of the measurement direction. According to Eq.~(8), to achieve nonreciprocity in the Laplacian matrix, i.e., $J_{mn}(\omega) \neq J_{nm}(\omega)$, we require circuit components whose admittance values depend on the measurement direction. This typically necessitates active elements, such as the negative impedance converter with current inversion (INIC) [**?**].

As shown in Fig.~1(b), the INIC consists of an operational amplifier and several linear elements. Using Kirchhoff' s current law, we can easily derive the input currents at its two ports:

$$I_l = -\frac{i\omega C_I}{Z_-}(V_l - V_r), \quad I_r = \frac{i\omega C_I}{Z_+}(V_r - V_l)$$

This shows that the admittances in different directions are generally unequal: $J_l(\omega) = -\frac{i\omega C_I}{Z_-}$ and $J_r(\omega) = \frac{i\omega C_I}{Z_+}$. For convenience, we theoretically choose the impedances in the INIC to satisfy $Z_+ = Z_-$, making the currents flowing into the amplifier from both ends equal in magnitude but opposite in direction, i.e., $I_l = I_r$, which yields admittances with opposite signs: $J_l(\omega) = -J_r(\omega)$.

Additionally, we will use components with negative values (such as negative resistors), which can be realized in two ways as shown in Fig.~1(c) [**?**]. They implement effective negative impedance (admittance) for grounded one-port and floating two-port configurations through operational amplifiers. Using Kirchhoff' s laws, the input impedance of the grounded one-port circuit [left panel of Fig.~1(c)] is $(V_g - 2V_g)/Z = -Z$. Similarly, the input impedances of the floating two-port circuit [right panel of Fig.~1(c)] are $Z_{ij} = [V_i - (2V_i - V_j)]/Z = -Z$ and $Z_{ji} = [V_j - (2V_j - V_i)]/Z = -Z$, i.e., $Z_{ji} = Z_{ij} = -Z$.

Using these key elements, we design the circuit simulation schematic for the nonreciprocal AA model shown in Fig.~1(a), based on the lumped-element circuit model for a left-handed transmission line composed of inductors L_0 and capacitors C_0 [**?**]. The circuit consists of N effective voltage nodes V_n $(n = 1, \cdots, N)$ corresponding to the N lattice sites of the model. According to Eq.~(8), we can easily write the Laplacian matrix for this circuit:

$$\mathbf{J}(\omega) = -i\omega \begin{pmatrix} -C_1 & C_0 - C_I & 0 & \cdots & C_0 + C_I \\ C_0 + C_I & -C_2 & C_0 - C_I & \cdots & 0 \\ 0 & C_0 + C_I & -C_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & C_0 - C_I \\ C_0 - C_I & 0 & 0 & C_0 + C_I & -C_N \end{pmatrix} + \left[i\omega(2C_0 + C_r) + \frac{1}{i\omega L_0} + \frac{1}{R_0} \right] \mathbf{E} \equiv -i\omega \mathbf{A} +$$

where $C_n \equiv C_{n,I} + C_{0,r,n}$, and

$$X_{R/L} = C_0 \pm C_I - \frac{1}{\omega^2 L_{a,b}} \mp \frac{1}{\omega^2 L_{b,a}} + \frac{1}{i\omega R_{a,b}} \pm \frac{1}{i\omega R_{b,a}}$$

The capacitors $C_{n,I}$, inductors $L_{a,b}$, resistors $R_{a,b}$, and R_0 are defined in Fig.~1, and $\mathbf{E}$ is the identity matrix. Boundary conditions are controlled by the X-related modules and switches in Fig.~1(a): when both end switches are connected to port o and all X-related modules ($X_{a,b}$ and $X_{l,r}$) are open-circuited, the system corresponds to open boundary conditions; when both end switches are connected to port p, we will see below that by adjusting the parameters of the X-related modules we can simulate periodic boundary conditions with magnetic flux.

Comparing this circuit Laplacian matrix (15) with the Hamiltonian matrix (2) of the nonreciprocal AA model, we can establish a mapping between them (except for the $\mathbf{D}$ term, which is proportional to the identity matrix $\mathbf{E}$): $\mathbf{A} \Leftrightarrow \mathbf{H}$. Using corresponding dimensionless parameters, we obtain the following relationships.

For the main body of the circuit (excluding boundaries), from

$$\frac{C_0 + C_I}{C_0 - C_I} = e^{2\alpha}, \quad \frac{C_{n,I}}{C_0} = -\frac{2\lambda \cos(2\pi\beta n)}{\cosh \alpha}$$

we get

$$\frac{C_I}{C_0} = \tanh \alpha, \quad \frac{C_{n,I}}{C_0} = -\frac{2\lambda \cos(2\pi\beta n)}{\cosh \alpha}$$

Here we take C_0 as the reference capacitance. This allows us to understand the role of each circuit component: (C_0, C_r, L_0) constitute the overall reference potential of the tight-binding model (the $\mathbf{D}$ part). C_0 also provides the reciprocal coupling between sites, while the nonreciprocal coupling and varying on-site potentials are implemented by the capacitor C_I in the INIC and the grounded capacitors C_n, respectively. As we will see later, the introduction of resistor R_0 and capacitor C_r is to prevent divergent circuit responses; they only cause a global shift of the admittance spectrum in the complex plane. Note that C_n may be required to be negative depending on node n; effective negative capacitance can be realized using the scheme in Fig.~1(c).

For the boundary part, using the corresponding dimensionless parameter relationships

$$\frac{X_R + X_L}{2C_0} = \frac{e^{\alpha} e^{-i\Phi} + e^{-\alpha} e^{i\Phi}}{e^{\alpha} + e^{-\alpha}}, \quad \frac{X_R - X_L}{2C_0} = \frac{e^{\alpha} e^{-i\Phi} - e^{-\alpha} e^{i\Phi}}{e^{\alpha} - e^{-\alpha}}$$

we obtain

$$\frac{1}{\omega^2 L_a} = \frac{1-\cos\Phi}{L_0 C_0}, \quad \frac{1}{\omega^2 L_b} = \frac{1}{L_a}\coth\alpha; \quad \frac{1}{R_a} = \frac{\omega\sin\Phi}{L_0 C_0}, \quad \frac{1}{R_b} = \frac{1}{R_a}\coth\alpha$$

Here we take L_0 and R_0 as reference inductance and resistance, respectively, and define the reference frequency $\omega_0 \equiv 1/\sqrt{L_0 C_0}$ and the dimensionless quantity $\gamma \equiv R_0^{-1}\sqrt{L_0/C_0}$. When Φ is an integer multiple of 2π, $L_{a,b}$ and $R_{a,b}$ all diverge, representing open circuits and corresponding to periodic boundary conditions without magnetic flux. As Φ varies, $R_{a,b}$ may become negative, which can also be realized using the scheme in Fig.~1(c).

IV. Simulation of Energy Spectrum and Winding Number Under Periodic Boundary Conditions

It is well known that even under periodic boundary conditions, disordered systems no longer have translational invariance, so the winding number cannot be calculated by transforming the Hamiltonian to momentum space. The usual approach is to insert a magnetic flux Φ at the center of the ring, making the system a periodic function of Φ (with period 2π) for calculation. For non-Hermitian systems, since energies are generally complex, one can define the winding number of complex energy in the complex plane to characterize the topological phase of non-Hermitian systems [**?**, **?**]:

$$\nu = \frac{1}{2\pi i}\int_0^{2\pi} d\Phi \frac{\partial}{\partial\Phi}\ln\det\mathbf{H}(\Phi) = \frac{1}{2\pi}\int_0^{2\pi} d\Phi \frac{\partial}{\partial\Phi}\theta(\Phi)$$

where $\theta(\Phi)$ is the argument of $\det\mathbf{H}(\Phi)$. For the nonreciprocal AA model, Ref.~[**?**] shows that different winding numbers represent different topological phases: under periodic boundary conditions, $\nu = 0$ indicates a topologically trivial localized phase, while $\nu = \pm 1$ indicates two topologically nontrivial extended phases.

To simulate and measure the winding number ν of the nonreciprocal AA model using circuits, we can replace $\mathbf{H}$ in definition (21) with $\mathbf{A}/C_0$ from the Laplacian matrix (15) (dividing by C_0 ensures the argument of the logarithm is dimensionless; overall scaling does not affect ν). Therefore, as long as we can experimentally measure the matrix $\mathbf{A}(\Phi)$ for different values of Φ, we can calculate the corresponding winding number.

We perform SPICE simulations using the circuit' s Green' s function form (9). For a circuit with periodic boundary conditions (both switches in Fig.~1 set to port p), we connect an AC current source with frequency ω only to node n, measure the voltage responses at all N nodes, and divide by the input current intensity to obtain the n-th column of the circuit' s Green' s function matrix $\mathbf{G}(\omega)$. Repeating this operation for each node yields the entire Green' s function matrix. The Laplacian matrix and corresponding matrix $\mathbf{A}$ can then be

obtained from the relation $\mathbf{J}(\omega) = \mathbf{G}^{-1}(\omega)$ [**?**]. With the experimentally measured Laplacian matrix $\mathbf{J}(\omega)$, we can calculate its admittance spectrum and corresponding left/right eigenvectors, as well as any physical quantities defined therefrom, for comparison with theory.

It should be noted that the above analysis is based on the steady-state response of the circuit system under AC current drive, i.e., Eq.~(7). In reality, when the driving frequency approaches an eigenfrequency, besides producing a pronounced resonant response, it can also excite other eigenmodes of the circuit system. Such responses are typically called transient responses. The experimental method for obtaining steady-state response generally involves delayed measurement, waiting for the transient response to decay before using a lock-in amplifier to capture the steady-state signal. Under periodic boundary conditions, the eigenenergies of the nonreciprocal AA model become complex, and correspondingly, the circuit' s eigenfrequencies ω_c are generally also complex [Fig.~2(b,c) left panels], meaning that transient responses at these frequencies will diverge over time ($\mathrm{Im}[\omega_c] < 0$) or decay ($\mathrm{Im}[\omega_c] > 0$). For a system driven at frequency $\omega \in \mathbb{R}$, divergence is detrimental to stable response, so we must consider suppressing the transient response, which we achieve by selecting appropriate R_0.

We can calculate the imaginary parts of the eigenfrequencies using a flux-free circuit ($\Phi = 0$, where $\mathbf{A}$ is independent of ω) to estimate the required R_0. Writing Eq.~$\mathbf{J}(\omega_c)\mathbf{V} = 0$ in eigenvalue form yields the eigenfrequency condition:

$$\left[(2+r) - \frac{\omega_c^2}{\omega_0^2}\right]^2 - a_n \left[(2+r) - \frac{\omega_c^2}{\omega_0^2}\right] - \gamma^2 = 0$$

where $r = C_r/C_0$ and a_n denotes the n-th eigenvalue of $\mathbf{A}/C_0$, which is generally complex under periodic boundary conditions [**?**]. Therefore, by choosing appropriate R_0 and C_r such that all eigenfrequency imaginary parts are nonnegative, i.e., $\min_n(\mathrm{Im}[\omega_{c,n}]) \geq 0$, the system response will not diverge over time. When the system is driven at a specific frequency ω, the stable response will be dominated by the ω mode [**?**]. As derived in Appendix A, the conditions for non-divergent circuit response are:

$$r \geq \max(\mathrm{Re}[a_n]) - 2 \quad \text{and} \quad \gamma \geq \max_n \frac{|\mathrm{Im}[a_n]|}{\sqrt{2 + r - \mathrm{Re}[a_n]}}$$

Additionally, to excite as many eigenmodes as possible, the driving frequency should be positioned within the eigen-spectrum. Since the energy spectrum of the nonreciprocal AA model $\mathbf{H}$ is distributed near the origin of the complex plane, the eigenvalues a_n of $\mathbf{A}(\Phi = 0)/C_0$ share this characteristic. Therefore, setting $\gamma = 0$ and $a_n = 0$ in Eq.~(23) yields a suitable driving frequency $\omega = \omega_0/\sqrt{2+r}$. This choice also ensures that $\mathbf{D} = R_0^{-1}\sqrt{L_0/C_0}(2+r)\mathbf{E}$ in Eq.~(15)

takes a simple form. Unless otherwise specified, all calculations below use $\omega = \omega_0/\sqrt{2+r}$.

Using SPICE, we simulate a circuit with $N = 21$ nodes under periodic boundary conditions [Fig.~1(a)]. The basic component values are $(L_0, C_0, R_0, C_r) = (10\,\mu\text{H}, 0.4\,\mu\text{F}, 5\,\Omega, 1.6\,\mu\text{F})$, giving $\omega_0 = 0.5\,\text{MHz}$ and $(r, \gamma) = (4, 1)$. The driving frequency is chosen as $\omega = \omega_0/\sqrt{6} \approx 0.2\,\text{MHz}$. Other components $(C_I, C_n, L_{a,b}, R_{a,b})$ are determined from model parameters using Eqs.~(18) and (20), with the quasiperiodic potential parameter $\beta = 13/21$. Using this scheme, we perform SPICE simulations for three typical regions in the theoretical phase diagram [Fig.~2(a)]. In the topological region with $\nu = \pm 1$, the simulated eigen-spectrum of $\mathbf{A}/C_0$ forms a loop winding around the origin in the complex plane [Fig.~2(b) right panel], while in the localized region with winding number $\nu = 0$, it collapses into a line on the real axis [Fig.~2(c) right panel]. Using the simulated $\mathbf{A}(\Phi)/C_0$ to calculate the phase $\theta(\Phi) = \arg\det[\mathbf{A}(\Phi)/C_0]$ as a function of Φ [Fig.~2(d)], we obtain the simulated winding numbers. The results show excellent agreement between simulation and theory.

V. Simulation of Competition Between Skin and Localization Modes Under Open Boundary Conditions

Reference~[**?**] proved that the nonreciprocal AA model has identical phase diagrams under open and periodic boundary conditions, except that states in the topological phase region exhibit skin modes under open boundary conditions while showing extended states under periodic boundary conditions. This section uses the same circuit Laplacian method to simulate the competition between skin and localization modes under open boundary conditions.

In terms of circuit design, open boundary conditions are realized by connecting both end switches in Fig.~1(a) to port o and leaving all X-related modules open-circuited. Similarly, we can reconstruct the circuit Laplacian matrix and corresponding $\mathbf{A}/C_0$ under open boundary conditions through SPICE simulation, then calculate the left/right eigenvectors to observe the competition between skin and localization modes under different parameters. Here we adopt a simpler method that achieves this without sequentially connecting the AC current source to each node.

The circuit' s Green' s function (9) can be expressed in terms of the left/right eigenvectors of the Laplacian as:

$$\mathbf{V} = \mathbf{G}(\omega)\mathbf{I} = \sum_n \frac{\psi_n^{(l)}(\omega)[\psi_n^{(l)}(\omega)]^\dagger \mathbf{I}}{j_n(\omega)} \psi_n^{(r)}(\omega)$$

Here, skin or localization modes are manifested by whether the distribution of matrix elements of the left/right eigenvectors tends toward one edge or is

localized at some intermediate position. By simply connecting an AC current source to a single node and measuring the corresponding voltage response, we can observe the competition. According to the above equation, this voltage response is a linear superposition of all eigen-right vectors $\psi_n^{(r)}(\omega)$ in the same phase region with coefficients $j_n^{-1}(\omega)[\psi_n^{(l)}(\omega)]^\dagger \mathbf{I}$, so it must exhibit either skin or localization effects.

We use the same component parameters as for periodic boundary conditions, except with $\gamma = 0$ (i.e., R_0 open-circuited). This is because under open boundary conditions, all eigenfrequencies are real, as shown in Fig.~2(b,c), so there is no divergence problem and no need for resistive suppression. Additionally, we require $\mathbf{D}(\omega) = 0$ in the Laplacian matrix; otherwise, when this term is large, all eigenvalues $j_n(\omega)$ tend toward a constant j, and from Eq.~(25) we would have $\mathbf{V} \to j^{-1}\mathbf{I}$, which is proportional to the input current and cannot reflect the competition. This is another reason for choosing $\gamma = 0$.

We simulate a circuit system with $N = 21$ nodes under open boundary conditions using SPICE. An AC current source with frequency $\omega = \omega_0/\sqrt{6}$ is connected to node $n_i = 11$, and the voltage amplitude at each node at frequency ω is measured. The results are shown in Figs.~3(a) and 3(b), clearly demonstrating that in the topological phase region, the response voltage distribution is near the right/left boundary, exhibiting right/left skin modes, while in the non-topological phase region, the response voltage always remains near the driving node, showing localized states.

To characterize the degree of localization of the corresponding voltage, we define the inverse participation ratio (IPR):

$$\mathrm{IPR} = \frac{\sum_n |V_n|^4}{\left(\sum_n |V_n|^2\right)^2}$$

Figures~3(c) and 3(d) show that the points with minimum IPR are close to the theoretical phase transition points. This is because both skin and localized states have strong localization, corresponding to large IPR values, while near the phase transition point the states are most extended, yielding small IPR values. At the phase boundary between left and right skin modes (i.e., the boundary between $\nu = +1$ and $\nu = -1$), corresponding to the reciprocal model, the skin effect disappears and the eigenstates become extended. At the boundary between skin and localized phases (i.e., between $\nu = \pm 1$ and $\nu = 0$), corresponding to the balance between skin and localization competition, the states are also extended.

VI. Summary

In this paper, by constructing classical electrical circuits and mapping their Laplacian matrices to the Hamiltonian matrix of the nonreciprocal AA model, we successfully simulate important steady-state properties of this model using

SPICE. These include the complex energy spectrum and spectral winding number that characterize the non-Hermitian topological properties under periodic boundary conditions, as well as the competition between non-Hermitian skin effects and quasi-disorder localization under open boundary conditions. We discuss in detail the principles and theoretical basis for setting circuit parameters, providing specific guidance for future experimental implementations. Due to the universality of our scheme, the design principles and theories discussed here can be directly applied to the simulation and experimentation of other quantum tight-binding models, such as the dual model of the nonreciprocal AA model mentioned in Ref.~[**?**]. This would only require appropriate modifications to the circuit scheme in Fig.~1, removing the INIC elements between nodes to achieve reciprocal hopping and adjusting the corresponding grounded elements to implement quasiperiodic complex on-site potentials.

The SPICE simulations in this paper were performed using LTspice software. To better approximate theoretical results, we used ideal linear components for capacitors, inductors, and resistors, and set the open-loop gain and input impedance of the INIC amplifiers to 500 G and 500 GΩ, respectively, to simulate ideal amplifiers. Due to the non-ideal nature of real components, simulation or experimental results may have some deviations that require case-specific analysis.

This paper focuses only on simulating the steady-state properties of quantum tight-binding models. In fact, classical circuits can also be used to simulate dynamical properties, as discussed for the nonreciprocal AA model in Ref.~[**?**]. Additionally, due to the rich characteristics of circuit components, nonlinear circuit elements can be utilized to simulate nonlinear quantum systems [**?**]. Therefore, classical circuits represent a powerful platform for simulating quantum systems that is low-cost, technologically mature, and broadly applicable.

Appendix A: Conditions for Non-Divergent Circuit Response

Based on the eigenfrequency expression (23), to ensure non-divergent circuit response we require $\mathrm{Im}[\omega_c^{(n)}/\omega_0] \geq 0$, which gives:

$$0 \geq 2\mathrm{Im}\sqrt{(2+r)-a_n-\gamma^2} = -\gamma - \mathrm{Im}\sqrt{(2+r)+a_n} - \mathrm{Im}\sqrt{(2+r)-a_n}$$

$$\Rightarrow -\gamma \leq \mathrm{Im}\sqrt{(2+r)+a_n} + \mathrm{Im}\sqrt{(2+r)-a_n}$$

Here we treat the square root as a complex number. Since the left side of the inequality is non-positive ($\gamma \geq 0$), we only need to consider the condition when

the right side (twice the real part of the square root) is non-positive. Therefore, squaring both sides of inequality (A1) and simplifying yields:

$$\gamma^2 + (2+r) - a_n + (2+r) - a_n^* \geq 2\sqrt{(2+r)+a_n}\sqrt{(2+r)-a_n}$$

Squaring both sides again and simplifying gives:

$$[(2+r) - \mathrm{Re}(a_n)]\gamma^2 \geq [\mathrm{Im}(a_n)]^2$$

Since the right side of the inequality is non-negative, we must require $(2 + r) - \mathrm{Re}(a_n) \geq 0$, which gives $r \geq \mathrm{Re}(a_n) - 2$. Additionally, we obtain $\gamma \geq |\mathrm{Im}(a_n)|/\sqrt{2 + r - \mathrm{Re}(a_n)}$. Because all eigenfrequencies must satisfy these conditions, we require:

$$r \geq \max_n(\mathrm{Re}[a_n]) - 2 \quad \text{and} \quad \gamma \geq \max_n \frac{|\mathrm{Im}[a_n]|}{\sqrt{2 + r - \mathrm{Re}[a_n]}}$$

Note: Figure translations are in progress. See original paper for figures.