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Abstract

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Full Text

Preamble

Some Properties of a Hyperbolic Model of Complex Networks for the Small Parameter

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We analyze properties of degree and clustering in a hyperbolic geometric model of complex networks for the small parameter case $\tau < 1$, $2\sigma < 1$. We find that the probability of k-degree goes to 0 and the global clustering coefficient goes to 0 in probability as the number of nodes N $\to \infty$ for some specific growth R(N) of the region radius. Here the scale-free degree property fails and connections between neighbors become very weak. The transition of model properties with parameter σ changes seems to show that mobility is important for maintaining full and stable communication in society; otherwise, a silent society emerges. Some analysis techniques and methods are first applied to such a model.

Keywords: hyperbolic geometry model; complex network; k-degree; mean degree; clustering.

1. Introduction

Many models of complex networks have been proposed to capture common properties of real-world systems. Perhaps the most well-known is the Barabási-Albert model of preferential attachment [?], which is based on two generic mechanisms: (i) networks expand continuously by the addition of new vertices, and (ii) new vertices attach preferentially to sites that are already well-connected. This model reproduces the observed stationary scale-free distributions and indicates that the development of large networks is governed by robust self-organizing phenomena that go beyond the particulars of individual systems. Another common property of social networks is the formation of cliques, representing circles of friends or acquaintances in which every member knows every other member. This inherent tendency to cluster is quantified by the clustering coefficient or the global clustering coefficient. Although the Barabási–Albert model effectively captures the scale-free degree sequence, it fails to generate networks with clustering phenomena. There are growing indications that the tendency for affiliation among neighbors of a node is a manifestation of the network having a geometric structure [?], based upon the notion of a geometric random graph. Statistical mechanical models of complex networks were introduced in [?], and in particular applied to random geometric graphs in hyperbolic space [?].

The model employs a random mapping of graph nodes to points in a ball centered at the origin in the hyperbolic space $\mathbb{H}^{d+1}$, giving these nodes hyperbolic coordinates, and connection probability based on hyperbolic geometry. Therefore, random geometric graphs in hyperbolic geometry provide extremely promising models for the structure of complex networks.

In [?], we extended the hyperbolic geometric random graph model of Krioukov et al. [?] to arbitrary dimension and considered this model in five regions of the parameter space for any dimension. We found power-law (scale-free) expected degree distributions for two regions, while for others the power law disappears and the probability of k-degree goes to zero. The case $\tau < 1$ and $2\sigma < 1$ remained unresearched since the main difficulty lies in that the probability of connection between a fixed node u and a random node v is mostly contributed by $r_v < R - r_u + \omega(N)$, and the previous explicit formula for hyperbolic distance is powerless for this case.

In this article, we continue our study for the remaining parameter region $\tau < 1$ and $2\sigma < 1$. We explore the hyperbolic distance formula deeply, overcome some technical difficulties, and obtain estimations for the probability integral of connection, enabling us to analyze the degree and clustering properties of the model. We find that the probability of k-degree goes to 0 and the global clustering coefficient goes to 0 in probability as the number of nodes $N \to \infty$ for some specific growth $R(N)$ of the region radius. Some analysis methods and techniques are first applied to such a model. For the clustering study, we partly borrow methods from Fountoulakis for $\tau < 1$, $2\sigma > 1$ and $d = 1$ in [?].

In generalization of analysis results in [?, ?, ?] and this article, for N-node random geometric graphs $G(N, \zeta, \tau, \sigma, \nu, d)$ in the hyperbolic ball with graph Hamiltonian energy parameter $\tau < 1$ and rescaled hyperbolic ball radius $R_H = \ln N^{\nu}$, we have the following classification. There exists a power law of degrees for $2\sigma > 1, d \geq 1$ in [?, ?]; and there don' t exist scale-free degree distributions and the probability of k-degrees goes to zero as $N \to \infty$ for $2\sigma \leq 1, d \geq 1$ in [?] and this article. The global clustering coefficient will converge to some nontrivial constant in probability for $2\sigma > 2, d = 1$, which means the connection ratio of neighbors is scale-free in probability, and the global clustering coefficient will go to zero in probability as $N \to \infty$ for $1 < 2\sigma \leq 2, d = 1$ (we believe such results are correct for $d > 1$) in [?]. We prove that the global clustering coefficient has the same zero-tendency in probability for $\tau < 1, 2\sigma < 1, d \geq 1$ in this article, which means that connections among neighbors become increasingly unlikely. We don' t study the global clustering coefficient for the $2\sigma = 1$ case due to extensive computation, but we believe it has the same zero-tendency. Altogether, we can see that the density function (1.3) with $2\sigma = 1$ is exactly a critical probability for the scale-free degree in the random graph, and so is $\sigma = 1$ for the clustering, as Albert and Barabási mentioned in [?].

There are also many papers such as [?, ?, ?] that study the hyperbolic geometric random graph model with numerical methods, whose approximate results prompt the development of analytical methods.

Why do the degree and clustering of networks become weaker for the growing speed of the radius $R_H = \ln N^{\nu}$ as the parameter σ becomes smaller in the hyperbolic geometric random graph model? We may observe that the probability of nodes (1.3) staying in the interval $[r, r + dr]$ of small radius r increases when the parameter σ becomes smaller, and correspondingly nodes are more likely to stay in the region closer to the origin. For example, with $d = 1$ and $r \leq R/\sqrt{2}$, the density of this region increases when σ becomes smaller, so two nodes in this region are possibly closer and easier to connect since formula (1.2). On the other hand, we can observe that the probability of nodes lying in the region close to the origin decreases when the hyperbolic ball radius increases. For example, with $d = 1$ and $r \leq R/\sqrt{2}$, the probability of nodes staying in this region decreases as the radius grows. Hence these two opposite tendencies interact: the willingness of nodes to stay in the interval $[r, r+dr]$ of small radius (and thus become dense) as σ becomes smaller is resisted by the radius growth at speed $R_H = \ln N^{\nu}$. In contrast, the willingness of nodes to stay in regions far from the origin as σ becomes larger is consistent with the influence of radius growth. Finally, a critical point is achieved at $2\sigma = 1$ for the scale-free degree and at $\sigma = 1$ for the clustering property of networks at the fixed growth speed of

$R_H = \ln N^{\nu}$. This phenomenon in the model is similar to a principle in human society: the willingness of nodes to stay in the interval $[r, r+dr]$ of small radius represents the willingness of people to be immobile and stay within small circles, while mobility represents the opposite.

If people dislike moving, communication among them will become less and less as society expands at some speed of population growth, finally forming a silent society. Conversely, if the likelihood of mobility increases, communication among people becomes easier and more frequent, so mobility makes society' s communication full and stable, i.e., the scale-free property occurs.

Based on the above analysis, should mobility be another generic mechanism for networks, especially social networks?

1.1 Model Introduction

We analyzed a class of exponential random graph models in [**?**], in which a random graph $G = G(N, \zeta, \tau, \sigma, \nu, d)$ with N vertices, denoting the vertex set by V_N, has positive model parameters ζ, σ, ν, and τ. Its elements are randomly distributed into a ball of radius R centered at the origin in the hyperbolic space $\mathbb{H}^{d+1}$ (integer $d \geq 1$) with probability density

$$\rho_H(\mathbf{x}) = \rho(r)\rho_\theta(\ ) = \rho(r)\rho_1(\theta_1)\cdots\rho_d(\theta_d),$$

and the probability of an edge occurring between vertices u and v is

$$p_{u,v} = \frac{1}{1 + e^{\zeta(d_{u,v} - R)}}. \tag{1.1}$$

Here $\mathbf{x} = (r,\ )$ represents spherical coordinates on $\mathbb{H}^{d+1}$ with the usual coordinates $\ = (\theta_1, \theta_2, \cdots, \theta_d)$ on S^d, where $\theta_k \in [0, \pi)$ for $k = 1 \ldots d-1$ and $\theta_d \in [0, 2\pi)$, and $r \in [0, R)$. The angular density functions are $\rho_k(\theta_k) = \sin^{d-k}\theta_k / I_{d,k}$ with $I_{d,k} = \int_0^\pi \sin^{d-k}\theta\, d\theta$ for $k = 1 \ldots d-1$, and $\rho_d(\theta_d) = \frac{1}{2\pi}$. The radial density is

$$\rho(r) = \frac{\sinh^d(\sigma\zeta r)}{C_d} \tag{1.3}$$

with $C_d = \int_0^R \sinh^d(\sigma\zeta r)\, dr$. The distance function $d_{u,v}$ is the hyperbolic distance between two points $u = (r_u,\ _u)$ and $v = (r_v,\ _v)$, given by the hyperbolic law of cosines:

$$\cosh(\zeta d_{u,v}) = \cosh(\zeta r_u)\cosh(\zeta r_v) - \sinh(\zeta r_u)\sinh(\zeta r_v)\cos\theta_{u,v}, \tag{1.4}$$

where $\theta_{u,v}$ is the angular distance (relative angle) between $_u$ and $_v$ on S^d. The parameter σ governs the radial node distribution, ν the node density, τ

the 'temperature' , and $-\zeta^2$ governs the curvature of the hyperbolic space, respectively.

The conditional probability that a node u at a fixed position in the hyperbolic ball connects to an angularly random node v at fixed radius r_v is

$$\hat{p}_{u,v} = \int_0^\pi \rho_1(\theta_{u,v}) p_{u,v} \, d\theta_{u,v} = \int_0^\pi \sin^{d-1}\theta \, p_{u,v} \, d\theta_{u,v}, \tag{1.5}$$

which we call the angular integral. Let $I_{u,v}$ be an indicator random variable that equals 1 when there is an edge between u and v in the graph, and 0 otherwise. The conditional probability that a node u at a fixed position connects to a random node v is

$$\Pr[I_{u,v} = 1 \mid r_u, \ _u] = \int_0^R \int_0^\pi \sin^{d-1}\theta_{u,v} \, p_{u,v} \, d\theta_{u,v} \, \rho(r_v) \, dr_v. \tag{1.6}$$

The integrals in (1.5) and (1.6) depend only on the relative angle $\theta_{u,v}$, so the probabilities are independent of the specific angular position θ_u. We will sometimes omit this specific angular position. The expected degree of node u at a fixed position is

$$\langle k_u \rangle = \langle k(r_u) \rangle = (N-1) \Pr[I_{u,v} = 1 \mid r_u]. \tag{1.7}$$

If we average over the random position of node u, i.e., computing the node expected degree, we obtain the mean degree

$$\langle k \rangle = \int_0^R \langle k(r_u) \rangle \rho(r) \, dr. \tag{1.8}$$

We also let $D_u = \sum_{v \in V_N \setminus \{u\}} I_{u,v}$ be the number of connections to vertex u, i.e., its degree. We refer to the connection number equaling a positive integer k as k-degree, and denote its probability by $\Pr[D_u = k]$. Throughout, we let $R = R(N) \to \infty$, $\omega(N) = o(R(N)) \to \infty$, $o(1) \to 0$ as $N \to \infty$, and $o_p(1)$ means convergence to 0 in probability as $N \to \infty$. We also use Big Theta Θ notation for some function $f(x)$, i.e., $\Theta(f(x))$, which means $\exists c_1, c_2 > 0$ such that $c_1 f(x) \le \Theta(f(x)) \le c_2 f(x)$.

In this article, the function $f(x)$ is related to N and $N \to \infty$, so we always assume the Big Theta function $\Theta(f(x))$ holds for sufficiently large N. To simplify notation, we rescale variables as follows: $(\eta, R_H, \tau) = (r, R, T)$, $\omega_1(N) = \omega(N)$.

There are also some useful computational results:

$$C_d = (1+o(1))\frac{e^{d\sigma\zeta R}}{\sigma\zeta d 2^d} \quad \text{and} \quad \int_0^r \sinh^d(\sigma\zeta r)\,dr = (1+o(1))e^{\sigma d\xi(r-R)} = (1+o(1))e^{2\sigma(\eta-R_H)} \tag{1.9}$$

for $r \gg 1$, and $\rho(r) = (1+o(1))\sigma\zeta d(e^{d\eta} - e^{-2\sigma}e^{2\sigma R_H d\eta})^d$ from [**?**].

We introduce some notation about clustering in the network model:

- $\Lambda(u, v; w)$ represents the event that the triple u, v, w forms an incomplete triangle pivoted at w, i.e., u, v, w forms a path of length 2 with w being the middle vertex.
- $\Delta(u, v, w)$ represents the event that the triple u, v, w forms a complete triangle.
- $T = T(G)$, $\Lambda = \Lambda(G)$ denote the number of complete triangles or incomplete triangles in graph G, respectively.

1.2 Main Results

We study the degree and clustering properties of the random graph $G = G(N, \zeta, \tau, \sigma, \nu, d)$ with N vertices and parameters $\tau < 1$ and $2\sigma < 1$ in arbitrary dimensional hyperbolic space $\mathbb{H}^{d+1}$. The main results of this article are as follows.

THEOREM 1.1 Let $\tau < 1$, $2\sigma < 1$ and $d \geq 1$. Then

$$\Pr[I_{u,v} = 1 \mid r_u,\ _u] = \int_0^R \hat{p}_{u,v}\rho(r_v)\,dr_v = \Theta(e^{-2\sigma\eta_u}) \quad \text{for any } r_u. \tag{1.10}$$

Mean degree

$$\langle k\rangle = \Theta\left((N-1)R_H e^{-2\sigma R_H}\right). \tag{1.11}$$

The probability of k-degree

$$\Pr[D_u = k] \to 0 \quad \text{as } N \to \infty,\ \text{for any given } k \geq 0,\ \text{if we let } R_H = \ln N^\nu.$$

THEOREM 1.2 Let $\tau < 1$, $2\sigma < 1$ and $d \geq 1$. Then

$$E(\Lambda) = 3\binom{N}{3}\binom{e^{-2\sigma R_H}}{1}^2. \tag{1.12}$$

There exists a lower bound estimation and order estimation for $E(T)$:

$$E(T) \gtrsim \binom{N}{3} e^{-3\sigma R_H} \quad \text{and} \quad E(T) = o(E(\Lambda)). \tag{1.13}$$

THEOREM 1.3 Let $\tau < 1$, $2\sigma < 1$ and $d \geq 1$. Also let $R_H = \ln N^{\nu}$. The global clustering coefficient $C_2(G)$ of the graph $G(N, \zeta, \tau, \sigma, \nu, d)$ will converge to zero in probability, i.e.,

$$C_2(G) = \frac{3T(G)}{\Lambda(G)} = \frac{3E(T)(1+o_p(1))}{E(\Lambda)(1+o_p(1))} \longrightarrow 0, \quad \text{as } N \to \infty.$$

The paper is organized as follows. In Section 1.1 we introduce the model. In Section 2 we provide a useful preliminary result. We give degree estimations and prove Theorem 1.1 in Section 3. We discuss incomplete and complete triangles and prove Theorem 1.2 in Section 4. Finally, we estimate the variance and prove Theorem 1.3 in Sections 5 and 6.

1.3 Simulations

We adopt R software (https://www.r-project.org/) for statistical simulation. According to the law of large numbers in probability theory, we use the mean to approximate the expectation in R programs. We simulate the connection probability of a fixed node in two kinds of dimensions, k-degree, and the global clustering coefficient of incomplete and complete triangles. One can also use some usual ComplexNetworks toolkit within Cactus [**?**] for simulation.

2. Preliminary Lemma

The hyperbolic distance $d_{u,v}$ in (1.4) can be explicitly expressed according to the following lemma.

LEMMA 2.1 Let $h, h_1 : \mathbb{N} \to \mathbb{R}^+$ such that $h(N), h_1(N) \to \infty$ as $N \to \infty$. Let u, v be two distinct points in $\mathbb{H}^{d+1}$ with $\theta_{u,v}$ denoting their relative angle. Let also $\hat{\theta}_{u,v} := (e^{-2\zeta r_v} + e^{-2\zeta r_u})/2$. If we assume u, v with $r_u, r_v \geq h(N)$, then as $h_1(N)\hat{\theta}_{u,v} \leq \theta_{u,v} \leq \pi$, we have

$$d_{u,v} = r_u + r_v + \ln \sin\left(\frac{\theta_{u,v}}{2}\right) - \ln \hat{\theta}_{u,v} + O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right) \tag{2.1}$$

uniformly for all u, v satisfying the above condition. Note that we can take $h_1(N) = h(N)$, for example. Thus we may choose $h_1(N)$ to make $h_1(N)\hat{\theta}_{u,v} < \pi$.

Proof. For $r_u, r_v \geq h(N)$, we have $\hat{\theta}_{u,v} \ll \pi$, hence the condition $h_1(N)\hat{\theta}_{u,v} \leq \theta_{u,v} \leq \pi$ is well-defined. Obviously, we can take $h_1(N) = h(N)$ so that $h(N)\hat{\theta}_{u,v} \ll \pi$. Next, we prove (2.1). The right-hand side of (1.4) gives

$$\cosh(\zeta r_u)\cosh(\zeta r_v) - \sinh(\zeta r_u)\sinh(\zeta r_v)\cos\theta_{u,v} = \frac{e^{\zeta(r_u+r_v)}}{2}\left(\sin^2\frac{\theta_{u,v}}{2} + e^{-2\zeta(r_u+r_v)}\cos^2\frac{\theta_{u,v}}{2}\right) + e^{-\zeta(r_u+r_v)}\sin^2$$

Here, $0 < O((\hat{\theta}_{u,v}/\theta_{u,v})^2) \leq \frac{2\pi^2}{(\hat{\theta}_{u,v})^2}$, and we also use Young' s inequality to get $e^{-\zeta(r_u+r_v)} = e^{-2\zeta\frac{r_u+r_v}{2}} \leq \frac{1}{2}(e^{-2\zeta r_u} + e^{-2\zeta r_v}) \leq \hat{\theta}_{u,v}$.

Thus,

$$\cosh(\zeta r_u)\cosh(\zeta r_v) - \sinh(\zeta r_u)\sinh(\zeta r_v)\cos\theta_{u,v} \geq \frac{1}{2}\left(\hat{\theta}_{u,v}\right)^2\left(1 + O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right)\right),$$

so from (1.4) we get $e^{\zeta d_{u,v}} \geq \frac{1}{\pi^2}(\hat{\theta}_{u,v})^2 h_1^2(N)$. Furthermore, from (1.4) and the above expansion, we have

$$\zeta d_{u,v} + \ln(1 + e^{-2\zeta d_{u,v}}) = \zeta(r_u + r_v) + \ln\sin^2\frac{\theta_{u,v}}{2} + \ln\left(1 + O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right)\right),$$

and from the lower bound we obtain the result

$$d_{u,v} = (r_u + r_v) + \ln\sin\frac{\theta_{u,v}}{2} - \ln\hat{\theta}_{u,v} + O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right),$$

where $|O((\hat{\theta}_{u,v}/\theta_{u,v})^2)| \leq \frac{3\pi^2}{2}(\hat{\theta}_{u,v}/\theta_{u,v})^2$.

Especially, for all u, v with $r_u + r_v - R \geq \omega(N)$ where $\omega(N) = o(R(N)) \to \infty$ as $N \to \infty$, we may choose $\tilde{\theta}_{u,v} = 2e^{(R-r_u-r_v)/2}$ such that $\tilde{\theta}_{u,v} \leq \theta_{u,v} \leq \pi$. Then (2.1) is achieved, since $r_u, r_v \geq \omega(N)$ and $\tilde{\theta}_{u,v} \geq h_1(N)\hat{\theta}_{u,v}$, where $h_1(N) = 2\omega(N)$. Further, we have

$$d_{u,v} - R = r_u + r_v - R + 2\ln\sin\left(\frac{\theta_{u,v}}{2}\right) + O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right),$$

and

$$p_{u,v} = \frac{1}{1 + e^{\zeta(d_{u,v}-R)}} = \frac{1}{1 + C_N e^{\eta_u + \eta_v - R_H}},$$

where $C_N = 1 + \hat{\theta}_{u,v}$.

3. Degree Estimation

In this section, we prove Theorem 1.1.

3.1.1 $\Pr[I_{u,v} = 1 \mid r_u, \ _u]$ **under the condition** $r_u \geq \omega(N)$

By definition,

$$\Pr[I_{u,v} = 1 \mid r_u, \ _u] = \int_0^R \hat{p}_{u,v}\rho(r_v)\, dr_v = \int_0^{R-r_u+\omega(N)} \hat{p}_{u,v}\rho(r_v)\, dr_v + \int_{R-r_u+\omega(N)}^{R} \hat{p}_{u,v}\rho(r_v)\, dr_v. \tag{3.1}$$

The second integral has the estimation from Appendix A.2:

$$\int_{R-r_u+\omega(N)}^{R} \hat{p}_{u,v}\rho(r_v)\, dr_v = (1+o(1)) \int_{R-r_u+\omega(N)}^{R} c^*(\tau, d)e^{R_H-\eta_u-\eta_v} \cdot (1+o(1))2\sigma d\eta_v - e^{-2\sigma} e^{2\sigma R_H d\eta_v})^d = o(1)e^{-2\sigma\eta_u} \tag{3.2}$$

The primary difficulty is estimating the first integral

$$\int_0^{R-r_u+\omega(N)} \hat{p}_{u,v}\rho(r_v)\, dr_v, \tag{3.3}$$

which requires more refined analysis. We decompose it into two subparts. The first subpart has the estimation

$$\int_0^{R-r_u} \hat{p}_{u,v}\rho(r_v)\, dr_v = \int_0^{R-r_u} \int_0^{\pi} \rho_1(\theta_{u,v}) \frac{1}{1+e^{\zeta(d_{u,v}-R)}}\, d\theta_{u,v}\rho(r_v)\, dr_v \quad (d_{u,v} \leq r_u + r_v \leq R) = \Theta(1) \int_0^{R-r_u} \int_0 \tag{3.4}$$

For the second subpart

$$\int_{R-r_u}^{R-r_u+\omega(N)} \hat{p}_{u,v}\rho(r_v)\, dr_v = \int_{R-r_u}^{R-r_u+\omega(N)} \int_0^{\pi} \rho_1(\theta_{u,v}) \frac{1}{1+e^{\zeta(d_{u,v}-R)}}\, d\theta_{u,v}\rho(r_v)\, dr_v, \tag{3.5}$$

we consider the angular integral $\hat{p}_{u,v}$ in two cases since $R - r_u \leq r_v \leq R - r_u + \omega(N)$.

Case (1): If $r_u \leq R - \omega(N)$, then $r_v \geq \omega(N)$. Together with the primary condition $r_u \geq \omega(N)$, we have the explicit distance expression from Lemma 2.1:

$$d_{u,v} = r_u + r_v + \ln \sin\left(\frac{\theta_{u,v}}{2}\right) - \ln \hat{\theta}_{u,v} + O\left(\left(\frac{\hat{\theta}_{u,v}}{\theta_{u,v}}\right)^2\right), \qquad (3.6)$$

hence the connection probability

$$p_{u,v} = \frac{1}{1+e^{\zeta(d_{u,v}-R)}} = \frac{1}{1+C_N e^{\eta_u+\eta_v-R_H}}$$

for $\tilde{\theta}_{u,v} \le \theta_{u,v} \le \pi$. So the angular integral

$$\hat{p}_{u,v} = \int_0^{\pi} \rho_1(\theta) p_{u,v}\, d\theta = \int_0^{\tilde{\theta}_{u,v}} \rho_1(\theta) p_{u,v}\, d\theta + \int_{\tilde{\theta}_{u,v}}^{\pi} \rho_1(\theta) p_{u,v}\, d\theta = o(e^{-(\eta_u+\eta_v-R_H)}) + \int_{\tilde{\theta}_{u,v}}^{\pi} \sin^{d-1}\theta \frac{1}{1+C_N e^{\eta_u+\eta_v-R}}$$

where θ_δ is defined by $\sin^2(\theta_\delta/2) = e^{-(\eta_u+\eta_v-R_H)}$. The last equality follows from the process: if $\theta_\delta \le \pi$, then $\int_0^{\theta_\delta} \theta^{d-1}\, d\theta = \Theta(\theta_\delta^d) = \Theta\left(e^{-(\eta_u+\eta_v-R_H)}\right)$ for $\tau \le 1$; and if $\pi \le \theta_\delta$, then $\int_0^{\pi} \sin^{d-1}\theta\, d\theta = \Theta(1)$. So it is obvious that $\int_0^{\theta_\delta} \sin^{d-1}\theta\, d\theta = \Theta\left(e^{-(\eta_u+\eta_v-R_H)}\right)$, and there always has $\int_0^{\pi} \sin^{d-1}\theta\, d\theta = \Theta\left(e^{-(\eta_u+\eta_v-R_H)}\right)$ for $0 < \theta_\delta < \pi$.

Case (2): If $r_u \ge R - \omega(N)$, we luckily have the lower bound estimation of the distance from (1.4). Notice that $R - 3\omega(N) \le r_u - r_v \le d_{u,v} \le r_u + r_v$, so $d_{u,v} \to \infty$ as $N \to \infty$. We expand both sides of the hyperbolic distance formula (1.4) and get

$$e^{\zeta d_{u,v}}(1+e^{-2\zeta d_{u,v}}) = e^{\zeta r_u + \zeta r_v}\left((1-\cos\theta_{u,v}) + e^{-2\zeta r_u}(1+\cos\theta_{u,v}) + e^{-2\zeta r_v}(1+\cos\theta_{u,v}) + e^{-2\zeta(r_u+r_v)}(1-\cos\theta\right.$$

so

$$e^{\zeta d_{u,v}} = (1+e^{-2\zeta d_{u,v}})^{-1} e^{\zeta r_u + \zeta r_v}\left(\sin^2\frac{\theta_{u,v}}{2} + e^{-2\zeta r_u}\cos^2\frac{\theta_{u,v}}{2} + e^{-2\zeta r_v}\cos^2\frac{\theta_{u,v}}{2} + e^{-2\zeta(r_u+r_v)}\sin^2\frac{\theta_{u,v}}{2}\right).$$

Hence

$$e^{\zeta(d_{u,v}-R)} = (1+e^{-2\zeta d_{u,v}})^{-1} e^{\zeta(r_u+r_v-R)}\left(\sin^2\frac{\theta_{u,v}}{2} + e^{-2\zeta r_u}\cos^2\frac{\theta_{u,v}}{2} + e^{-2\zeta r_v}\cos^2\frac{\theta_{u,v}}{2} + e^{-2\zeta(r_u+r_v)}\sin^2\frac{\theta_{u,v}}{2}\right) = \qquad (3.7)$$

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Set θ_δ as before, i.e., $\sin^2(\theta_\delta/2)e^{\zeta(r_u+r_v-R)} = 1$. For $0 \le \theta_{u,v} \le \theta_\delta$ we have $e^{\zeta(d_{u,v}-R)} \le 4(1+o(1))$, while for $0 \le \theta_{u,v} \le \pi$ we have $d_{u,v} - R \ge (1+o(1))(r_u + r_v - R)$. Hence we have the upper bound estimation

$$\hat{p}_{u,v} \le \int_0^{\theta_\delta} \rho_1(\theta)\frac{1}{1+e^{d_{u,v}-R}}\,d\theta + \int_{\theta_\delta}^{\pi} \rho_1(\theta)\frac{1}{1+e^{d_{u,v}-R}}\,d\theta \le \int_0^{\theta_\delta} \rho_1(\theta)\,d\theta + \int_{\theta_\delta}^{\pi} (1+o(1))e^{\eta_u+\eta_v-R_H}\sin^{d-1}\theta\,d\theta \lesssim e$$

and the lower bound estimation

$$\hat{p}_{u,v} = \int_0^{\pi} \rho_1(\theta)p_{u,v}\,d\theta \ge \int_0^{\theta_\delta} \rho_1(\theta)\frac{1}{1+e^{d_{u,v}-R}}\,d\theta,$$

for the same process as before.

Finally, from the above two cases we generalize

$$\hat{p}_{u,v} = \int_0^{\pi} \rho_1(\theta)p_{u,v}\,d\theta = \Theta\left(e^{-(\eta_u+\eta_v-R_H)}\right). \tag{3.8}$$

Further, for the second subpart (3.5), we have

$$\int_{R-r_u}^{R-r_u+\omega(N)} \hat{p}_{u,v}\rho(r_v)\,dr_v = \int_{R-r_u}^{R-r_u+\omega(N)} \Theta\left(e^{-(\eta_u+\eta_v-R_H)}\right)\rho(r_v)\,dr_v = \Theta\left(\int_{R-r_u}^{R-r_u+\omega(N)} e^{-(\eta_u+\eta_v-R_H)}\rho(r_v)\,dr_v\right) \tag{3.10}$$

Combining the estimations of the two subparts (3.4) and (3.5), we have the estimation for the first part integral (3.3):

$$\int_0^{R-r_u+\omega(N)} \hat{p}_{u,v}\rho(r_v)\,dr_v = \Theta\left(e^{-2\sigma\eta_u}\right).$$

Further, together with the estimation of the second part integral (3.2), we get

$$\Pr[I_{u,v} = 1 \mid r_u,\ _u] = \int_0^R \hat{p}_{u,v}\rho(r_v)\,dr_v = \Theta\left(e^{-2\sigma\eta_u}\right).$$

We see that the connection probability $\Pr[I_{u,v} = 1 \mid r_u,\ _u]$ is mainly contributed by $0 \le r_v \le R - r_u + \omega(N)$ for $r_u > \omega(N)$. This property differs from the case $2\sigma > 1$ in [**?**].

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3.1.2 $\Pr[I_{u,v}=1 \mid r_u,\ {}_u]$ under the condition $r_u \le \omega(N)$

We decompose formula (3.1) into two parts as follows:

$$\Pr[I_{u,v}=1 \mid r_u,\ {}_u] = \int_0^R \hat{p}_{u,v}\rho(r_v)\,dr_v = \int_0^{R-r_u} \hat{p}_{u,v}\rho(r_v)\,dr_v + \int_{R-r_u}^{R} \hat{p}_{u,v}\rho(r_v)\,dr_v. \tag{3.11}$$

For the first integral, we have the estimation

$$\int_0^{R-r_u} \hat{p}_{u,v}\rho(r_v)\,dr_v = \int_0^{R-r_u}\int_0^{\pi} \rho_1(\theta_{u,v})\frac{1}{1+e^{\zeta(d_{u,v}-R)}}\,d\theta_{u,v}\rho(r_v)\,dr_v \quad (d_{u,v} \le r_u + r_v \le R) = \Theta(1)\int_0^{R-r_u}\int_0^{\pi}$$

For the second integral, since $R - 2\omega(N) \le r_v - r_u \le d_{u,v} \le r_u + r_v$, we have $d_{u,v} \to \infty$ as $N \to \infty$. Using the same analysis as in the second case of (3.5), we have $\hat{p}_{u,v} = \Theta(e^{-(\eta_u+\eta_v-R_H)})$, so

$$\int_{R-r_u}^{R} \hat{p}_{u,v}\rho(r_v)\,dr_v = \int_{R-r_u}^{R}\int_0^{\pi} \rho_1(\theta_{u,v})\frac{1}{1+e^{\zeta(d_{u,v}-R)}}\,d\theta_{u,v}\rho(r_v)\,dr_v = \Theta(e^{-(\eta_u+\eta_v-R_H)})\rho(r_v)\,dr_v = \Theta(e^{-2\sigma\eta_u}). \tag{3.12}$$

Combining both parts, we have

$$\Pr[I_{u,v}=1 \mid r_u,\ {}_u] = \int_0^R \hat{p}_{u,v}\rho(r_v)\,dr_v = \Theta(e^{-2\sigma\eta_u}).$$

3.1.3 $\Pr[I_{u,v}=1 \mid r_u,\ {}_u]$

From Sections 3.1.1 and 3.1.2, we have the first conclusion (1.10) in Theorem 1.1:

$$\Pr[I_{u,v}=1 \mid r_u,\ {}_u] = \int_0^R \hat{p}_{u,v}\rho(r_v)\,dr_v = \Theta(e^{-2\sigma\eta_u}) \quad \text{for any } r_u.$$

[Figure 1: see original paper] shows simulation results consistent with our analysis. On the other hand, we also obtain a fact: for $0 \le r_v \le R - r_u$, $\hat{p}_{u,v} = \Theta(1)$; for $R - r_u \le r_v \le R$, $\hat{p}_{u,v} = \Theta(e^{-(\eta_u+\eta_v-R_H)})$.

3.2 Mean Degree

By simple computation, we have the second conclusion (1.11) in Theorem 1.1:

$$\langle k\rangle = (N-1)\int_0^R \rho(r)\Theta(e^{-2\sigma\eta_u})\,dr = \Theta\left((N-1)R_H e^{-2\sigma R_H}\right).$$

[Figure 1: see original paper] Plot from simulation of $\Pr[I_{u,v}=1\mid r_u,\ _u]$ by the law of large numbers. The vertical axis is $\log(\Pr)$ and the horizontal axis is radius r_u. The data points with $N=2^{12}$ from lower to upper take parameters $d=3, 2\sigma=1/2, \tau=1/2$; $d=1, 2\sigma=2/3, \tau=1/3$; $d=1, 2\sigma=1/2, \tau=1/2$ respectively, and $\zeta=1$. Comparison of three almost straight lines is consistent with equation (1.10).

3.3 The Probability of k-Degree

Next we analyze the probability of $D_u=k$, where k is a positive integer. Let constant $\lambda\in(0,1)$. Then

$$\Pr[D_u=k] = \binom{N-1}{k}\int_0^R (\Pr[I_{u,v}=1\mid r_u])^k(1{-}\Pr[I_{u,v}=1\mid r_u])^{N-1-k}\rho(r_u)\,dr_u = \binom{N-1}{k}\int_{\lambda R}^R(\Pr[I_{u,v}=1$$

Now to analyze the first integral, let $t=\Pr[I_{u,v}=1\mid r_u]=\Theta(e^{-2\sigma\eta_u})$ and we can see $t\to 0$ uniformly for $\lambda R\le r_u\le R$, and

$$(1-t)^{N-1-k} = (1+o(1))\exp[-(N-1)t]\times\exp\left[-\frac{N-1}{2(1-\xi)^2}t^2\right],\quad 0\le\xi\le t,$$

with $o(1)=O(e^{-2\sigma\lambda R_H})$ when $r_u\ge\lambda R$. So

$$\binom{N-1}{k}\int_{\lambda R}^R(\Pr[I_{u,v}=1\mid r_u])^k(1{-}\Pr[I_{u,v}=1\mid r_u])^{N-1-k}\rho(r_u)\,dr_u = \binom{N-1}{k}\int_{\lambda R}^R(\Pr[I_{u,v}=1\mid r_u])^k(1{+}o(1$$

Since $(N-1)\Theta(e^{-2\sigma R_H})\le(N-1)\Pr[I_{u,v}=1\mid r_u,\ _u]=(N-1)\Theta(e^{-2\sigma\eta_u})\le(N-1)\Theta(e^{-2\sigma\lambda R_H})$, when we take $R_H=\ln N^\nu$, then $(N-1)\Pr[I_{u,v}=1\mid r_u,\ _u]\to\infty$ as $N\to\infty$, and since $e^{-x}x^k$ is decreasing to 0 for $x>k$, we have

$$\frac{((N-1)\Pr[I_{u,v}=1\mid r_u,\ _u])^k}{k!}\exp[-(N-1)\Pr[I_{u,v}=1\mid r_u,\ _u]]\to 0.$$

Thus we get

$$\binom{N-1}{k}\int_{\lambda R}^{R}(\Pr[I_{u,v}=1\mid r_u,\ _u])^k(1-\Pr[I_{u,v}=1\mid r_u,\ _u])^{N-1-k}\rho(r_u)\,dr_u\to 0,$$

as $N\to\infty$. Finally, we have the third conclusion in Theorem 1.1: $\Pr[D_u = k]\to 0$ as $N\to\infty$ for any given $k\geq 0$. Simulation (Figure 2) also shows the same tendency.

4. Expectation of Incomplete and Complete Triangles

In this section we compute longer links and prove Theorem 1.2.

4.1 Incomplete Triangle for Arbitrary Dimension

In this subsection, we compute the probability $P(\Lambda(u,v;w))$ of the event $\Lambda(u,v;w)$. First we define the angular integral $\hat{p}_{u,v;w}$ as a conditional probability describing that nodes u,v at fixed radial coordinates r_u,r_v connect to a node w at fixed radius r_w. We pivot the radial direction of vertex w as the center axis, and since the connections $\{u,w\}$ and $\{v,w\}$ are independent events, we may compute $\hat{p}_{u,v;w}$ by integrating over the relative angles $\theta_{u,w},\theta_{v,w}$ of u,v about w respectively (Figure 3):

$$\hat{p}_{u,v;w}=\hat{p}_{u,w}\hat{p}_{v,w}=\int_0^{\pi}\int_0^{\pi}\rho_1(\theta_{u,w})\rho_1(\theta_{v,w})p_{u,w}p_{v,w}\,d\theta_{u,w}d\theta_{v,w}. \tag{4.1}$$

For $d=1$, we have $\hat{p}_{u,w}=\int_0^{\pi}p_{u,w}\,d\theta_{u,w}$ due to angular symmetry for the hyperbolic distance. So

$$P(\Lambda(u,v;w))=\int_0^R\int_0^R\int_0^R\hat{p}_{u,v;w}\rho(r_u)\rho(r_v)\rho(r_w)\,dr_u dr_v dr_w=\int_0^R\int_0^R\int_0^R\hat{p}_{u,w}\hat{p}_{v,w}\rho(r_u)\rho(r_v)\rho(r_w)\,dr_u dr_v dr$$

Hence we obtain formula (1.12) in Theorem 1.2:

$$E(\Lambda)=3\binom{N}{3}\binom{e^{-2\sigma R_H}}{1}^2.$$

4.2 Complete Triangle for Arbitrary Dimension

In this subsection, we consider complete triangles and complete the proof of the second part of Theorem 1.2.

If we choose a triangle formed by (u,v,w) pivoted at w, for $d=1$ there is a similar angular integral definition

$$\hat{p}_{u,v,w} = P(\Delta(u,v,w) \mid r_u, r_v, r_w) = \int_0^{\pi}\int_0^{\pi} p_{u,w}p_{v,w}p_{u,v}\, d\theta_{u,w}d\theta_{v,w}, \tag{4.2}$$

where $\theta_{u,v} = |\theta_{u,w} - \theta_{v,w}|$ or $\min\{\theta_{u,w} + \theta_{v,w}, 2\pi - (\theta_{u,w} + \theta_{v,w})\}$ (refer to A in Figure 3). For $d > 1$, the angular integral is

$$\hat{p}_{u,v,w} = P(\Delta(u,v,w) = 1 \mid r_u, r_v, r_w) = \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} p_{u,w}p_{v,w}p_{u,v}\rho_1(\theta_{u,w})\rho_1(\theta_{v,w})\rho_2(\theta_{v,2})\, d\theta_{u,w}d\theta_{v,w}d\theta_{v,2},$$

where $\rho_1(\theta_1) = \sin^{d-1}\theta_1/I_{d,1}$, $\rho_2(\theta_2) = \sin^{d-2}\theta_2/I_{d,2}$, and $\theta_{v,2}$ is the angle between the uw-plane and vw-plane (refer to B in Figure 3). It is obvious that $\hat{p}_{u,v,w} \le \hat{p}_{u,w}\hat{p}_{v,w}$.

The probability of the event $\Delta(u,v,w)$ is

$$P(\Delta(u,v,w)) = \int_0^{R}\int_0^{R}\int_0^{R} \hat{p}_{u,v,w}\rho(r_u)\rho(r_v)\rho(r_w)\, dr_u dr_v dr_w.$$

If $r_w > \omega(N)$, there is

$$P(\Delta(u,v,w) \mid r_w > \omega(N)) \le \int_0^{R}\int_0^{R}\int_0^{R} \hat{p}_{u,v,w}\rho(r_u)\rho(r_v)\rho(r_w)\, dr_u dr_v dr_w \le \int_0^{R}\int_0^{R}\int_0^{R} \hat{p}_{u,w}\hat{p}_{v,w}\rho(r_u)\rho(r_v)\rho(\tag{4.3}$$

In the following, we consider $P(\Delta(u,v,w) \mid r_w \le \omega(N))$. Starting from the range $r_w \le \omega(N)$, $r_u, r_v \le R$, we have $\hat{p}_{u,v,w} = \Theta(1)$ since $p_{u,w}, p_{v,w}, p_{u,v}$ are $\Theta(1)$. So

$$P(\Delta(u,v,w) \mid r_w \le \omega(N), r_u, r_v \le R) = \int_0^{\omega(N)}\int_0^{R}\int_0^{R} \hat{p}_{u,v,w}\rho(r_u)\rho(r_v)\rho(r_w)\, dr_u dr_v dr_w = \int_0^{\omega(N)}\int_0^{R}\int_0^{R} \Theta(1 \tag{4.4}$$

On the other hand, we have

$$P(\Delta(u,v,w) \mid r_w \le \omega(N), r_u \le R, r_v \le R) \lesssim \int_0^{\omega(N)}\int_0^{R}\int_0^{R} \hat{p}_{u,v,w}\rho(r_u)\rho(r_v)\rho(r_w)\, dr_u dr_v dr_w \lesssim \int_0^{\omega(N)}\int_0^{R}\int_0^{R} \tag{4.5}$$

For the triangle in the situation below, we choose to pivot the triangle at v:

$$P(\Delta(u,v,w) \mid r_w \le \omega(N), r_u, r_v \le R) \lesssim \int_0^{\omega(N)}\int_0^R\int_0^R \widehat{p}_{u,v,w}\rho(r_u)\rho(r_v)\rho(r_w)\, dr_u dr_v dr_w = \int_0^{\omega(N)}\int_0^R\int_0^R \widehat{p}_{u,v}$$

(4.6)

Together (4.4), (4.5), (4.6), and (4.7), we have

$$P(\Delta(u,v,w) \mid r_w \le \omega(N)) = o(P(\Lambda(u,v;w))). \tag{4.8}$$

Hence from (4.3) and (4.8), we have the order estimation (1.13) in Theorem 1.2:

$$P(\Delta(u,v,w)) = o(P(\Lambda(u,v;w))), \quad E(T) = o(E(\Lambda)).$$

Next, we give the lower bound estimation for $P(\Delta(u,v,w)) \gtrsim e^{-3\sigma R_H}$, which will deduce the lower bound estimation (1.13) in Theorem 1.2:

$$E(T) = \binom{N}{3} P(\Delta(u,v,w)) \gtrsim \binom{N}{3} e^{-3\sigma R_H}.$$

From a similar process as (4.4), we get

$$P(\Delta(u,v,w) \mid r_w, r_u, r_v \le R) = \int_0^R\int_0^R\int_0^R \widehat{p}_{u,v,w}\rho(r_u)\rho(r_v)\rho(r_w)\, dr_u dr_v dr_w = \int_0^R\int_0^R\int_0^R \Theta(1)\rho(r_v)\rho(r_u)\rho($$

(4.9)

On the other hand, we consider $P(\Delta(u,v,w) \mid r_w \le R, r_u, r_v \le R)$:

$$\int_0^R\int_0^R\int_0^R \widehat{p}_{u,v,w}\rho(r_u)\rho(r_v)\rho(r_w)\, dr_u dr_v dr_w = \int_0^R\int_0^R\int_0^R \left(\int_0^{R-r_w} \widehat{p}_{u,w}\rho(r_u)\, dr_u + \int_{R-r_w}^R \widehat{p}_{u,w}\rho(r_u)\, dr_u\right)$$

(4.10)

where the third equality from last (4.11) involves the following details:

$$\int_0^{R-r_w} \Theta(1)\rho(r_u)\, dr_u = \Theta\left(\int_0^{R-r_w} (e^{d\eta_u} - e^{-2\sigma}e^{2\sigma R_H d\eta_u})^d\, d\eta_u\right) = \Theta\left(e^{-2\sigma\eta_w} f(\eta_w)\right), \tag{4.11}$$

where $f(\eta_w) = e^{-2\sigma(R_H-\eta_w)}\int_0^{R_H-\eta_w}(e^{d\eta} - e^{-2\sigma}e^{2\sigma R_H d\eta})^d\, d\eta$, and

$$\int_0^R \Theta(e^{-(\eta_w+\eta_u-R_H)})\rho(r_u)\, dr_u = \Theta\left(\int_0^R e^{-(\eta_w+\eta_u-R_H)}(e^{d\eta_u} - e^{-2\sigma}e^{2\sigma R_H d\eta_u})^d\, d\eta_u\right) = \Theta\left(e^{-2\sigma\eta_w} g(\eta_w)\right),$$

where $g(\eta_w) = e^{(1-2\sigma)(R_H-\eta_w)} \int_0^{R_H-\eta_w} e^{-\eta_u}(e^{d\eta_u} - e^{-2\sigma}e^{2\sigma R_H d\eta_u})^d \, d\eta_u$. By the general monotonic method, we can get $f(\eta_w) + g(\eta_w)$ is initially increasing then decreasing when η_w varies from 0 to R_H, so we have $f(\eta_w) + g(\eta_w) \geq \min\{f(R_H/2), g(R_H)\} \geq c_0(d,\sigma)$, where the constant $c_0(d,\sigma)$ depends only on σ, d for sufficiently large N. On the other hand, the upper bound of $f(\eta_w)+g(\eta_w)$ is obvious, so we have

$$\int_0^{R-r_w} \Theta(1)\rho(r_u)\, dr_u + \int_0^{R} \Theta(e^{-(\eta_w+\eta_u-R_H)})\rho(r_u)\, dr_u = \Theta(e^{-2\sigma\eta_w}).$$

Combining (4.10) and (4.12), we have

$$\int_0^R \int_0^R \int_0^R \hat{p}_{u,v,w}\rho(r_u)\rho(r_v)\rho(r_w)\, dr_u dr_v dr_w = \Theta\left(e^{-3\sigma R_H}\right), \tag{4.13}$$

hence $P(\Delta(u,v,w)) \gtrsim e^{-3\sigma R_H}$.

Question: Is $P(\Delta(u,v,w)) = \Theta(e^{-3\sigma R_H})$?

[Figure 4: see original paper] Two Complete Triangles

5. Variance Estimation of Incomplete and Complete Triangles

We first estimate $E(T^2)$ and $E(\Lambda^2)$. We need to specify $R_H = \ln N^\nu$.

5.1 Estimation of $E(T^2)$

LEMMA 5.1 $E(T^2) = (1+o(1))E^2(T)$.

We mark two complete triangle events $\Delta(u_1,v_1;w_1)$ and $\Delta(u_2,v_2;w_2)$ formed by two groups of vertices u_1,v_1,w_1 and u_2,v_2,w_2 respectively. First we have

$$E(T^2) = E\left[\left(\sum_{(u,v,w)} 1_{\{\Delta(u,v,w)\}}\right)^2\right] = E(T) + \sum_{(u_1,v_1;w_1)\neq(u_2,v_2;w_2)} E(1_{\{\Delta(u_1,v_1;w_1)\}} 1_{\{\Delta(u_2,v_2;w_2)\}}), \tag{5.1}$$

where for the second part, there are three cases (Figure 4):

1. $\{u_1,v_1,w_1\} \cap \{u_2,v_2,w_2\} = \emptyset$.
2. $\{u_1,v_1,w_1\}$ and $\{u_2,v_2,w_2\}$ share exactly one common vertex.
3. $\{u_1,v_1,w_1\}$ and $\{u_2,v_2,w_2\}$ share exactly two common vertices, i.e., they share a common edge.

We denote the event by T_i corresponding to the i-th case, $i = 1, 2, 3$.

Case 1 (refer to A in Figure 4): There are $\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$ and $\binom{N-3}{3}$ possible choices for $\{u_2, v_2, w_2\}$. So

$$E(T_1) = \binom{N}{3}\binom{N-3}{3}P(\Delta(u_1, v_1, w_1))P(\Delta(u_2, v_2, w_2)) = (1 + o(1))E^2(T).$$

Case 2 (refer to B in Figure 4): There are $\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$, and if w_1 is the common vertex, there remain $\binom{N-3}{2}$ possible choices for $\{u_2, v_2\}$. Then

$$P(\Delta(u_1, v_1, w_1), \Delta(u_2, v_2, w_1)) \leq \int_0^R\int_0^R\int_0^R\int_0^R\int_0^R \hat{p}_{u_1,v_1,w_1}\hat{p}_{u_2,v_2,w_1}\rho(r_{u_1})\rho(r_{v_1})\rho(r_{u_2})\rho(r_{v_2})\rho(r_{w_1})\, dr_{u_1} dr_{v_1}$$

Considering the lower bound of $E(T)$ in (4.9), we have

$$E(T_2) = 3\binom{N}{3}\binom{N-3}{2}P(\Delta(u_1, v_1, w_1), \Delta(u_2, v_2, w_1)) \leq 3\binom{N}{3}\binom{N-3}{2}O(R_H^2)e^{-4\sigma R_H} = o(E^2(T)).$$

Case 3 (refer to C in Figure 4): There are $\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$, and if v_1, w_1 are common vertices, there remain $\binom{N-3}{1}$ possible choices for u_2. Then

$$P(\Delta(u_1, v_1, w_1), \Delta(u_2, v_1, w_1)) \leq \int_0^R\int_0^R\int_0^R\int_0^R \hat{p}_{u_1,v_1}\hat{p}_{v_1,u_2}\hat{p}_{u_2,w_1}\rho(r_{u_1})\rho(r_{v_1})\rho(r_{v_2})\rho(r_{w_1})\, dr_{u_1} dr_{v_1} dr_{v_2} dr_{w_1}$$

so

$$E(T_3) = 3\binom{N}{3}\binom{N-3}{1}P(\Delta(u_1, v_1, w_1), \Delta(u_2, v_1, w_1)) \leq 3\binom{N}{3}\binom{N-3}{1}O(R_H^2)e^{-4\sigma R_H} = o(E^2(T)).$$

Combining all cases, we obtain Lemma 5.1.

5.2 Estimation of $E(\Lambda^2)$

LEMMA 5.2 $E(\Lambda^2) = (1 + o(1))E^2(\Lambda)$.

Similarly, we have

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$$E(\Lambda^2) = E\left[\left(\sum_{(u,v;w)} 1_{\{\Lambda(u,v;w)\}}\right)^2\right] = E(\Lambda) + \sum_{(u_1,v_1;w_1)\neq(u_2,v_2;w_2)} E(1_{\{\Lambda(u_1,v_1;w_1)\}}1_{\{\Lambda(u_2,v_2;w_2)\}}). \tag{5.2}$$

We classify the second part events into the following eight cases (Figure 5):

1. $\{u_1, v_1, w_1\} \cap \{u_2, v_2, w_2\} = \emptyset$.

The following cases involve $\{u_1, v_1, w_1\}$ and $\{u_2, v_2, w_2\}$ sharing exactly one common vertex:

2. $w_1 = w_2$, $\{u_1, v_1\} \cap \{u_2, v_2\} = \emptyset$.
3. $u_1 = u_2$, $\{v_1, w_1\} \cap \{v_2, w_2\} = \emptyset$.
4. $u_1 = w_2$, $\{v_1, w_1\} \cap \{u_2, v_2\} = \emptyset$.

The following cases involve $\{u_1, v_1, w_1\}$ and $\{u_2, v_2, w_2\}$ sharing exactly two common vertices:

5. $u_1 = u_2$, $v_1 = v_2$, $w_1 \neq w_2$.
6. $u_1 = w_2$, $v_1 = v_2$, $w_1 \neq u_2$.
7. $u_1 = u_2$, $w_1 = w_2$, $v_1 \neq v_2$.
8. $u_1 = w_2$, $w_1 = u_2$, $v_1 \neq v_2$.

We denote the event by Λ_i corresponding to the i-th case, $i = 1, 2, \ldots, 8$.

Case 1 (refer to I in Figure 5): There are $3\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$ and $3\binom{N-3}{3}$ possible choices for $\{u_2, v_2, w_2\}$, so

$$E(\Lambda_1) = 9\binom{N}{3}\binom{N-3}{3}P(\Lambda(u_1, v_1; w_1))P(\Lambda(u_2, v_2; w_2)) = (1 + o(1))E^2(\Lambda).$$

Case 2 (refer to II in Figure 5): There are $3\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$ and $\binom{N-3}{2}$ possible choices for $\{u_2, v_2\}$, so

$$P(\Lambda(u_1, v_1; w_1)\Lambda(u_2, v_2; w_1)) \leq \int_0^R\int_0^R\int_0^R\int_0^R\int_0^R \hat{p}_{u_1,w_1}\hat{p}_{v_1,w_1}\hat{p}_{u_2,w_1}\hat{p}_{v_2,w_1}\rho(r_{u_1})\rho(r_{v_1})\rho(r_{u_2})\rho(r_{v_2})\rho(r_{w_1})\, dr$$

Thus

$$E(\Lambda_2) = 3\binom{N}{3}\binom{N-3}{2}P(\Lambda(u_1, v_1; w_1)\Lambda(u_2, v_2; w_1)) \leq 3\binom{N}{3}\binom{N-3}{2}\left(e^{-2\sigma R_H}\right)^2 = o(E^2(\Lambda)).$$

Case 3 (refer to III in Figure 5): There are $3\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$ and $\binom{N-3}{1}\binom{N-3}{1}$ possible choices for $\{v_2, w_2\}$, so there are totally $6\binom{N}{3}\binom{N-3}{1}\binom{N-3}{1}$ choices for this case.

$$P(\Lambda(u_1, v_1; w_1)\Lambda(u_1, v_2; w_2)) \leq \int_0^R \int_0^R \int_0^R \int_0^R \int_0^R \hat{p}_{u_1,w_1}\hat{p}_{v_1,w_1}\hat{p}_{u_1,w_2}\hat{p}_{v_2,w_2}\rho(r_{u_1})\rho(r_{v_1})\rho(r_{w_1})\rho(r_{v_2})\rho(r_{w_2})\, dr$$

where we use $\int_0^R \hat{p}_{u,v}\Theta(e^{-2\sigma\eta_v})\rho(r_v)\, dr_v \leq O(R_H)e^{-2\sigma R_H}$. Thus

$$E(\Lambda_3) = 6\binom{N}{3}\binom{N-3}{1}\binom{N-3}{1}P(\Lambda(u_1, v_1; w_1)\Lambda(u_1, v_2; w_2)) \leq 6\binom{N}{3}\binom{N-3}{1}\binom{N-3}{1}O(R_H^2)e^{-4\sigma R_H} = o$$

Case 4 (refer to IV in Figure 5): There are $3\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$ and $\binom{N-3}{1}\binom{N-3}{1}$ possible choices for $\{u_2, v_2\}$, so there are totally $6\binom{N}{3}\binom{N-3}{1}\binom{N-3}{1}$ choices for this case.

$$P(\Lambda(u_1, v_1; w_1)\Lambda(u_2, v_2; u_1)) \leq \int_0^R \int_0^R \int_0^R \int_0^R \int_0^R \hat{p}_{u_1,w_1}\hat{p}_{v_1,w_1}\hat{p}_{u_2,u_1}\hat{p}_{v_2,u_1}\rho(r_{u_1})\rho(r_{v_1})\rho(r_{w_1})\rho(r_{v_2})\rho(r_{u_2})\, dr_u$$

where we use $\int_0^R \hat{p}_{u_1,w_1}\Theta(e^{-2\sigma\eta_{u_1}})\Theta(e^{-2\sigma\eta_{u_1}})\rho(r_{u_1})\, dr_{u_1} \leq O(e^{-2\sigma R_H})$. Thus

$$E(\Lambda_4) = 6\binom{N}{3}\binom{N-3}{1}\binom{N-3}{1}P(\Lambda(u_1, v_1; w_1)\Lambda(u_2, v_2; u_1)) \leq 6\binom{N}{3}\binom{N-3}{1}\binom{N-3}{1}O(R_He^{-4\sigma R_H}) = o$$

Case 5 (refer to V in Figure 5): There are $3\binom{N}{3}\binom{N-3}{1}$ possible choices for w_2, so there are totally $3\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$ and $\binom{N-3}{1}$ choices for this case.

$$P(\Lambda(u_1, v_1; w_1)\Lambda(u_1, v_1; w_2)) \leq \int_0^R \int_0^R \int_0^R \int_0^R \hat{p}_{v_1,w_1}\hat{p}_{w_1,u_1}\hat{p}_{u_1,w_2}\rho(r_{u_1})\rho(r_{v_1})\rho(r_{w_1})\rho(r_{w_2})\, dr_{w_2}dr_{u_1}dr_{w_1}dr_{v_1}$$

Thus

$$E(\Lambda_5) = 3\binom{N}{3}\binom{N-3}{1}P(\Lambda(u_1, v_1; w_1)\Lambda(u_1, v_1; w_2)) \leq 3\binom{N}{3}\binom{N-3}{1}O(R_H^2e^{-4\sigma R_H}) = o(E^2(\Lambda)).$$

Case 6 (refer to VI in Figure 5): There are $3\binom{N}{3}\binom{N-3}{1}$ possible choices for w_2, so there are totally $3\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$ and $\binom{N-3}{1}$ choices for this case.

$$P(\Lambda(u_1, v_1; w_1)\Lambda(u_2, v_1; u_1)) \leq \int_0^R \int_0^R \int_0^R \int_0^R \hat{p}_{v_1,w_1}\hat{p}_{w_1,u_1}\hat{p}_{u_1,u_2}\rho(r_{u_1})\rho(r_{v_1})\rho(r_{w_1})\rho(r_{u_2})\, dr_{u_2}dr_{u_1}dr_{w_1}dr_{v_1} \leq$$

Thus

$$E(\Lambda_6) = 3\binom{N}{3}\binom{N-3}{1}P(\Lambda(u_1, v_1; w_1)\Lambda(u_2, v_1; u_1)) \leq 3\binom{N}{3}\binom{N-3}{1}O(R_H^2 e^{-4\sigma R_H}) = o(E^2(\Lambda)).$$

Case 7 (refer to VII in Figure 5): There are $3\binom{N}{3}\binom{N-3}{1}$ possible choices for v_2, so there are totally $3\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$ and $\binom{N-3}{1}$ choices for this case.

$$P(\Lambda(u_1, v_1; w_1)\Lambda(u_1, v_2; w_1)) = \int_0^R \int_0^R \int_0^R \int_0^R \hat{p}_{v_1,w_1}\hat{p}_{u_1,w_1}\hat{p}_{v_2,w_1}\rho(r_{u_1})\rho(r_{v_1})\rho(r_{v_2})\rho(r_{w_1})\, dr_{u_1}dr_{v_1}dr_{v_2}dr_{w_1} =$$

Thus

$$E(\Lambda_7) = 3\binom{N}{3}\binom{N-3}{1}P(\Lambda(u_1, v_1; w_1)\Lambda(u_1, v_2; w_1)) \leq 3\binom{N}{3}\binom{N-3}{1}O(e^{-2\sigma R_H}) = o(E^2(\Lambda)).$$

Case 8 (refer to VIII in Figure 5): There are $3\binom{N}{3}$ possible choices for $\{u_1, v_1, w_1\}$ and $\binom{N-3}{1}$ choices for v_2, so there are totally $6\binom{N}{3}\binom{N-3}{1}$ choices for this case.

$$P(\Lambda(u_1, v_1; w_1)\Lambda(w_1, v_2; u_1)) \leq \int_0^R \int_0^R \int_0^R \int_0^R \hat{p}_{v_1,w_1}\hat{p}_{w_1,u_1}\hat{p}_{u_1,v_2}\rho(r_{u_1})\rho(r_{v_1})\rho(r_{w_1})\rho(r_{v_2})\, dr_{v_2}dr_{u_1}dr_{w_1}dr_{v_1} \leq$$

Thus

$$E(\Lambda_8) = 3\binom{N}{3}\binom{N-3}{1}P(\Lambda(u_1, v_1; w_1)\Lambda(w_1, v_2; u_1)) \leq 6\binom{N}{3}\binom{N-3}{1}O(R_H^2 e^{-4\sigma R_H}) = o(E^2(\Lambda)).$$

Together with all cases, we have Lemma 5.2.

5.3 Variance and Chebyshev’ s Inequality

From the previous subsections, we have

$$\mathrm{Var}(\Lambda) = E(|\Lambda - E(\Lambda)|^2) = E(\Lambda^2) - E^2(\Lambda) = o(E^2(\Lambda)), \mathrm{Var}(T) = E(|T - E(T)|^2) = E(T^2) - E^2(T) = o(E^2(T)),$$

if $R_H = \ln N^\nu$. Further, by Chebyshev’s inequality $P(|X - E(X)| \geq \varepsilon) \leq \frac{\mathrm{Var}(X)}{\varepsilon^2}$,
if we replace ε with $\varepsilon E(X)$, we get

$$P\left(\left|\frac{\Lambda - E(\Lambda)}{E(\Lambda)}\right| \geq \varepsilon\right) \leq \frac{\mathrm{Var}(\Lambda)}{\varepsilon^2 E^2(\Lambda)} = o(1)\varepsilon,$$

i.e.,

$$\Lambda = E(\Lambda)(1 + o_p(1)).$$

Similarly,

$$T = E(T)(1 + o_p(1)).$$

[Figure 6: see original paper] Plot of the tendency of the global clustering coefficient against network size N, for $d = 1$, $2\sigma = 1/2$, $\tau = 1/2$, $\nu = 1$ and $\zeta = 1$. One can see the tendency of $C_2(G) \to 0$ as $N \to \infty$.

6. Global Clustering Coefficient

The global clustering coefficient $C_2(G)$ of a graph G is defined as

$$C_2(G) := \frac{3T(G)}{\Lambda(G)}.$$

So from the previous section, we have Theorem 1.3:

$$C_2(G) = \frac{3T(G)}{\Lambda(G)} = \frac{3E(T)(1 + o_p(1))}{E(\Lambda)(1 + o_p(1))} \longrightarrow 0, \quad \text{as } N \to \infty.$$

Simulation results (refer to Figure 6) also support this conclusion.

7. Conclusions

In this article, we give a precise characterization of degree and clustering in random geometric graphs on the hyperbolic ball of any dimension for the small parameter $\tau < 1$, $2\sigma < 1$ and the given relation $R_H = \ln N^\nu$. We obtain the zero-tendency for the k-degree and the global clustering coefficient as $N \to \infty$. Although this may not meet the desire for power-law behavior with small parameters in the real world, the interaction between the density parameter σ and the region expansion speed is reflected by this article and the previous article [**?**], which reveals that mobility may have important significance as a generic mechanism in networks. Some techniques and analysis methods are first used in this article, and we hope that our results and methods may be applied in machine learning since hyperbolic geometry has been widely used in this field for the quasi-isometric embedding proof of Sarkar [**?**, **?**]. Here we use the global clustering coefficient to describe the clustering phenomenon, but there are also other definitions for clustering tendency, so it is interesting to try other clustering definitions for this model. We recently learned that Fountoulakis gave a finer characterization about clustering in [**?**]. It is a natural step to follow their work for the case $\tau < 1$, $2\sigma < 1$. There are many other models of hyperbolic geometry, for example in [**?**, **?**], and some researchers advise us that the Klein model in [**?**] is well-suited for calculation, so it is a good choice to apply our method to such models. We also think that the random geometric graph model on the hyperbolic ball can analyze social networks since there exist some common principles as we mentioned in the Introduction.

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Appendix

A.1

For $r_u + r_v - R \geq \omega(N)$ we divide the integral (1.5) as follows:

$$\hat{p}_{u,v} = \int_0^{\pi} \rho_1(\theta) p_{u,v} \, d\theta = \int_0^{\tilde{\theta}_{u,v}} \rho_1(\theta) p_{u,v} \, d\theta + \int_{\tilde{\theta}_{u,v}}^{\pi} \rho_1(\theta) p_{u,v} \, d\theta.$$

For the first part of the integral,

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$$\int_0^{\tilde{\theta}_{u,v}} \rho_1(\theta) p_{u,v}\, d\theta \le \int_0^{\tilde{\theta}_{u,v}} \rho_1(\theta)\, d\theta \le \frac{d}{I_{d,1}} e^{R_H-\eta_u-\eta_v} \tilde{\theta}_{u,v}^d = o(e^{R_H-\eta_u-\eta_v}), \tag{A.1}$$

and for the second part of the integral,

$$\int_{\tilde{\theta}_{u,v}}^{\pi} \rho_1(\theta) p_{u,v}\, d\theta = \int_{\tilde{\theta}_{u,v}}^{\pi} \rho_1(\theta) \frac{1}{1+e^{\eta_u+\eta_v-R_H}\tau(\theta)}\, d\theta + o(e^{R_H-\eta_u-\eta_v}) = \int_{\tilde{\theta}_{u,v}}^{\pi} \rho_1(\theta) \frac{1}{1+e^{\eta_u+\eta_v-R_H}\tau(\theta)}\, d\theta + o(e^{R_H-$$

and the last equation is from the estimation $I = \int_\delta^\pi \frac{\sin^{d-1}\theta}{\tau(\theta)}\, d\theta = c^*(\tau, d)\delta^\tau + o(\delta^\tau)$ in Appendix A.1 of [**?**] with $\delta = e^{R_H-\eta_u-\eta_v}$. Finally, we have

$$\hat{p}_{u,v} = \int_0^\pi \rho_1(\theta) p_{u,v}\, d\theta = (1+o(1)) c^*(\tau, d) e^{R_H-\eta_u-\eta_v}. \tag{A.2}$$

Note: Figure translations are in progress. See original paper for figures.

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