

## Some Remarks on the Hausdorff Metric

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### Abstract

In this paper, we discuss the properties of functions generated using Hausdorff metric.

### Full Text

#### Some Discussions on Hausdorff Metric

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### Abstract

In this paper, we discuss the properties of functions generated using Hausdorff metric.

**Keywords:** Fuzzy sets; Hausdorff metric;  $\alpha$ -cut

## 1. Introduction

Let  $(X, d_X)$  be a metric space. For  $u \in F(X)$ , let  $[u]_\alpha$  denote the  $\alpha$ -cut of  $u$ , i.e.  $[u]_\alpha = \text{cl}(\{x \in X : u(x) \geq \alpha\})$ ,  $\text{supp } u = \{u > 0\}$ ,  $\alpha \in (0, 1]$ ,  $\alpha = 0$ , where  $S$  denotes the closure of  $S$  in  $(X, d)$ .

The set of upper semi-continuous fuzzy sets in  $(X, d_X)$  is denoted by  $F_{USC}(X)$ , i.e.

$$F_{USC}(X) = \{u \in F(X) : [u]_\alpha \text{ is closed in } (X, d_X) \text{ for } \alpha \in (0, 1]\}.$$

$U_{SC}(X)$  is the set of normal fuzzy sets in  $F_{USC}(X)$ , i.e.

$$U_{SC}(X) = \{u \in F(X) : [u]_\alpha \in C(X) \text{ for } \alpha \in (0, 1]\},$$

where  $C(X)$  is the set of nonempty closed sets of  $(X, d_X)$ .

$$U_{SCG}(X) = \{u \in F(X) : [u]_\alpha \in K(X) \text{ for } \alpha \in (0, 1]\},$$

where  $K(X)$  is the set of nonempty compact sets of  $(X, d_X)$ .

Let  $(X, d_X)$  be a metric space. We use  $H$  to denote the Hausdorff metric on  $C(X)$  induced by  $d_X$ , i.e.,  $H(U, V) = \max\{H^*(U, V), H^*(V, U)\}$  for arbitrary  $U, V \in C(X)$ , where  $H^*(U, V) = \sup_{u \in U} d_X(u, V) = \sup_{u \in U} \inf_{v \in V} d_X(u, v)$ .

**Remark 1.1.**  $\rho$  is said to be an extended metric on  $Y$  if  $\rho$  is a function from  $Y \times Y$  into  $\mathbb{R} \cup \{+\infty\}$  satisfying positivity, symmetry and triangle inequality.

The Hausdorff metric  $H$  on  $C(X)$  induced by  $d_X$  on  $X$  an extended metric, but probably not a metric, because  $H(A, B)$  could be equal to  $+\infty$  for some  $A, B \in C(X)$ .

The following inequality should be a known result:

$$H^*(U, W) \leq H^*(U, V) + H^*(V, W) \quad \text{for } U, V, W \in C(X).$$

Let  $\mathbb{R}^m$  be the  $m$ -dimensional Euclidean space. See [?] for the symbols in this paper.

In this paper, we uniformly use  $H$  to denote the Hausdorff metric on  $C(X)$  induced by  $d_X$ , where  $(X, d_X)$  is a certain metric space. The meaning of  $H$  can be judged according to the context.

We have obtained the following statements on the measurability of the function  $H([u]_\alpha, [v]_\alpha)$  (See [?], which was submitted on 2019.07.06):

- For  $u \in F_{USC}^1(X)$  and  $x_0 \in X$ ,  $H([u]_\alpha, \{x_0\})$  is a measurable function of  $\alpha$  on  $[0, 1]$ .
- For  $u, v \in F_{USC}^1(\mathbb{R}^m)$ ,  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .
- For  $u, v \in F_{USCG}^1(X)$ ,  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .
- There exists a metric space  $X$  and  $u, v \in F_{USC}^1(X)$  such that  $H([u]_\alpha, [v]_\alpha)$  is a non-measurable function of  $\alpha$  on  $[0, 1]$ .

In [?], we submitted the proofs of the first three statements. The proofs of the first three statements and the example given in chi-naXiv:202108.00116v1 which shows the last statement were recorded in a handwritten material before 2019.07.06. In this version, a very small change is made to the example.

## 2. Properties of $H([u]_\alpha, [v]_\alpha)$

In this section, we give the proofs of the first three statements and the example to show the last statement.

**Proposition 2.1.** For  $u \in F_{USC}^1(X)$  and  $x_0 \in X$ ,  $H([u]_\alpha, \{x_0\})$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

*Proof.* We can see that for  $0 \leq \alpha \leq \beta \leq 1$ ,

$$H([u]_\alpha, \{x_0\}) = \sup_{x \in [u]_\alpha} d(x, x_0) \geq \sup_{x \in [u]_\beta} d(x, x_0) = H([u]_\beta, \{x_0\}).$$

So the desired result follows from the fact that  $H([u]_\alpha, \{x_0\})$  is a monotone function of  $\alpha$  on  $[0, 1]$ .

For  $u, v \in F_{USC}^1(X)$  and  $r \in \mathbb{R}$ , we use the symbol  $\{H(u, v) > r\}$  to denote the set  $\{\alpha \in [0, 1] : H([u]_\alpha, [v]_\alpha) > r\}$ .

**Proposition 2.2.** For  $u, v \in F_{USC}^1(\mathbb{R}^m)$ ,  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

*Proof.* We only need to show that for each  $r \in \mathbb{R}$ , the set  $\{H(u, v) > r\}$  is a measurable set.

**Step (i)** For each  $r \in \mathbb{R}$ , if  $\alpha > 0$  and  $\alpha \in \{H(u, v) > r\}$ , then there exists  $\delta(\alpha) > 0$  such that  $[\alpha - \delta(\alpha), \alpha] \subseteq \{H(u, v) > r\}$ .

We proceed by contradiction. If for each  $\delta > 0$ ,  $[\alpha - \delta, \alpha] \not\subseteq \{H(u, v) > r\}$ , then there exists an increasing sequence  $\{\gamma_n\}$  such that  $\gamma_n \rightarrow \alpha$  and  $H([u]_{\gamma_n}, [v]_{\gamma_n}) \leq r$ .

Given  $x \in [u]_\alpha$ , then  $d(x, [v]_{\gamma_n}) \leq H([u]_{\gamma_n}, [v]_{\gamma_n}) \leq r$ . Therefore there exist  $y_n \in [v]_{\gamma_n}$  such that  $d(x, y_n) = d(x, [v]_{\gamma_n}) \leq r$ . Hence there is a subsequence  $\{y_{n_i}\}$  of  $\{y_n\}$  such that  $\{y_{n_i}\}$  converges to  $y \in \mathbb{R}^m$ . Note that  $d(x, y) \leq r$  and  $y \in \cap [v]_{\gamma_{n_i}} = [v]_\alpha$ , so we have  $d(x, [v]_\alpha) \leq r$ .

From the arbitrariness of  $x$ ,  $H^*([u]_\alpha, [v]_\alpha) \leq r$ . Similarly, we can deduce that  $H^*([v]_\alpha, [u]_\alpha) \leq r$ . Thus  $H([u]_\alpha, [v]_\alpha) \leq r$ , which is a contradiction.

**Step (ii)** For each  $r \in \mathbb{R}$ , if  $\{H(u, v) > r\} \setminus \{0\} \neq \emptyset$ , then  $\{H(u, v) > r\} \setminus \{0\}$  is a union of disjoint positive length intervals.

Suppose that  $\{H(u, v) > r\} \setminus \{0\} \neq \emptyset$ . For  $x \in \{H(u, v) > r\} \setminus \{0\}$ , let  $(cid : 122)(cid : 125)(cid : 124)(cid : 123)_x = (cid : 83)\{[a, b] : x \in [a, b] \subseteq \{H(u, v) > r\} \setminus \{0\}\}$ , i.e.  $(cid : 122)(cid : 125)(cid : 124)(cid : 123)_x$  is the largest interval in  $\{H(u, v) > r\} \setminus \{0\}$  which contains  $x$ . Then by step (i),  $(cid : 122)(cid : 125)(cid : 124)(cid : 123)_x$  is a positive length interval. Note that for  $x, y \in \{H(u, v) > r\} \setminus \{0\}$ , if  $(cid : 122)(cid : 125)(cid : 124)(cid : 123)_x \cap (cid : 122)(cid : 125)(cid : 124)(cid : 123)_y \neq \emptyset$ , then  $(cid : 122)(cid : 125)(cid : 124)(cid : 123)_x = (cid : 122)(cid : 125)(cid : 124)(cid : 123)_y$ . Thus  $\{H(u, v) > r\} \setminus \{0\}$  is a union of disjoint positive length intervals.

**Step (iii)** For each  $r \in \mathbb{R}$ ,  $\{H(u, v) > r\}$  is a measurable set.

Clearly, if positive length intervals are disjoint, then these positive length intervals are at most countable. Thus, from step (ii),  $\{H(u, v) > r\} \setminus \{0\}$  is a measurable set. So  $\{H(u, v) > r\}$  is a measurable set.

**Remark 2.3.** Let  $u, v \in F_{USC}^1(X)$ . For each  $r \in \mathbb{R}$ , if  $0 \in \{H(u, v) > r\}$ , then there exists  $\delta > 0$  such that  $[0, \delta] \subseteq \{H(u, v) > r\}$ .

The above fact is equivalent to the following fact: Let  $u, v \in F_{USC}^1(X)$ . Then  $H([u]_0, [v]_0) \leq \liminf_{\alpha \rightarrow 0^+} H([u]_\alpha, [v]_\alpha)$ , where  $\liminf_{\alpha \rightarrow 0^+} H([u]_\alpha, [v]_\alpha) = +\infty$  is possible.

Combined this fact with the proof of Proposition 2.2, we have the following conclusion: Let  $u, v \in F_{USC}^1(\mathbb{R}^m)$  and let  $r \in \mathbb{R}$ . If  $\{H(u, v) > r\} (cid : 54) = \emptyset$ , then  $\{H(u, v) > r\}$  is a union of disjoint positive length intervals (Obviously,  $\{H(u, v) > r\}$  could be an interval. It is easy to see that for fixed  $r \geq 0$ , the possible forms of the maximal intervals in  $\{H(u, v) > r\}$  are as follows:  $[0, \alpha]$ ,  $[0, \alpha]$ ,  $(\beta, \alpha)$  and  $(\beta, \alpha]$ , where  $\alpha \in (0, 1]$  and  $\beta \in [0, 1]$ ).

**Remark 2.4.** Let  $u, v \in F_{USC}^1(X)$  and let  $\alpha > 0$ . The following two statements are equivalent: (i) For each  $r \in \mathbb{R}$ , if  $\alpha \in \{H(u, v) > r\}$ , then there exists  $\delta(\alpha) > 0$  such that  $[\alpha - \delta(\alpha), \alpha] \subseteq \{H(u, v) > r\}$ . (ii)  $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \rightarrow \alpha^-} H([u]_\gamma, [v]_\gamma)$  (where  $\liminf_{\gamma \rightarrow \alpha^-} H([u]_\gamma, [v]_\gamma) = +\infty$  is possible).

So the statement “ $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \rightarrow \alpha^-} H([u]_\gamma, [v]_\gamma)$  for all  $\alpha \in (0, 1]$ ” is equivalent to the statement proved by step (i) of the proof of Proposition 2.2, which is listed below: - For each  $r \in \mathbb{R}$ , if  $\alpha > 0$  and  $\alpha \in \{H(u, v) > r\}$ , then there exists  $\delta(\alpha) > 0$  such that  $[\alpha - \delta(\alpha), \alpha] \subseteq \{H(u, v) > r\}$ .

**Remark 2.5.** From the proof of Proposition 2.2 and Remark 2.4, we know that for  $u, v \in F_{USC}^1(X)$ , if  $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \rightarrow \alpha^-} H([u]_\gamma, [v]_\gamma)$  for all  $\alpha \in (0, 1]$ , then  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

The following Proposition 2.6 is Lemma 4.4 in [?].

**Proposition 2.6.** Let  $U_n \in K(X)$  for  $n = 1, 2, \dots$ . If  $U_1 \supseteq U_2 \supseteq \dots \supseteq U_n \supseteq \dots$  and  $U = \bigcap_{n=1}^{+\infty} U_n \in K(X)$ , then  $H(U_n, U) \rightarrow 0$  as  $n \rightarrow +\infty$ . If  $U_1 \subseteq U_2 \subseteq \dots \subseteq U_n \subseteq \dots$  and  $V = \bigcup_{n=1}^{+\infty} U_n \in K(X)$ , then  $H(U_n, V) \rightarrow 0$  as  $n \rightarrow +\infty$ .

*Proof.* (i) is easy to show. We only prove (ii). Suppose that  $H(U_n, U) \not\rightarrow 0$ . Then there is an  $\varepsilon_0 > 0$  such that  $H(U_n, U) > \varepsilon_0$ . Hence there exists  $x_n \in U$  such that  $d(x_n, U_n) > \varepsilon_0$ .

Since  $U$  is compact, there is a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  such that  $x_{n_i}$  converges to  $x \in U$ . Note that there exists  $\{y_n\}$  such that  $y_n \in U_n$  and  $y_n \rightarrow x$ . Thus  $d(x, U_n) \rightarrow 0$ , which contradicts (3).

**Proposition 2.7.** For  $u, v \in F_{USC}^1(X)$ ,  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

*Proof.* Note that  $H([u]_\alpha, [v]_\alpha)$  is finite at  $\alpha \in (0, 1]$  and for  $\alpha, \beta \in (0, 1]$ ,  $|H([u]_\alpha, [v]_\alpha) - H([u]_\beta, [v]_\beta)| \leq H([u]_\alpha, [u]_\beta) + H([v]_\alpha, [v]_\beta)$ . Then by Propo-

sition 2.6 (i),  $H([u]_\beta, [v]_\beta) = H([u]_\alpha, [v]_\alpha)$ , i.e.  $H([u]_\alpha, [v]_\alpha)$  is left-continuous at  $\alpha \in (0, 1]$ .

Thus from Remark 2.5,  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

**Remark 2.8.** From Proposition 2.6, we know that for  $u, v \in F_{USCG}^1(X)$ ,  $H([u]_\alpha, [v]_\alpha)$  is left-continuous at  $\alpha \in (0, 1]$ , and that for  $u, v \in F_{USCB}^1(X)$ ,  $H([u]_\alpha, [v]_\alpha)$  is right-continuous at  $\alpha = 0$ .

To give the example which shows the last statement presented in Section 1, we need some conclusions at first.

The following representation theorem should be a known conclusion.

**Theorem 2.9.** Let  $X$  be a set. Given  $u \in F(X)$ , then for all  $\alpha \in (0, 1]$ ,  $[u]_\alpha = \bigcap_{\beta < \alpha} [u]_\beta$ .

Conversely, suppose that  $\{u(\alpha) : \alpha \in (0, 1]\}$  is a family of sets in  $X$  satisfying  $u(\alpha) = \bigcap_{\beta < \alpha} u(\beta)$  for all  $\alpha \in (0, 1]$ . Define  $v \in F(X)$  by  $v(x) := \sup\{\alpha : x \in u(\alpha)\}$  ( $\sup \emptyset = 0$ ). Then  $[v]_\alpha = u(\alpha)$  for all  $\alpha \in (0, 1]$ .

$\rho$  is said to be a metric on  $Y$  if  $\rho$  is a function from  $Y \times Y$  into  $\mathbb{R}$  satisfying positivity, symmetry and triangle inequality. At this time,  $(Y, \rho)$  is said to be a metric space.  $\rho$  is said to be an extended metric on  $Y$  if  $\rho$  is a function from  $Y \times Y$  into  $\mathbb{R} \cup \{+\infty\}$  satisfying positivity, symmetry and triangle inequality. At this time,  $(Y, \rho)$  is said to be an extended metric space.

Let  $(Y, \rho)$  be an extended metric space. For  $y \in Y$  and  $\varepsilon > 0$ , let  $B(y, \varepsilon)$  denote the set  $\{z \in Y : \rho(y, z) < \varepsilon\}$ .  $\{B(y, \varepsilon) : y \in Y, \varepsilon > 0\}$  is a basis for the topology induced by  $\rho$  on  $Y$ . The closure of a set  $A$  in  $(Y, \rho)$ , denoted by  $\bar{A}$ , refers to the closure of  $A$  in  $Y$  according to the topology induced by  $\rho$  on  $Y$ . Then  $x \in \bar{A}$  if and only if there is a sequence  $\{x_n\}$  in  $Y$  such that  $\rho(x_n, x) \rightarrow 0$ . So  $x \in \bar{A}$  if and only if  $\rho(x, A) = 0$ .

Here we mention that if  $(Y, \rho)$  is an extended metric space, then the Hausdorff distance  $H$  on  $C(Y)$  induced by  $\rho$  using (1) is an extended metric on  $C(Y)$ , where  $C(Y)$  denotes the set of nonempty closed sets in  $(Y, \rho)$ . It can be seen that  $H$  satisfies positivity and symmetry. To show that  $H$  satisfies the triangle inequality, we only need to show that  $H^*(U, W) \leq H^*(U, V) + H^*(V, W)$  for  $U, V, W \in C(Y)$ . To do this, let  $x \in U$ . Then

$$\rho(x, W) \leq \inf_{y \in V, z \in W} \{\rho(x, y) + \rho(y, z)\} \leq \inf_{y \in V} \{\rho(x, y) + \rho(y, W)\} \leq \inf_{y \in V} \rho(x, y) + H^*(V, W) = \rho(x, V) + H^*(V, W) \leq$$

From the arbitrariness of  $x$  in  $U$ , we obtain (4). So the Hausdorff distance  $H$  on  $C(Y)$  is the Hausdorff extended metric.

For simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric in this paper.

For an extended metric space  $(Y, \rho)$ , we define

$$F_{USC}(Y) = \{u \in F(Y) : [u]_\alpha \text{ is closed in } (Y, \rho) \text{ for } \alpha \in (0, 1]\}.$$

Let  $\Gamma$  be a set, and for each  $\gamma \in \Gamma$ , let  $(X_\gamma, d_\gamma)$  be a metric space. Define an extended metric  $d$  on  $\prod_{\gamma \in \Gamma} X_\gamma$  as

$$d(x, y) := \sup\{d_\gamma(x_\gamma, y_\gamma) : \gamma \in \Gamma\} \quad \text{for } x = (x_\gamma)_{\gamma \in \Gamma} \text{ and } y = (y_\gamma)_{\gamma \in \Gamma}.$$

We use the symbol  $\prod_{\gamma \in \Gamma} (X_\gamma, d_\gamma)$  to denote the extended metric space  $(\prod_{\gamma \in \Gamma} X_\gamma, d)$ . If not mentioned specially, we suppose by default that the extended metric on  $\prod_{\gamma \in \Gamma} X_\gamma$  is the  $d$  given by (5).

Let  $u_\gamma \in F(X_\gamma)$ ,  $\gamma \in \Gamma$ . Define  $u \in F(\prod_{\gamma \in \Gamma} X_\gamma)$  as

$$[u]_\alpha = \prod_{\gamma \in \Gamma} [u_\gamma]_\alpha \quad \text{for each } \alpha \in (0, 1].$$

We use  $\otimes_{\gamma \in \Gamma} u_\gamma$  to denote the fuzzy set  $u$  given by (6).

From Theorem 2.9,  $u$  is well-defined because for each  $\alpha \in (0, 1]$ ,

$$[u]_\alpha = \prod_{\gamma \in \Gamma} [u_\gamma]_\alpha = \bigcap_{\beta < \alpha} \prod_{\gamma \in \Gamma} [u_\gamma]_\beta = \bigcap_{\beta < \alpha} [u]_\beta.$$

In this paper, if not mentioned specially, we use  $\bar{S}$  to denote the closure of  $S$  in a certain extended metric space  $(X, d_X)$ . For a set  $S \subseteq X_\gamma$ ,  $\gamma \in \Gamma$ , we use  $\bar{S}$  to denote the closure of  $S$  in  $(X_\gamma, d_\gamma)$ . For a set  $S \subseteq \prod_{\gamma \in \Gamma} X_\gamma$ , we also use  $\bar{S}$  to denote the closure of  $S$  in  $(\prod_{\gamma \in \Gamma} X_\gamma, d)$ . The readers can judge the meaning of  $\bar{S}$  according to the context.

**Lemma 2.10.** Let  $\Gamma$  be a set, and for each  $\gamma \in \Gamma$ , let  $(X_\gamma, d_\gamma)$  be a metric space. If  $A_\gamma \subseteq X_\gamma$  for  $\gamma \in \Gamma$ , then

$$\overline{\prod_{\gamma \in \Gamma} A_\gamma} = \prod_{\gamma \in \Gamma} \overline{A_\gamma}.$$

*Proof.* Clearly  $\prod_{\gamma \in \Gamma} A_\gamma \subseteq \prod_{\gamma \in \Gamma} \overline{A_\gamma}$ . Conversely, if  $x = (x_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} \overline{A_\gamma}$ , then for each  $\varepsilon > 0$ , there exists  $y = (y_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} A_\gamma$  such that  $d_\gamma(x_\gamma, y_\gamma) \leq \varepsilon$  for all  $\gamma \in \Gamma$ . So  $d(x, y) \leq \varepsilon$ . From the arbitrariness of  $\varepsilon > 0$ , we have  $x \in \overline{\prod_{\gamma \in \Gamma} A_\gamma}$ . Thus  $\prod_{\gamma \in \Gamma} \overline{A_\gamma} \subseteq \overline{\prod_{\gamma \in \Gamma} A_\gamma}$ . In summary,  $\overline{\prod_{\gamma \in \Gamma} A_\gamma} = \prod_{\gamma \in \Gamma} \overline{A_\gamma}$ .

**Theorem 2.11.** Let  $\Gamma$  be a set, and for each  $\gamma \in \Gamma$ , let  $(X_\gamma, d_\gamma)$  be a metric space. If  $u_\gamma \in F_{USC}(X_\gamma)$  for each  $\gamma \in \Gamma$ , then  $u = \bigotimes_{\gamma \in \Gamma} u_\gamma$  is a fuzzy set in  $F_{USC}(\prod_{\gamma \in \Gamma} X_\gamma)$ .

*Proof.* By (6) and Lemma 2.10, for each  $\alpha \in (0, 1]$ ,  $[u]_\alpha = \prod_{\gamma \in \Gamma} [u_\gamma]_\alpha = \prod_{\gamma \in \Gamma} \overline{[u_\gamma]_\alpha} = \overline{[u]_\alpha}$ , thus  $u \in F_{USC}(\prod_{\gamma \in \Gamma} X_\gamma)$ .

In the following theorem, we use  $H$  to denote the Hausdorff metric on  $C(X_\gamma)$  induced by  $d_\gamma$ . We also use  $H$  to denote the Hausdorff metric on  $C(\prod_{\gamma \in \Gamma} X_\gamma)$  induced by  $d$ .

**Theorem 2.12.** Let  $\Gamma$  be a set, and for each  $\gamma \in \Gamma$ , let  $(X_\gamma, d_\gamma)$  be a metric space. If  $A_\gamma$  and  $B_\gamma$  are elements in  $C(X_\gamma)$  for  $\gamma \in \Gamma$ , then

$$H\left(\prod_{\gamma \in \Gamma} A_\gamma, \prod_{\gamma \in \Gamma} B_\gamma\right) = \sup_{\gamma \in \Gamma} H(A_\gamma, B_\gamma).$$

*Proof.* From Lemma 2.10,  $\prod_{\gamma \in \Gamma} A_\gamma$  and  $\prod_{\gamma \in \Gamma} B_\gamma$  are elements in  $C(\prod_{\gamma \in \Gamma} X_\gamma)$ . Note that  $d(x, \prod_{\gamma \in \Gamma} B_\gamma) = \sup_{\gamma \in \Gamma} d_\gamma(x_\gamma, B_\gamma)$  for each  $x = (x_\gamma)_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} X_\gamma$ . Thus

$$H^*\left(\prod_{\gamma \in \Gamma} A_\gamma, \prod_{\gamma \in \Gamma} B_\gamma\right) = \sup_{x \in \prod_{\gamma \in \Gamma} A_\gamma} d\left(x, \prod_{\gamma \in \Gamma} B_\gamma\right) = \sup_{x \in \prod_{\gamma \in \Gamma} A_\gamma} \sup_{\gamma \in \Gamma} d_\gamma(x_\gamma, B_\gamma) = \sup_{\gamma \in \Gamma} \sup_{x_\gamma \in A_\gamma} d_\gamma(x_\gamma, B_\gamma) = \sup_{\gamma \in \Gamma} H^*(A_\gamma, B_\gamma).$$

Now, we give an example to show that there exists a metric space  $X$  and  $u, v \in F_{USC}^1(X)$  such that  $H([u]_\alpha, [v]_\alpha)$  is a non-measurable function of  $\alpha$  on  $[0, 1]$ .

**Example 2.13.** We see  $[0, 100] \setminus \{10\}$  as a metric subspace of  $\mathbb{R}$ . Let  $z \in (0, 1]$ . Define  $u_z \in F_{USC}^1([0, 100] \setminus \{10\})$  as

$$[u_z]_\alpha = \begin{cases} \{3\}, & \alpha \in [z, 1], \\ \{3\} \cup (10, 10 + \varepsilon], & \alpha = z(1 - \varepsilon), 0 < \varepsilon \leq 1. \end{cases}$$

Let  $z \in (0, 1]$ . Define  $v_z \in F_{USC}^1([0, 100] \setminus \{10\})$  as

$$[v_z]_\alpha = \begin{cases} \{73\}, & \alpha \in (z, 1], \\ [71, 81], & \alpha \in [0, z]. \end{cases}$$

Then for  $z \in (0, 1]$ ,

$$H([u_z]_\alpha, [v_z]_\alpha) = \begin{cases} 71 - \varepsilon, & \alpha = z(1 - \varepsilon), 0 < \varepsilon \leq 1, \\ 78, & \alpha \in (z, 1], \\ 70, & \alpha = z, \end{cases}$$

where  $H$  is the Hausdorff metric on  $C([0, 100] \setminus \{10\})$  induced by the metric on  $[0, 100] \setminus \{10\}$ .

We see  $[0, 9]$  as a metric subspace of  $\mathbb{R}$ . Define  $w \in F([0, 9])$  as  $w(t) = 1$  for all  $t \in [0, 9]$ .

Let  $A$  be a non-measurable set in  $(0, 1]$ . Let  $u := \bigotimes_{z \in [0, 1]} u_z$  and let  $v := \bigotimes_{z \in [0, 1]} v_z$ , where

$$X_z = \begin{cases} [0, 100] \setminus \{10\}, & z \in A, \\ [0, 9], & z \in [0, 1] \setminus A. \end{cases}$$

Then by Theorem 2.11,  $u$  and  $v$  are fuzzy sets in  $F_{USC}^1(\prod_{z \in [0, 1]} X_z)$ , where

$$X_z = \begin{cases} [0, 100] \setminus \{10\}, & z \in A, \\ [0, 9], & z \in [0, 1] \setminus A. \end{cases}$$

Here we mention that  $(\prod_{z \in [0, 1]} X_z, d)$  is a metric space with  $d$  given by (5). By Theorem 2.12,

$$H([u]_\alpha, [v]_\alpha) = \sup_{z \in A} H([u_z]_\alpha, [v_z]_\alpha) \vee \sup_{z \in [0, 1] \setminus A} H([0, 9], [0, 9]) = \sup_{z \in A} H([u_z]_\alpha, [v_z]_\alpha) = \begin{cases} 78, & \alpha \in A, \\ \leq 71, & \alpha \in [0, 1] \setminus A. \end{cases}$$

So  $\{\alpha \in [0, 1] : H([u]_\alpha, [v]_\alpha) > 73\} = A$ , and thus  $H([u]_\alpha, [v]_\alpha)$  is a non-measurable function of  $\alpha$  on  $[0, 1]$ .

### 3. Some Discussions

In [?] (Lemma 6.3) and [?] (Lemma 6.5), we pointed out that for  $u \in U_{SCG}(X)$ , the cut-function  $[u](\alpha) = [u]_\alpha$  from  $[0, 1]$  to  $(C(X), H)$  is left-continuous on  $(0, 1]$ . Then it follows immediately that for  $u, v \in F_{USC}^1(X)$ ,  $H([u]_\alpha, [v]_\alpha)$  is left-continuous at  $\alpha \in (0, 1]$  (see Proposition 2.7). From this fact, it's natural to realize that for  $u, v \in F_{USC}^1(X)$ ,  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

Let  $(X, d_X)$  be a metric space. We say that  $S \subseteq F_{USC}^1(X)$  satisfies condition  $(X, d_X)$ -I if  $[u]_\alpha \cap B(x, r)$  is compact in  $(X, d_X)$  for all  $u \in S$ ,  $\alpha \in (0, 1]$ ,  $x \in X$  and  $r \in \mathbb{R}^+$ , where  $B(x, r) := \{y \in X : d_X(x, y) \leq r\}$ .

Clearly,  $S = F_{USC}^1(\mathbb{R}^m)$  satisfies condition  $\mathbb{R}^m$ -I and  $S = F_{USCG}^1(X)$  satisfies condition  $(X, d_X)$ -I.

If  $S \subseteq F_{USC}^1(X)$  satisfies condition  $(X, d_X)$ -I, then proceeding similarly as in step (i) of the proof of Proposition 2.2, we have that  $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \rightarrow \alpha^-} H([u]_\gamma, [v]_\gamma)$  for all  $u, v \in S$  and  $\alpha \in (0, 1]$ . Thus as mentioned in Remark 2.5, for all  $u, v \in S$ ,  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

There exists a metric space  $(X, d_X)$  and  $S \subseteq F_{USC}^1(X)$  which satisfies a condition weaker than condition  $(X, d_X)$ -I. By using this weaker condition, we can proceed similarly as in step (i) of the proof of Proposition 2.2 to show that  $H([u]_\alpha, [v]_\alpha) \leq \liminf_{\gamma \rightarrow \alpha^-} H([u]_\gamma, [v]_\gamma)$  for all  $u, v \in S$  and  $\alpha \in (0, 1]$ .

#### 4. Improvements

In this section, we give some improvements of Propositions 2.1, 2.2 and 2.7, which are the statements on measurability of  $H([u]_\alpha, [v]_\alpha)$  presented in [?]. We first prove Theorem 4.1 which is an improvement of Propositions 2.1 and 2.7. Then we show Theorem 4.3 and use it to improve Theorem 4.1 and Proposition 2.2.

Let  $v \in F_{USC}^1(X)$  and let  $0 \leq \alpha < \beta \leq 1$ . The “variation”  $w_v(\alpha, \beta)$  is defined as

$$w_v(\alpha, \beta) := \sup\{H([v]_\xi, [v]_\eta) : \xi, \eta \in (\alpha, \beta)\}.$$

**Theorem 4.1.** Let  $u \in F_{USC}^1(X)$  and let  $v \in F_{USCG}^1(X)$ . Then  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

*Proof.* The proof is divided into three steps.

**Step (I)**  $H^*([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

Let  $\xi \in \mathbb{R}$  and let  $n \in \mathbb{N}$ . Define  $S_\xi := \{\alpha \in [0, 1] : H^*([u]_\alpha, [v]_\alpha) \geq \xi\}$ ,  $S_{\xi, n} := S_\xi \cap (\frac{1}{n}, 1]$ .

To show that  $H^*([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ , it suffices to show that for each  $\xi \in \mathbb{R}$  and  $n \in \mathbb{N}$ ,  $S_{\xi, n}$  is a measurable set.

Since  $v \in F_{USCG}^1(X)$ , from Lemma 6.5 in [?] for each  $k = 1, 2, \dots$ , there exist  $0 = \alpha_0^{(k)} < \alpha_1^{(k)} < \dots < \alpha_{l_k}^{(k)} = 1$  such that  $w_v(\alpha_i^{(k)}, \alpha_{i+1}^{(k)}) \leq \frac{1}{k}$  for all  $i = 0, 1, \dots, l_k - 1$ .

Let  $T_{k, i} := \{x : \text{there exists } s \in S_\xi \text{ such that } \alpha_i^{(k)} < x \leq s \leq \alpha_{i+1}^{(k)}\}$ . Put  $T_k := \bigcup_{i=0}^{l_k-1} T_{k, i}$ . We affirm that (i)  $T_k$  is a measurable set, (ii)  $T_k \supseteq S_{\xi, n}$ , and (iii)  $T_k \subseteq S_{\xi - \frac{1}{k}, n}$ .

If  $T_{k, i} \neq \emptyset$ , then  $T_{k, i}$  is an interval. Thus (i) is true. (ii) follows from the definition of  $T_k$ .

For each  $i = 1, \dots, l_k - 1$  and each  $x \in T_{k,i}$ , there exists an  $s \in S_\xi$  such that  $\alpha_i^{(k)} < x \leq s \leq \alpha_{i+1}^{(k)}$ , and thus

$$H^*([u]_x, [v]_x) \geq H^*([u]_s, [v]_x) \geq H^*([u]_s, [v]_s) - H^*([v]_x, [v]_s) \geq \xi - \frac{1}{k}.$$

Clearly,  $T_k \subseteq (\frac{1}{n}, 1]$ . So (iii) is proved. Hence  $T_k \subseteq S_{\xi - \frac{1}{k}, n}$ .

By affirmations (ii) and (iii), we have

$$S_{\xi, n} \subseteq \bigcap_{k=1}^{\infty} T_k \subseteq \bigcap_{k=1}^{\infty} S_{\xi - \frac{1}{k}, n} = S_{\xi, n}.$$

From affirmation (i),  $\bigcup_{k=1}^{\infty} T_k$  is measurable, and thus by (8),  $S_{\xi, n} = \bigcup_{k=1}^{\infty} T_k$  is measurable.

**Step (II)**  $H^*([v]_\alpha, [u]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

The proof of Step (II) is similar to that of Step (I).

Let  $\xi \in \mathbb{R}$  and let  $n \in \mathbb{N}$ . Define  $S_\xi := \{\alpha \in [0, 1] : H^*([v]_\alpha, [u]_\alpha) \geq \xi\}$ ,  $S_{\xi, n} := S_\xi \cap (\frac{1}{n}, 1]$ .

To show that  $H^*([v]_\alpha, [u]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ , it suffices to show that for each  $\xi \in \mathbb{R}$  and  $n \in \mathbb{N}$ ,  $S_{\xi, n}$  is a measurable set.

Let  $T'_{k,i} := \{x : \text{there exists } s \in S_\xi \text{ such that } \alpha_i^{(k)} < s \leq x \leq \alpha_{i+1}^{(k)}\}$ . Put  $T'_k := \bigcup_{i=0}^{l_k-1} T'_{k,i}$ . We affirm that (i')  $T'_k$  is a measurable set, (ii')  $T'_k \supseteq S_{\xi, n}$ , and (iii')  $T'_k \subseteq S_{\xi - \frac{1}{k}, n}$ .

(i') is true because if  $T'_{k,i} \neq \emptyset$ , then  $T'_{k,i}$  is a point or an interval. (ii') follows from the definition of  $T'_k$ .

For each  $i = 1, \dots, l_k - 1$  and each  $x \in T'_{k,i}$ , there exists an  $s \in S_\xi$  such that  $\alpha_i^{(k)} < s \leq x \leq \alpha_{i+1}^{(k)}$ , and thus

$$H^*([v]_x, [u]_x) \geq H^*([v]_x, [u]_s) \geq H^*([v]_s, [u]_s) - H^*([v]_s, [v]_x) \geq \xi - \frac{1}{k}.$$

Hence  $T'_k \subseteq S_{\xi - \frac{1}{k}}$ . Clearly,  $T'_k \subseteq (\frac{1}{n}, 1]$ . So (iii') is proved.

From affirmations (ii') and (iii'), we have

$$S_{\xi, n} \subseteq \bigcap_{k=1}^{\infty} T'_k \subseteq \bigcap_{k=1}^{\infty} S_{\xi - \frac{1}{k}, n} = S_{\xi, n}.$$

So by affirmation (i') and (9),  $S_{\xi,n} = \bigcup_{k=1}^{\infty} T'_k$  is measurable.

**Step (III)**  $H([u]_{\alpha}, [v]_{\alpha})$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

Since  $H([u]_{\alpha}, [v]_{\alpha}) = \max\{H^*([u]_{\alpha}, [v]_{\alpha}), H^*([v]_{\alpha}, [u]_{\alpha})\}$ , then the desired result follows immediately from the fact that both  $H^*([u]_{\alpha}, [v]_{\alpha})$  and  $H^*([v]_{\alpha}, [u]_{\alpha})$  are measurable functions of  $\alpha$  on  $[0, 1]$ , which is proved in steps (I) and (II).

**Remark 4.2.** Theorem 4.1 is an improvement of Proposition 2.7. Since a singleton set is a compact set, Theorem 4.1 is also an improvement of Proposition 2.1.

Obviously, if  $\xi \leq 0$ , then  $S_{\xi} = S_{\xi} = [0, 1]$  and  $S_{\xi,n} = S_{\xi,n} = [\frac{1}{n}, 1]$ .

Let  $(X, d_X)$  be a metric subspace of  $(Y, d_Y)$ . To distinguish from the closure of  $S$  in  $(X, d_X)$ , we use  $\overline{S}^Y$  to denote the closure of  $S$  in  $(Y, d_Y)$ .

For  $u \in F_{USC}^1(X)$ , define  $u^Y \in F_{USC}^1(Y)$  as  $[u^Y]_{\alpha} = \bigcap_{\beta < \alpha} [u]_{\beta}$  for  $\alpha \in (0, 1]$ . Note that  $[u^Y]_{\alpha} = \bigcap_{\beta < \alpha} [u^Y]_{\beta}$  for all  $\alpha \in (0, 1]$ , then by Theorem 2.9,  $u^Y$  is well-defined.

For each  $u \in F_{USC}^1(X)$ , define  $\Gamma(u)^Y := \{\alpha \in (0, 1] : [u^Y]_{\alpha} \text{ (cid : 37)} [u]_{\alpha}\}$ . If there is no confusion, we will write  $\Gamma(u)^Y$  as  $\Gamma(u)$  for simplicity.

We use  $H$  to denote the Hausdorff metric on  $C(X)$  induced by  $d_X$ , and we also use  $H$  to denote the Hausdorff metric on  $C(Y)$  induced by  $d_Y$ .

We will use the following Theorem 4.3 to improve Theorem 4.1 and Proposition 2.2.

**Theorem 4.3.** Let  $(X, d_X)$  be a metric subspace of  $(Y, d_Y)$  and let  $u, v \in F_{USC}^1(X)$ . Then

- (i)  $[u^Y]_{\alpha} \supseteq [u]_{\alpha}$  for all  $\alpha \in (0, 1]$ , and  $[u^Y]_0 = [u]_0$ .
- (ii) For each  $\alpha \in [0, 1] \setminus (\Gamma(u) \cup \Gamma(v))$ ,  $H([u^Y]_{\alpha}, [v^Y]_{\alpha}) = H([u]_{\alpha}, [v]_{\alpha})$ .
- (iii) The cardinality of  $\Gamma(u)$  is less than the cardinality of  $Y \setminus X$ .

*Proof.* (i) follows from the definition of  $u^Y$ . From (i) and the definition of  $\Gamma(u)$ , for each  $\alpha \in [0, 1] \setminus (\Gamma(u) \cup \Gamma(v))$ ,  $H([u^Y]_{\alpha}, [v^Y]_{\alpha}) = H(\overline{[u]_{\alpha}}^Y, \overline{[v]_{\alpha}}^Y) = H([u]_{\alpha}, [v]_{\alpha})$ , and thus (ii) is proved.

To show that (iii) is true, it suffices to construct an injection  $j : \Gamma(u) \rightarrow Y \setminus X$ .

Let  $\gamma \in \Gamma(u)$ . Then there is an  $x_{\gamma} \in Y$  such that  $x_{\gamma} \in [u^Y]_{\gamma} \setminus [u]_{\gamma}$ . Define  $j(\gamma) = x_{\gamma}$  for each  $\gamma \in \Gamma(u)$ . Since  $x_{\gamma} \notin [u]_{\gamma} = \bigcap_{\beta < \gamma} [u]_{\beta}$ , there is a  $\beta < \gamma$  such that  $x_{\gamma} \notin [u]_{\beta}$ . On the other hand, since  $x_{\gamma} \in [u^Y]_{\gamma}$ , we have  $x_{\gamma} \in \overline{[u]_{\beta}}^Y$ . Thus  $x_{\gamma} \in Y \setminus X$ . Hence  $j$  is a function from  $\Gamma(u)$  to  $Y \setminus X$ .

Let  $\xi, \eta \in \Gamma(u)$  with  $\xi < \eta$ . Since  $x_{\xi} \notin [u]_{\xi}$ , then  $x_{\xi} \notin [u^Y]_{\lambda}$  when  $\lambda > \xi$ . Hence  $x_{\xi} \notin [u^Y]_{\eta}$ . Notice that  $x_{\eta} \in [u^Y]_{\eta}$ , and therefore  $x_{\xi} \neq x_{\eta}$ . Thus  $j$  is an injection. So (iii) is proved.

**Corollary 4.4.** Let  $(X, d_X)$  be a metric subspace of  $(Y, d_Y)$  and  $Y \setminus X$  an at most countable set. Then for  $u, v \in F_{USC}^1(X)$ ,  $H([u^Y]_\alpha, [v^Y]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$  is equivalent to  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

*Proof.* By (ii), (iii) of Theorem 4.3, we have that  $H([u^Y]_\alpha, [v^Y]_\alpha) = H([u]_\alpha, [v]_\alpha)$  on  $[0, 1]$  except at most countable  $\alpha \in [0, 1]$ . Thus we obtain the desired result.

Let  $S \subseteq \mathbb{R}^m$ . We see  $\mathbb{R}^m \setminus S$  as a metric subspace of  $\mathbb{R}^m$ .

**Corollary 4.5.** Let  $S$  be an at most countable subset of  $\mathbb{R}^m$ . For  $u, v \in F_{USC}^1(\mathbb{R}^m \setminus S)$ ,  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

*Proof.* The desired result follows from Proposition 2.2 and Corollary 4.4.

**Corollary 4.6.** Let  $(X, d_X)$  be a metric subspace of  $(Y, d_Y)$  and  $Y \setminus X$  an at most countable set. Let  $u, v \in F_{USC}^1(X)$ . If  $u^Y \in F_{USCG}^1(Y)$ , then  $H([u]_\alpha, [v]_\alpha)$  is a measurable function of  $\alpha$  on  $[0, 1]$ .

*Proof.* The desired result follows from Theorem 4.1 and Corollary 4.4.

**Remark 4.7.** Clearly, if  $u \in F_{USCG}^1(X)$ , then  $[u]_\alpha = [u^Y]_\alpha$  for  $\alpha \in (0, 1]$  and thus  $u^Y \in F_{USCG}^1(Y)$ . So Corollary 4.6 is an improvement of Theorem 4.1. Corollary 4.5 is an improvement of Proposition 2.2. Theorem 4.1 is the special case of Corollary 4.6 when  $Y = X$ . Proposition 2.2 is the special case of Corollary 4.5 when  $S = \emptyset$ .

In essence, contents including Theorem 4.3, Corollaries 4.4 and 4.5 have already been proved in chinaXiv:202108.00116v1, which is a previous version of this paper.

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*Note: Figure translations are in progress. See original paper for figures.*

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