The Relationship Between Nonsymbolic Quantity Representation and Symbolic Fraction Representation

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Abstract

A critical aspect of individuals learning symbolic fractions is forming accurate representations of their numerical values. Existing research hypothesizes that the cognitive basis for symbolic fraction representation is the non-symbolic quantity representation that humans possess from infancy (such as representing the quantities of two sets, or the ratio between two quantities). Evidence includes correlations at both behavioral and brain neural activity levels between the representation of non-symbolic quantities (especially non-symbolic quantity ratio relationships) and the representation of symbolic fractions. However, to establish that non-symbolic quantity representation is the cognitive basis for symbolic fraction representation, additional research is needed to demonstrate their unique correlation and causal relationship at the conceptual level of quantity, and to elucidate the cognitive mechanisms underlying the formation of symbolic fraction representation.

Full Text

The Relation Between Non-Symbolic Magnitude Representation and Symbolic Fraction Representation

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Abstract

A key to learning symbolic fractions is forming accurate representations of their numerical values. Current research hypothesizes that the cognitive foundation for symbolic fraction representation lies in non-symbolic magnitude representation—the ability to represent quantities and their proportions that emerges in early infancy. Evidence supporting this hypothesis includes correlations between non-symbolic magnitude representation (particularly for proportional relationships) and symbolic fraction representation at both behavioral and neural levels. However, to establish non-symbolic magnitude representation as the cognitive basis for symbolic fraction representation, further research is needed to demonstrate their unique conceptual connection and causal relationship, and to elucidate the specific cognitive mechanisms underlying the formation of symbolic fraction representation.

Keywords: non-symbolic magnitude representation, ratio processing system, non-symbolic proportional representation, non-symbolic fraction representation, symbolic fraction representation

1. Introduction

"The thirsty child drank half a bottle of mineral water in one breath." "Last week' s oranges were forgotten and a quarter had gone bad by the time I remembered." Fractions are ubiquitous in daily life and occupy a crucial position in children' s elementary mathematics education. Fractions represent a key extension of children' s numerical understanding from integers to rational numbers and provide a foundation for learning more complex mathematical concepts such as algebra (see Siegler et al., 2013). Empirical research has shown that children' s fraction knowledge predicts their later performance in algebra and general mathematics (Bailey et al., 2012; Mou et al., 2016; Siegler et al., 2011; Siegler & Pyke, 2013). However, fraction learning presents significant challenges for many children. Although students begin learning fractions around third grade, a substantial proportion of American middle and high school students cannot distinguish fraction magnitudes, order them by value, or convert between fractions and decimals (Bailey et al., 2014; Braithwaite & Siegler, 2018; Braithwaite et al., 2018; Jordan, Resnick, et al., 2017). In China, many upper elementary students also struggle to correctly understand fraction concepts or accurately represent fraction magnitudes (Gao et al., 2018; Zhang et al., 2014; Gao et al., 2016; Liu & Xin, 2010; Sun et al., 2016; Zhang et al., 2012). These difficulties may arise because fractions differ from integers in several fundamental ways: fraction magnitude depends on the ratio between numerator and denominator rather than either component alone; unlike integers, fractions do not have a unique successor (the integer after 2 is 3, but the successor of 1/2 is indeterminate) (Jordan, Rodrigues, et al., 2017; Ni & Zhou, 2005; Siegler et al., 2013; Liu & Xin, 2010; Yang & Liu, 2008).

While learning symbolically represented fractions (e.g., Arabic numerals) poses challenges for children, the ability to represent quantities and proportional relationships emerges much earlier, during infancy—before language acquisition (i.e., non-symbolic magnitude representation; Denison et al., 2013; Denison & Xu, 2014; McCrink & Wynn, 2007). Moreover, such quantity and proportion representation abilities are observed in non-human animal species (Starr & Brannon, 2015; Vallentin & Nieder, 2008). What, then, is the relationship between non-symbolic magnitude representation, particularly for proportions, and children's symbolic fraction representation? Both involve representing "magnitude," and the former emerges far earlier than the latter—could the former serve as the cognitive foundation for the latter? This question is crucial as it addresses the cognitive origins of how humans learn abstract, complex symbolic mathematical knowledge, a central issue in numerical cognition development research (Jordan, Resnick, et al., 2017; Matthews et al., 2016; Siegler et al., 2013). Furthermore, clarifying this relationship can provide theoretical guidance for mathematics education and help develop more effective teaching methods. If non-symbolic magnitude representation indeed constitutes a cognitive foundation for symbolic fraction representation, educators could design targeted training on non-symbolic magnitude skills to facilitate children' s fraction learning (Gouet et al., 2020; Matthews et al., 2016; Siegler et al., 2013).

This paper aims to review existing research exploring the relationship between non-symbolic magnitude representation and symbolic fraction representation. This line of inquiry represents a necessary component within the broader theoretical framework investigating the relationship between non-symbolic magnitude abilities and symbolic mathematics learning. Within this framework, numerous studies have examined the relationship between non-symbolic and symbolic representations of integers, and their findings and debates provide a valuable reference for fraction research. By examining whether patterns observed in integer research also appear in fraction research, we can develop a more comprehensive understanding of the relationship between non-symbolic and symbolic magnitude representations. This paper first reviews integer research, then introduces non-symbolic proportional representation abilities present from infancy, and finally examines the relationship between non-symbolic magnitude representation and symbolic fraction representation.

† In research examining the relationship between two non-symbolic quantities, some studies refer to the ratio between two quantities (e.g., the ratio of red to blue dots; termed "ratio" in English literature; DeWolf et al., 2015), representing a part-part relationship, while others refer to fractions (e.g., the proportion of pure juice relative to the total mixture of juice and water; termed "fraction" in English literature; DeWolf et al., 2015), representing a part-whole relationship. These two types of representations may differ in behavioral performance and underlying cognitive mechanisms (DeWolf et al., 2015), representing an important area for further investigation. However, many existing studies do not strictly distinguish between them; some examine the relationship between non-symbolic ratios and symbolic fractions, while others investigate non-symbolic fractions and symbolic fractions. To avoid introducing overly complex terminology, this paper uses "non-symbolic proportional representation" to encompass both part-part and part-whole representations. Here, "proportion" refers broadly to the ratio relationship between quantities, which includes the part-whole relationship denoted by fractions.

2. Non-Symbolic Magnitude Representation of Integers and Its Relation to Symbolic Mathematics Learning

Humans possess language-independent abilities to represent and process quantities from infancy, which we term non-symbolic magnitude representation (Feigenson et al., 2004; Leibovich & Ansari, 2016; Spelke, 2017). Behavioral research has shown that infants can discriminate or match different quantities and estimate the outcomes of simple arithmetic operations (McCrink & Wynn, 2009; vanMarle et al., 2016). Neuroimaging studies have further demonstrated that when infants view dot arrays of different numerosities (e.g., 8 vs. 16 dots), the posterior parietal cortex spontaneously processes and compares these numerical differences (Hyde & Spelke, 2011; Hyde et al., 2010). Beyond infancy, children and adults can compare or estimate quantities without relying on language (Hyde et al., 2016; Geary & vanMarle, 2016; Libertus et al., 2016; Wang et al., 2020). Comparative psychology research has also revealed that many non-human animals possess quantity representation and processing abilities, suggesting an evolutionary origin for non-verbal numerical cognition (Howard et al., 2019; see review by Starr & Brannon, 2015).

Some researchers propose that non-symbolic magnitude representation is based on a language-independent system called the Approximate Number System (ANS; Feigenson et al., 2004; Gallistel & Gelman, 2000). The ANS represents quantities approximately—that is, it provides only a rough estimate of a set' s magnitude. For example, when viewing 10 dots, the ANS might represent the quantity as 10, 9, 11, or other nearby values. Moreover, ANS representations become less precise as quantity increases. When the difference between two quantities is smaller, their corresponding ANS representations overlap more, making them harder to discriminate. Experimental data show that the difficulty of quantity discrimination depends on the ratio between quantities (the ratio effect; Feigenson et al., 2004), which is considered a hallmark of ANS-based representation. Additionally, ANS acuity varies across individuals—some people can discriminate ratios as close as 8:9, while others require larger ratio differences. Notably, the ANS typically represents larger quantities (>4). When a set contains four or fewer items, people may use another cognitive system—the Object Tracking System (OTS; Feigenson et al., 2004). The OTS can simultaneously and accurately represent and track each individual object in a small set ($ $4) and compare two sets using one-to-one correspondence. Because both ANS and OTS may be employed for quantities $ $4, most studies use quantities >4 to ensure that participants are using the ANS. This paper focuses on ANS-based non-symbolic magnitude representation. The ANS can represent both single non-symbolic quantities (as discussed in this section) and ratios between non-symbolic quantities (discussed below). It is worth noting that some researchers hypothesize that non-symbolic proportional representation may not rely on the ANS but on a specialized system for processing ratio information (Ratio Processing System; Matthews & Chesney, 2015; Matthews et al., 2016). However, current evidence does not allow us to determine whether such a distinct system exists (see discussion below).

Research has demonstrated several connections between the ANS and symbolic number learning. Both children and adults show distance effects when comparing non-symbolic (e.g., dot arrays) and symbolic (e.g., Arabic numerals) quantities (Mundy & Gilmore, 2009; Sasanguie et al., 2012; Schneider et al., 2017). Individuals activate the same posterior parietal regions when comparing non-symbolic and symbolic quantities, suggesting a common neural basis (Piazza, 2010). Furthermore, individual differences in ANS acuity correlate with individual differences in symbolic mathematics performance (vanMarle et al., 2014; Chu et al., 2016; Mou et al., 2018; Starr et al., 2018; Wang et al., 2020; see meta-analyses by Chen & Li, 2014; Fazio et al., 2014; Schneider et al., 2017). Training based on ANS operations (e.g., estimating the sum of two dot arrays) has been shown to improve subsequent symbolic arithmetic performance in both children and adults (Hyde et al., 2014; Szkudlarek & Brannon, 2018), suggesting a possible causal relationship.

However, some studies have found no significant correlation between ANS acuity and symbolic number or general mathematics learning (Negen & Sarnecka, 2015; see review by De Smedt et al., 2013). Reported correlations in some studies may be attributable to general cognitive abilities shared by ANS and mathematics tasks, with the relationship disappearing after controlling for these factors (Fuhs & McNeil, 2013; Gilmore et al., 2013; but see Keller & Libertus, 2015). Thus, whether ANS-based non-symbolic magnitude representation has a unique conceptual connection to symbolic numbers, or whether the two are relatively independent, remains controversial (Barner, 2017; Chen & Li, 2014; Gilmore & Cragg, 2018; Sasanguie et al., 2014).

Due to space limitations, relevant theoretical debates and experimental evidence can be found in the cited references. In short, the relationship between non-symbolic magnitude representation and symbolic numbers remains contentious. What, then, is the relationship between non-symbolic and symbolic representations for the more complex concept of fractions? The following sections introduce research on non-symbolic and symbolic fraction representations and their relationship, with reference to existing findings and debates from integer research (e.g., whether there is a unique correlation or causal relationship between non-symbolic and symbolic magnitude representations).

3. Non-Symbolic Proportional Representation

Humans and some non-human animals can represent not only single non-symbolic quantities (integers) but also ratios between two non-symbolic quantities, and such representations appear to be ANS-based. For example, rhesus monkeys can select, from alternative pairs, a pair of line segments whose length ratio matches that of a target pair, even when the absolute lengths differ (Vallentin & Nieder, 2008). In another study, monkeys viewed pairs of ratio images presented simultaneously, each composed of white and black shapes in particular proportions. The results showed that even after controlling for the absolute number of shapes, monkeys could determine which image contained the larger proportion (Drucker et al., 2016). Moreover, these animals showed distance effects when judging proportions, suggesting ANS-based representation.

Human infants also represent ratios between quantities from early in development. For instance, after being habituated to a particular ratio (e.g., 1:4, represented by a certain number of blue dots and four times as many yellow dots), 6-month-old infants looked longer at images containing a novel ratio (e.g., 1:2) than at the familiar ratio (1:4; McCrink & Wynn, 2007). These infants could discriminate between 1:4 and 1:2 (a twofold difference) but not between 1:3 and 1:2 (a 1.5-fold difference). Infants' discrimination of proportions is also modulated by the numerical distance between them, suggesting ANS-based representation. Additional research has found that 8-month-old infants, when shown a box containing many red and white balls with far more red than white balls, expect a randomly drawn sample to contain more red than white balls (Xu & Garcia, 2008), indicating that infants can predict sample proportions based on population proportions.

Children also demonstrate the ability to represent non-symbolic proportions (Xin & Liu, 2011). When 6-7-year-olds see a geometric figure with 1/2 of its area colored yellow, they can select another figure with 1/2 colored yellow to match the proportion (Goswami, 1989). Some studies have used juice concentration tasks to examine children' s non-symbolic proportional reasoning (Boyer & Levine, 2015; Boyer et al., 2008). In these tasks, the length of a red bar represents the amount of pure juice, while a blue bar represents water. The larger the proportion of red to total length, the greater the juice concentration. After viewing a target concentration, children can select a matching alternative even when the total bar length differs (i.e., different total lengths but the same proportion; Boyer et al., 2008), or they can approximately locate the target concentration on a number line (Mohring et al., 2016). Furthermore, 9-year-old children recognize that the same concentration can be represented by different quantity pairs—for example, that 3 parts juice and 9 parts water yields the same concentration as 2 parts juice and 6 parts water (i.e., 3:(3+9) = 2:(2+6); Boyer & Levine, 2012).

Interestingly, children' s performance is better when the juice and water bars are stacked vertically than when placed side-by-side (Möhring et al., 2016). This may be because vertical stacking better reflects the part-whole relationship, whereas side-by-side arrangement emphasizes part-part relationships, making concentration estimation more difficult (Möhring et al., 2016). Moreover, when juice and water amounts are represented by countable discrete units rather than continuous bars, many children struggle to accurately represent concentration, possibly because they tend to count the discrete units of each component rather than representing their proportional relationship (Boyer et al., 2008; similar findings in Begolli et al., 2020; Xin & Han, 2014). These findings suggest that children' s non-symbolic proportional representation is susceptible to the format of quantity presentation, demonstrating instability.

4. Relationship Between Non-Symbolic Magnitude Representation and Symbolic Fraction Representation

Children can use the ANS to represent both single quantities (integers) and proportions between quantities, and some researchers argue that these non-symbolic representations are closely related to symbolic fraction representation (Jordan, Rodrigues, et al., 2017; Siegler et al., 2013). In this hypothesis, the core link between non-symbolic and symbolic representations is that both fundamentally represent "magnitude." Understanding symbolic fractions requires comprehending their numerical value (i.e., their magnitude) and how this value relates to other fractions. ANS-based non-symbolic representation provides this foundational "magnitude" information, which individuals then match to symbolic fractions during learning. It is important to note that this hypothesis emphasizes the relationship between ANS representation and symbolic fraction representation but does not strictly distinguish whether symbolic fraction representation relates more to non-symbolic representation of single quantities or of proportions. Some researchers have recently proposed that symbolic fraction representation may not be based on non-symbolic representation of single quantities but only on non-symbolic proportional representation (Matthews et al., 2016). To clarify this issue, we examine the relationships between symbolic fraction representation and both types of non-symbolic representation.

One study found that fifth graders' accuracy in non-symbolic representation of single quantities (comparing dot array magnitudes) correlated with their symbolic fraction performance (comparing symbolic fractions and estimating fractions on number lines; Fazio et al., 2014), suggesting a relationship between single-quantity non-symbolic representation and symbolic fraction representation. However, other research reported that after controlling for general cognitive and language abilities, third graders' accuracy in non-symbolic representation of single quantities did not predict their conceptual understanding of symbolic fractions (shading figures to represent fractions, fraction comparison, marking fractions on number lines), procedural fraction knowledge (addition/subtraction calculations and word problems), or general mathematics achievement (Jordan et al., 2013). Overall, research examining the relationship between ANS-based non-symbolic representation of single quantities and symbolic fraction representation remains limited and has yielded inconsistent findings.

Regarding the relationship between non-symbolic proportional representation and symbolic fraction representation, both elementary school children and adults show distance effects when comparing two non-symbolic proportions (represented as ratios of vertical line lengths), two symbolic fractions, or a non-symbolic proportion and a symbolic fraction. Moreover, comparisons between two non-symbolic proportions are faster than the other two types of comparisons (Kalra et al., 2020). These findings suggest that individuals can translate between different formats of proportional information for comparison. Because non-symbolic proportion comparisons are fastest, researchers propose that individuals convert all proportional formats to non-symbolic representations for comparison, suggesting that non-symbolic proportional representation may underlie symbolic fraction representation (Kalra et al., 2020).

Additionally, 9-year-olds' accuracy in non-symbolic proportion matching (juice concentration tasks) correlates significantly with their composite symbolic fraction test scores (e.g., comparing symbolic fraction magnitudes, fraction addition/subtraction; Möhring et al., 2016). Another study reported that fifth graders' accuracy in juice concentration matching correlated with their conceptual understanding of symbolic fractions (e.g., fraction magnitude) but not with symbolic fraction computation (Hansen et al., 2015). Beyond correlational evidence, training fourth graders on non-symbolic proportional representation accuracy using juice concentration tasks for approximately one week improved their symbolic fraction performance (Gouet et al., 2020), suggesting a possible causal link between non-symbolic proportional and symbolic fraction representations.

Neuroimaging findings also support the relationship between non-symbolic proportional and symbolic fraction representations. ERP studies show that when adults compare non-symbolic proportions with symbolic fractions, distance effects emerge in frontal and parietal components (P2, N3, P3), indicating similar neural processing across different fraction formats (Zhang et al., 2013). fMRI research reveals that when participants are adapted to one proportion and then view another, significant activation occurs in the anterior intraparietal sulcus and prefrontal cortex, with activation strength varying according to the distance between the two proportions (an ANS signature; Jacob & Nieder, 2009a, 2009b). Critically, these neural distance effects are observed whether proportional information is presented non-symbolically (e.g., two horizontal lines with lengths in a 1:5 ratio, or red and blue dot arrays in a 5:10 ratio) or symbolically (e.g., "3/6," "halb" —German for "half"), suggesting that non-symbolic proportional and symbolic fraction representations share common neural substrates. Mock et al. (2018) also found that when adults compare non-symbolic proportions or symbolic fractions presented in different formats (pie charts and dot arrays for non-symbolic; Arabic fractions and decimals for symbolic), the intraparietal sulcus and prefrontal cortex are activated under both symbolic and non-symbolic conditions. Similar findings have been observed in children. Park et al. (2018) had second and fifth graders compare two non-symbolic proportions (ratios of vertical line lengths), two symbolic fractions, or one symbolic fraction and one non-symbolic proportion. In all cases, parietal and prefrontal cortex activation showed distance effects. These results suggest that the brain may employ common neural mechanisms to represent and process non-symbolic proportions and symbolic fractions, demonstrating their neural-level correlation.

Based on behavioral and neuroimaging evidence, non-symbolic proportional representation appears more closely related to symbolic fraction representation than does non-symbolic representation of single quantities. Recent research has specifically investigated this issue. Matthews and Lewis et al. (2016) found that adults' accuracy in non-symbolic proportion matching (ratios of line lengths and dot arrays) predicted their symbolic fraction performance (fraction magnitude comparison, number line estimation, and paper-and-pencil tests of conceptual and procedural fraction knowledge) as well as their algebra knowledge. Notably, this study also measured individuals' accuracy in non-symbolic representation of single quantities (e.g., comparing dot array magnitudes or line lengths—the conventional ANS acuity measure). When non-symbolic proportional representation accuracy was not included, non-symbolic representation of single quantities correlated with symbolic fraction comparison, conceptual/procedural knowledge, and algebra knowledge. However, after including non-symbolic proportional representation accuracy, this variable significantly predicted symbolic fraction and algebra performance, while the predictive power of single-quantity non-symbolic representation became non-significant. This study highlights that non-symbolic proportional, rather than single-quantity, representation is closely related to symbolic fraction representation.

These researchers have further proposed that non-symbolic proportional representation is not based on the ANS but on a dedicated Ratio Processing System (RPS; Matthews & Chesney, 2015; Matthews et al., 2016). This hypothesis arises from the argument that while the ANS may provide a cognitive foundation for symbolic representation of countable integers, fractions differ from countable integers (e.g., it is difficult to count out the quantity 3/5), making the ANS an inadequate basis for fraction learning. Instead, the RPS is specialized for representing non-symbolic proportional information. In this hypothesis, the RPS—not the ANS—constitutes the cognitive foundation for symbolic fraction representation (Matthews & Chesney, 2015; Matthews et al., 2016).

Additional evidence supporting the RPS comes from another experiment (Park et al., 2020). Preschool children, elementary school children, and adults judged the magnitudes of proportions represented non-symbolically (using line lengths, dot arrays, or geometric areas) and also judged the magnitudes of the individual quantities themselves (e.g., which line is longer—measuring ANS acuity). Results showed that performance on one type of non-symbolic proportion task correlated with performance on other types of non-symbolic proportion tasks, and these correlations were generally stronger than correlations between proportion comparison performance and single-quantity comparison performance using the same stimulus type (e.g., performance on line-based proportion tasks was better predicted by area-based proportion comparison than by single-line magnitude comparison). The researchers interpret these findings as evidence that non-symbolic proportion representation is supported by a specialized cognitive system for processing ratio relationships (RPS) rather than the general ANS.

Recent research has also examined non-symbolic proportional representation and symbolic fraction representation in typical children and adults as well as in adults with dyscalculia (Bhatia et al., 2020). Results suggest that all groups may use the RPS to represent non-symbolic proportional information, and this system may support the mapping between non-symbolic proportional and symbolic fraction representations. These findings indicate that the RPS may be present across diverse populations.

5. Key Unresolved Issues

The preceding review reveals that humans demonstrate the ability to represent and process non-symbolic proportions from infancy, though this ability may show instability during childhood depending on how quantities are presented. Moreover, non-symbolic magnitude representation correlates with symbolic fraction representation at both behavioral and neural levels. However, these correlations remain controversial. To more accurately understand this relationship, further exploration is needed in the following areas:

(1) Unique conceptual connection. While some studies have found correlations between non-symbolic magnitude representation (of single quantities or proportions) and symbolic fraction representation, further research is needed to determine whether such correlations reflect a unique conceptual connection. Existing integer research suggests that correlations between non-symbolic and symbolic magnitude representations may not stem from unique numerical conceptual links but rather from general cognitive or language factors (Fuchs & McNeil, 2013; Gilmore et al., 2013). Compared to integers, proportions and fractions (whether non-symbolic or symbolic) contain two components (a/b; Fazio et al., 2014) and may therefore require greater involvement of general cognitive abilities. Individuals must simultaneously represent both components (numerator and denominator), compute their ratio, and inhibit the tendency to respond based on either component alone. These processes extend beyond mere magnitude representation to encompass multiple cognitive operations and computations. Indeed, neuroimaging research has found that brain regions coactivated during non-symbolic proportional and symbolic fraction representation include areas associated with attention, memory, and inhibitory control (Mock et al., 2019). Cui et al. (2020) also found that processing symbolic fractions activates the left middle temporal gyrus, a region involved in semantic processing, suggesting that the brain may extract semantic knowledge about mathematical rules (e.g., that fraction magnitude depends on the numerator-denominator ratio) when representing fractions. These findings highlight the importance of examining the role of multiple general cognitive factors when considering correlations between non-symbolic and symbolic magnitude representations. However, most fraction studies reviewed here have not rigorously examined these potential influences (but see Matthews et al., 2016). Consequently, even though we observe correlations between non-symbolic magnitude and symbolic fraction representations, the nature of these correlations remains poorly understood—a critical issue for future research.

(2) Causality. If non-symbolic magnitude representation and symbolic fraction representation are indeed correlated, does this necessarily indicate that the former serves as the cognitive foundation for the latter—that is, that a causal relationship exists? Most current studies have examined only concurrent correlations, making it difficult to infer causality. Moreover, longitudinal and neurocognitive tracking studies on integers have found, contrary to predictions from the hypothesis that ANS-based non-symbolic magnitude representation underlies symbolic number representation, that individuals' ANS acuity does not predict their later symbolic number performance; instead, symbolic number performance predicts later ANS acuity (Lyons et al., 2018; Matejko & Ansari, 2016; Suárez-Pellicioni & Booth, 2018; but see Elliott et al., 2018, for bidirectional prediction). One possible explanation is that symbolic number representations are inherently more precise, and long-term symbolic training can enhance the precision of non-symbolic magnitude representation (Lyons et al., 2018; Matejko & Ansari, 2016). Does a similar relationship exist for fractions? To address this question, longitudinal studies are needed to examine the temporal sequence of influences between non-symbolic magnitude representation and symbolic fraction representation, and training studies should investigate whether training on non-symbolic magnitude skills can produce changes in symbolic representation (Gout et al., 2020). Moreover, such examinations must be bidirectional, testing both whether non-symbolic magnitude representation predicts symbolic fraction representation and vice versa. These rigorous tests have not yet been conducted in fraction research and represent an important direction for future studies.

(3) Cognitive mechanisms. If non-symbolic magnitude representation and symbolic fraction representation are indeed correlated (or even causally related), how exactly does the former support learning of the latter? Some researchers propose that the link between non-symbolic magnitude and symbolic fraction representations depends on understanding that both represent the concept of "magnitude" (Jordan, Resnick, et al., 2017; Siegler et al., 2013). However, symbolic fraction representation expresses a precise, unique numerical value, whereas non-symbolic magnitude representation (whether of single quantities or proportions) is inherently approximate and imprecise (e.g., non-symbolic representation cannot precisely represent the magnitude of 3/4). How, then, is symbolic fraction representation built upon non-symbolic representation? This question also arises in integer representation research (how understanding of exact integer magnitudes is built upon approximate magnitude representation; Gallistel, 2007). Some researchers have begun addressing this question. For example, vanMarle et al. (2016) found that children's learning of number word concepts requires both ANS-provided magnitude representation and OTS-provided precise representation of individuals, with these two systems' contributions gradually integrating through language (Spelke, 2011). This aligns with the hypothesis that number concept acquisition draws on multiple cognitive sources including magnitude-specific information (e.g., ANS representation), language, and general cognition (e.g., inhibitory control; Dehaene et al., 2003; LeFevre et al., 2010). These findings and theoretical hypotheses provide reference points for understanding the relationship between non-symbolic and symbolic fraction representations: non-symbolic magnitude representation provides basic magnitude information that may help individuals represent fractions as integrated wholes rather than as separate components (Siegler et al., 2013). However, this magnitude information is not precise; language or other general cognitive abilities may be required to refine the non-symbolic-to-symbolic mapping process, thereby forming accurate symbolic fraction representations. Of course, how symbolic fraction representation is gradually established on the basis of non-symbolic magnitude representation, how non-symbolic magnitude representation, general cognitive abilities, and language specifically interact during symbolic fraction learning, and whether their roles vary across different stages of fraction learning, remain largely unexplored—important questions for future research.

(4) The Ratio Processing System hypothesis. Current research suggests that non-symbolic proportional representation is more closely related to symbolic fraction representation than is non-symbolic representation of single quantities. However, can we therefore conclude that non-symbolic proportional representation is supported by an RPS distinct from the ANS, and further, that the cognitive foundation for symbolic fractions is the RPS (Matthews et al., 2016)? As noted above, some evidence supports the RPS hypothesis—for example, that RPS (indexed by non-symbolic proportional representation accuracy) rather than ANS (indexed by single-quantity non-symbolic representation accuracy) significantly predicts symbolic fraction performance (Matthews et al., 2016). However, one alternative explanation is that measured non-symbolic proportional representation accuracy may reflect a combination of ANS acuity and rapid computation abilities. For instance, in non-symbolic proportion tasks, individuals may use the ANS to represent two quantities, rapidly compute their ratio (forming a non-symbolic proportion representation), and then compare the magnitudes of two proportions. This could explain why measured RPS predicts symbolic fraction representation better than ANS alone (i.e., non-symbolic proportion responses may involve both ANS acuity and computational processing). This hypothesis could also explain why accuracies across different types of non-symbolic proportion tasks correlate more strongly with each other than with single-quantity representation accuracies using the same stimulus type (Park et al., 2020; but see Matthews & Chesney, 2015, for further discussion of response times in proportion representation tasks). Because current research has not accounted for individuals' computational abilities regarding quantitative relationships, such alternative explanations cannot be ruled out.

Furthermore, if the RPS is a representation system independent of the ANS, does it have distinct neural substrates? Current neuroimaging research on the ANS has found that both humans and non-human animals activate the intraparietal sulcus (IPS) when representing and processing single quantities (integers) and proportions (Hyde & Mou, 2015; Mock et al., 2018; Nieder, 2016; Piazza, 2010). However, there is currently no clear evidence that non-symbolic proportional and single-quantity representations activate distinct brain regions. Therefore, more definitive behavioral and neural evidence is needed to determine whether a representation system specialized for processing proportional information exists independently of the ANS.

In summary, current research indicates that non-symbolic magnitude representation abilities exist from infancy and that such representations, particularly of non-symbolic proportional relationships, are related to symbolic fraction representation. However, the stability and causal nature of this relationship require further investigation, and more research is needed to reveal the cognitive mechanisms through which symbolic fraction representation develops from non-symbolic magnitude representation. Findings about the relationship between non-symbolic and symbolic fraction processing will advance our understanding of how basic magnitude representation abilities relate to formal symbolic mathematics learning, thereby enhancing our knowledge of the cognitive origins of human mathematics learning and informing the development of more effective teaching methods.

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Note: Figure translations are in progress. See original paper for figures.