Complex-valued Renyi Entropy

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Abstract

Complex-valued expression models have been widely used in the application of intelligent decision systems. However, there is a lack of entropy to measure the uncertain information of the complex-valued probability distribution. Therefore, how to reasonably measure the uncertain information of the complex-valued probability distribution is a gap to be filled. In this paper, inspired by the Renyi entropy, we propose the Complex-valued Renyi entropy, which can measure uncertain information of the complex-valued probability distribution under the framework of complex numbers, and is also the first time to measure uncertain information in the complex space. The Complex-valued Renyi entropy contains the features of the classical Renyi entropy, i.e., the Complex-valued Renyi Entropy corresponds to different information functions with different parameters q. Meanwhile, the Complex-valued Renyi entropy has some properties, such as non-negativity, monotonicity, etc. Some numerical examples can demonstrate the flexibilities and reasonableness of the Complex-valued Renyi entropy.

Full Text

Preamble

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Abstract

Complex-valued expression models have been widely employed in intelligent decision systems. However, existing entropy measures are inadequate for quantifying the uncertain information inherent in complex-valued probability distributions, representing a significant gap in the literature. Inspired by Renyi entropy, this paper proposes Complex-valued Renyi entropy (CRE), which measures uncertain information within a complex-number framework. To our knowledge, this is the first attempt to measure uncertain information in complex space. CRE inherits key features of classical Renyi entropy, transforming into different information functions through its parameter q. We establish several important properties of CRE, including non-negativity and monotonicity. Numerical examples demonstrate both the flexibility and reasonableness of the proposed measure.

Keywords: complex-valued probability distribution, Renyi entropy, complex-valued Renyi entropy, uncertain information

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1. Introduction

Modeling and measuring uncertain information forms the foundation of expert and artificial intelligence decision-making. Researchers have developed numerous theoretical frameworks for this purpose, including probability theory [?], Dempster-Shafer theory [?, ?], fuzzy set theory [?], rough set theory [?], and various extensions [?, ?, ?]. Correspondingly, effective measures for quantifying uncertain information have been proposed, such as Shannon entropy [?], belief entropy [?], fuzzy entropy [?], Renyi entropy [?], Tsallis entropy [?], and correlation entropies [?]. These theoretical advances in real-valued space have enabled practical applications in areas like group decision-making [?, ?, ?], pattern classification [?, ?], cluster analysis [?, ?], reliability analysis [?, ?], and others [?, ?, ?, ?].

In recent years, complex-valued functions have been increasingly incorporated into uncertain information modeling. Ramot et al. extended real-valued membership to complex-valued membership and proposed complex-valued fuzzy sets [?]. Alkouri and Salleh introduced complex-valued intuitionistic fuzzy sets and discussed distance measures between them [?], while related theories based on belief functions have also emerged [?, ?]. Additionally, information carriers containing complex-valued signals are becoming more common, including image signals [?], audio signals [?], and physiological signals [?]. However, current uncertain information measurement methods in real space cannot adequately quantify the uncertainty in these complex-valued theories, limiting their practical application across many fields. Consequently, developing reasonable uncertain information measures within a complex-valued framework represents an urgent research challenge.

Inspired by Renyi entropy [?], this paper proposes a novel entropy measure for uncertain information in complex-valued frameworks, termed Complex-valued Renyi entropy (CRE). CRE generalizes classical Renyi entropy to complex-valued spaces. Through its parameter α, CRE can transform into different information functions while maintaining desirable mathematical properties. Numerical examples demonstrate both the high compatibility and reasonableness of CRE.

The remainder of this paper is organized as follows. Section 2 reviews the fundamentals of Renyi entropy. Section 3 presents the Complex-valued Renyi entropy. Section 4 validates the compatibility and reasonableness of CRE through numerical examples. Section 5 concludes the work.

2. Preliminaries

This section introduces the foundational concepts of Renyi entropy. In 1970, Renyi proposed a q-order generalized entropy in phase space, known as Renyi entropy, which represents a common form of generalized entropy [?]. The formal definition is as follows:

Definition 2.1 (Renyi entropy). Suppose p_j is the probability of the system state falling into the j-th phase frame, N is the number of phase frames, and $\sum_{j=1}^{N} p_j = 1$. Then Renyi entropy is defined as:

$$R_q = \frac{1}{1-q} \log_2 \sum_{j=1}^{N} p_j^q$$

Renyi entropy exhibits several important properties for measuring the uncertainty of probability distributions:

Definition 2.2 (Properties of Renyi entropy).

P.1: **Non-negativity**, i.e., $R_q \geq 0$.

P.2: **Monotonically decreasing** with respect to q.

P.3: **Convexity**, i.e., R_q is a concave function if and only if $0 < q \leq 1$.

Renyi entropy can transform into other information functions as the parameter α changes:

- **Shannon entropy**: When $\alpha = 1$, Renyi entropy reduces to Shannon entropy: $R_1 = -\sum_{j=1}^{N} p_j \log_2 p_j$.
- **Hartley entropy**: When $\alpha \to 0$, $R_{\alpha\to 0} = \log_2 N$.
- **Collision entropy**: When $\alpha = 2$, $R_2 = -\log_2 \sum_{j=1}^{N} p_j^2 = -\log_2 P(X = Y)$, where X and Y are independently and identically distributed.

3. The Complex-valued Renyi Entropy

This section proposes Complex-valued Renyi entropy (CRE) for measuring uncertain information in complex-valued space. CRE can be converted to other information functions and is defined as follows:

Definition 3.1 (Complex-valued Renyi entropy). Suppose there is a complex-valued probability distribution $\widehat{P} = \{\widehat{p}_1, \widehat{p}_2, \cdots, \widehat{p}_j, \cdots, \widehat{p}_n\}$ where $\widehat{p}_j = p_j e^{i2\pi\theta_j}$ and satisfies $\sum_{j=1}^{n} |\widehat{p}_j|^2 = 1$, $\theta_j \in [0,1]$. The uncertainty of the complex-valued probability distribution is measured by:

$$\check{R}_q = \frac{1}{1-q} \log_2 \sum_{j=1}^{n} (\widehat{p}_j \widehat{p}_j^*)^q$$

where $\widehat{p}_j^*$ is the complex conjugate of $\widehat{p}_j$.

Next, we examine the information functions corresponding to CRE when α takes specific values.

- **When** $\alpha = 1$, CRE degrades to Shannon entropy in the complex-valued framework.

Consider the limit as $q \to 1$:

$$\check{R}_1 = \lim_{q \to 1} \frac{1}{1-q} \log_2 \sum_{j=1}^{n} (\widehat{p}_j \widehat{p}_j^*)^q$$

Applying L' Hôpital' s rule and simplifying:

$$\check{R}_1 = -\sum_{j=1}^{n} \widehat{p}_j \widehat{p}_j^* \log_2 p_j - \sum_{j=1}^{n} \widehat{p}_j \widehat{p}_j^* \log_2 p_j + i \sum_{j=1}^{n} \widehat{p}_j \widehat{p}_j^* \log_2 e^{i2\pi\theta_j} - i \sum_{j=1}^{n} \widehat{p}_j \widehat{p}_j^* \log_2 e^{i2\pi\theta_j}$$

This simplifies to:

$$\check{R}_1 = -\sum_{j=1}^{n} \widehat{p}_j \widehat{p}_j^* \log_2 p_j - \sum_{j=1}^{n} \widehat{p}_j \widehat{p}_j^* \log_2 p_j$$

Therefore, we can express $\check{R}_1$ as:

$$\check{R}_1 = -\sum_{j=1}^{n} \widehat{p}_j \widehat{p}_j^* \log_2 p_j - \sum_{j=1}^{n} \widehat{p}_j \widehat{p}_j^* \log_2 p_j$$

- **When** $\alpha \to 0$, CRE degrades to Hartley entropy in the complex-valued framework:

$$\breve{R}_{\alpha\to 0} = \frac{1}{1-0}\log_2 \sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^0 = \log_2 N$$

- **When** $\alpha = 2$, CRE degrades to Collision entropy in the complex-valued framework:

$$\breve{R}_2 = \frac{1}{1-2}\log_2 \sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^2 = -\log_2 \sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^2$$

Next, we discuss the properties of CRE.

P.1: Non-negativity, i.e., $\breve{R}_q \geq 0$.

Proof: For $q > 1$, we have $0 \leq \hat{p}_j \hat{p}_j^* \leq 1$. Then $(\hat{p}_j \hat{p}_j^*)^q \leq \hat{p}_j \hat{p}_j^*$, so $\sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q \leq \sum_{j=1}^{n} \hat{p}_j \hat{p}_j^* = 1$. Therefore, $\log_2 \sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q \leq 0$, and finally $\breve{R}_q \geq 0$.

For $q < 1$, we have $\sum_{j=1}^{n} \hat{p}_j \hat{p}_j^* = 1$. Then $(\hat{p}_j \hat{p}_j^*)^q \geq \hat{p}_j \hat{p}_j^*$, so $\sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q \geq 1$. Therefore, $\log_2 \sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q \geq 0$, and $\breve{R}_q \geq 0$.

When $q = 1$, CRE degenerates to $\breve{R}_1$, and $\breve{R}_1 \geq 0$. Thus, $\breve{R}_q \geq 0$ for all q.

P.2: Monotonically decreasing, i.e., $\breve{R}_q$ decreases as q increases.

Proof: Taking the derivative of $\breve{R}_q$ with respect to parameter q yields:

$$\frac{d\breve{R}_q}{dq} = \frac{-1}{(1-q)^2}\log_2 \sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q + \frac{1}{1-q} \cdot \frac{\sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q \ln(\hat{p}_j \hat{p}_j^*)}{\sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q \ln 2}$$

Simplifying:

$$\frac{d\breve{R}_q}{dq} = \frac{1}{(1-q)^2 \ln 2}\left[\frac{\sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q \ln(\hat{p}_j \hat{p}_j^*)}{\sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q} - \ln \sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q\right]$$

By Jensen' s inequality, since $\ln(x)$ is a concave function, we have:

$$\frac{\sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q \ln(\hat{p}_j \hat{p}_j^*)}{\sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q} \leq \ln\left(\frac{\sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^{q+1}}{\sum_{j=1}^{n} (\hat{p}_j \hat{p}_j^*)^q}\right)$$

Thus, $\frac{d\breve{R}_q}{dq} \leq 0$, which means $\breve{R}_q$ decreases as q increases.

P.3: Convexity, i.e., $\breve{R}_q$ is a concave function if and only if $0 < q \leq 1$.

Proof: For $0 < q < 1$, both $\ln(x)$ and x^q are concave functions. Therefore, $\breve{R}_q = \frac{1}{1-q}\log_2\sum_{j=1}^{n}(\hat{p}_j\hat{p}_j^*)^q$ is a concave function.

When $q = 1$, $\breve{R}_1 = -\sum_{j=1}^{n}\hat{p}_j\hat{p}_j^*\log_2 p_j - \sum_{j=1}^{n}\hat{p}_j\hat{p}_j^*\log_2 p_j$ is also a concave function since both $-\log_2 p_j$ and $\|\cdot\|^2$ are concave.

When $q \geq 1$, $\ln(x)$ is concave but x^q is convex, making $\breve{R}_q$ neither convex nor concave.

In summary, $\breve{R}_q$ is a concave function if and only if $0 < q \leq 1$.

4. Numerical Examples

This section demonstrates the flexibility and reasonableness of CRE for measuring uncertain information in complex-valued frameworks through several numerical examples.

Example 1. Consider a set of elementary events $E = \{e_1, e_2, e_3\}$ with the following complex-valued probability distribution:

$$\hat{P} = \{1e^{i2\pi 0}, 0, 0\}$$

The uncertainty measurements for various parameters q are shown in Table 1 . As evident from the table, the measurement of probability distribution $\hat{P}$ is 0 for any q, which is intuitive since the probability of event e_1 occurring is 1.

Example 2. Consider a complex-valued probability distribution $\hat{P}$ corresponding to the set of fundamental events $E = \{e_1, e_2, e_3, e_4\}$:

$$\hat{P} = \{\sqrt{0.2}e^{i2\pi 0.3}, 0.1e^{i2\pi 0.5}, 0.4e^{i2\pi 0.4}, 0.3e^{i2\pi 0.9}\}$$

The CRE measurement results for $\hat{P}$ are presented in Table 2 . The table shows that different values of q yield different measurement results. For instance, when $q = 0$, $\breve{R}_0 = 2$, and when $q = 0.5$, $\breve{R}_{0.5} = 1.9175$. Clearly, when $q \in [0, 1)$, $\breve{R}_q$ decreases monotonically. When $q = 1$, $\breve{R}_1 = 2.1391$, which characterizes the uncertainty of both amplitude and phase angle. Additionally, $\breve{R}_8$ exhibits greater uncertainty than $\breve{R}_2$ for $\hat{P}$.

Example 3. Consider a complex-valued probability distribution $\hat{P} = \{\sqrt{0.4}e^{i2\pi 0.4}, 0.3e^{i2\pi 0.2}, 0.1e^{i2\pi 0.8}\}$ corresponding to the set of elementary events E. Figure 1 [Figure 1: see original paper] shows the uncertainty measurement of CRE for this probability distribution as q varies from 0 to 100. The figure demonstrates that $\breve{R}_q$ remains larger than 0 across the entire range and confirms the monotonically decreasing property of $\breve{R}_q$.

Example 4. Consider a complex-valued probability distribution $\widehat{P}$ in the set of basic events E, where the distribution satisfies $\widehat{p}_j = \sqrt{\frac{1}{|E|}} e^{i2\pi \frac{1}{|E|}}$, with $|E|$ representing the number of events. The variation is shown in Table 3 . The uncertainty measurement for this complex-valued probability distribution based on CRE is illustrated in Figure 2 [Figure 2: see original paper]. The figure shows that the uncertainty of this complex-valued distribution increases continuously as the number of basic events grows. When $|E| = 32$, $\check{R}_q$ reaches its maximum value for any q. This example demonstrates that CRE can effectively measure the uncertainty of complex-valued probability distributions under varying numbers of basic events.

Example 5. Consider a complex-valued probability distribution $\widehat{P} = \{\alpha e^{i2\pi 0.5}, \beta e^{i2\pi 0.2}, \sqrt{1-\alpha^2-\beta^2} e^{i2\pi 0.1}\}$ in $E = \{e_1, e_2, e_3\}$, where the relationship between α and β is depicted in Figure 3 Figure 3: see original paper. The figure shows that α and β satisfy $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and $0 \leq \alpha^2 + \beta^2 \leq 1$. Figure 3(b) shows the measurement of $\widehat{P}$ based on CRE when $q = 0$, revealing that $\check{R}_{q=0}$ does not vary with α and β. This is reasonable because, according to the definition, the factor affecting $\check{R}_{q=0}$ is the number of basic events. Figure 3(c) presents the CRE measurement when $q = 0.5$, showing that $\check{R}_{q=0.5} \leq \check{R}_{q=0}$ regardless of variations in α and β. When $q \to 1$, $\check{R}_{q\to 1}$ measures the uncertainty between phase angle and amplitude in $\widehat{P}$, as shown in Figure 3(d). The asymmetry in Figure 3(d) is intuitive because the unequal phase angles of the three fundamental events lead to asymmetric measurements. Figures 3(e) and 3(f) display the uncertainty measurements for $q = 2$ and $q = 8$, respectively, showing that $\check{R}_{q=2} \geq \check{R}_{q=8}$ in the same coordinate system. This example demonstrates that CRE yields reasonable results when the support of a proposition changes.

5. Conclusion

This paper addresses the limitation that current uncertainty measurement methods cannot effectively quantify the uncertainty of complex-valued probability distributions. Inspired by Renyi entropy, we propose Complex-valued Renyi entropy, which provides an effective measure of uncertainty for probability distributions within a complex-valued framework. We analyze the effect of parameter q variations and establish key properties of CRE. Numerical examples further verify that CRE can effectively and reasonably measure the uncertainty of complex-valued probability distributions under changes in propositional support and the number of underlying events.

Future research will pursue two main directions. First, we will investigate the physical meaning of parameter q in CRE under the complex-valued probability distribution framework. Second, we will explore related concepts such as divergence measures to further elucidate the physical meaning of CRE and facilitate its application in engineering practice.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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