Postprint: Study of Low-Frequency Quasi-Periodic Oscillation Phenomena in the Power Spectral Density of Stochastic Luminosity Oscillations of Accretion Disks

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Abstract

A generalized Langevin equation model incorporating frequency fluctuation noise and exponentially correlated random force is employed to describe the vertical oscillations of black hole accretion disks, derive the analytical expression for the luminosity power spectral density of the accretion disk' s stochastic oscillations, and discuss the influence of system parameters on the low-frequency quasi-periodic oscillations (LFQPOs) phenomenon in the power density spectrum. The research results reveal that when appropriate system parameters are selected, resonant double-peak LFQPOs consisting of a fundamental frequency and a second harmonic frequency appear on the power density spectrum curve; the central frequency of the fundamental peak corresponds to the characteristic frequency of the accretion disk oscillations; the random force correlation time determines the height and width of the fundamental peak; and the frequency noise intensity and viscous damping affect only the second harmonic peak. These results demonstrate that the stochastic oscillation model of accretion disks can serve as an explanation for the origin of LFQPOs.

Full Text

Low-frequency Quasi-periodic Oscillations in the Power Spectral Density of Stochastic Oscillating Luminosity of Accretion Disks

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Abstract

We employ a generalized Langevin equation model incorporating frequency fluctuation noise and exponentially correlated stochastic forces to describe the vertical oscillations of black hole accretion disks. We derive an analytical expression for the power spectral density of the stochastically oscillating luminosity and investigate how system parameters affect low-frequency quasi-periodic oscillations (LFQPOs) in the power density spectrum. Our findings reveal that, for appropriate parameter choices, resonant double-peaked LFQPOs consisting of a fundamental peak and a second harmonic peak appear in the power spectral density curves. The central frequency of the fundamental peak corresponds to the characteristic oscillation frequency of the accretion disk, while the correlation time of the stochastic force determines the height and width of this peak. In contrast, the frequency noise intensity and viscous damping affect only the second harmonic peak. These results demonstrate that the stochastic oscillation model of accretion disks can serve as a viable explanation for the origin of LFQPOs.

Keywords: low-frequency quasi-periodic oscillations; accretion disk; frequency fluctuation noise; stochastic resonance
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Low-frequency quasi-periodic oscillations (LFQPOs) represent a ubiquitous phenomenon in black hole X-ray transients, manifesting as broad peaks in X-ray variability power spectral density curves with central frequencies ranging from several mHz to 30 Hz [1-3]. Based on factors such as peak quality factor, noise properties, and phase lags, LFQPOs are typically classified into three types: A, B, and C. Type C LFQPOs constitute the most common variety observed in black hole transients, appearing frequently in the low-hard state and hard-intermediate state, with frequency ranges from several mHz to 10 Hz, though they can also be observed in the high-soft state reaching frequencies up to 30 Hz. Type C LFQPOs are accompanied by strong flat-top noise, exhibit an approximately truncated power-law spectral profile, and typically display a complex structure composed of a main peak and several additional peaks. The central frequency of the main peak relates to the fundamental oscillation frequency, while additional peaks appear at subharmonic (half-frequency) and harmonic (double-frequency) positions relative to the fundamental [2-5]. In rare cases, the harmonic (or subharmonic) peak amplitude becomes comparable to the fundamental peak, forming a special "cathedral-like" double-peaked LFQPO structure. Such double-peaked LFQPOs have been observed in the power spectra of the black hole transients XTE J1550-564 and XTE J1859+226, where two strong, narrow peaks appear at related frequencies [6,7].

Despite significant observational progress, the physical origin of LFQPOs remains controversial. One plausible mechanism involves oscillations of the accretion disk. In accretion disks, fluid elements undergo inertial oscillations in the horizontal direction when subjected to restoring forces, while executing simple harmonic oscillations vertically near the equatorial plane, with coupling between oscillations in different directions. These oscillations can be categorized based on the nature of the restoring force into g-modes (gravity modes), p-modes (pressure modes), and c-modes (corrugation modes). Research indicates that these oscillations propagate within confined regions of the accretion disk, forming so-called "trapped oscillations" [8,9]. The frequencies of trapped oscillations are inversely proportional to black hole mass and may account for the quasi-periodic oscillations observed in black hole transients. Specifically, g-mode and p-mode oscillation frequencies fall within the high-frequency QPO range, whereas c-modes represent low-frequency oscillations confined between the inner radius and the inner vertical resonance radius, with frequencies comparable to those of LFQPOs [2]. Additionally, Titarchuk and Osherovich proposed a global vertical oscillation model for accretion disks, suggesting that vertical oscillations of the entire disk could produce persistent LFQPOs around 0.1 Hz [9].

However, the aforementioned oscillation models cannot fit LFQPOs directly from observed spectra. Furthermore, the properties of Type C LFQPOs and the shape and intensity of broadband noise suggest that quasi-periodic oscillation behavior should be associated with stochastic systems containing additional noise. Consequently, Harko and Leung et al. proposed using generalized Langevin equations to describe stochastic oscillations of accretion disks, obtaining through numerical calculations the power spectral density of oscillating luminosity and explaining some observational results [11,12]. Long et al. (2016) utilized this model to compute light curves and power spectra under three different conditions, obtaining spectral indices through power-law fitting of the power spectra, and found that variations in the observed spectral index arise from transitions between different oscillation regimes of the disk [13].

For such stochastic systems, stochastic resonance phenomena can emerge under certain conditions. Stochastic resonance represents a nonlinear phenomenon in multiplicative stochastic dynamical systems and has been widely applied to astrophysical research [14]. Our team previously investigated stochastic resonance in the power spectral density curves of stochastically oscillating accretion disk luminosity [15-17], interpreting the resonance peaks as a physical explanation for QPO phenomena in black hole X-ray binaries. In these previous works, stochastic terms existed only in the viscous coefficient and external forces, while the accretion disk oscillation frequency remained fixed, and no harmonic peaks appeared in the power spectral density curves. Building upon this foundation, the present study proposes a generalized Langevin equation model incorporating both frequency fluctuations and stochastic forces to describe vertical oscillations of black hole accretion disks. We calculate the power spectral density of stochastically oscillating luminosity and discuss the resulting double-peaked stochastic resonance phenomenon, providing a theoretical basis for interpreting Type C LFQPOs.

1.1 Accretion Disk Oscillation Model Equation

Consider a standard thin accretion disk surrounding a compact object. When subjected to external stochastic forces and internal viscous forces, the vertical oscillations of a unit mass particle in the disk can be described by a generalized Langevin equation [12]:

$$\ddot{z} + \xi\dot{z} + \omega_\perp^2 z = f(t)$$

where z and $\omega_\perp$ represent the displacement and frequency of the particle' s vertical oscillation, respectively, and ξ is the viscous dissipation coefficient within the accretion disk, taken as a constant. The external force $f(t)$ represents a turbulently acting stochastic force with exponential correlation, satisfying the following statistical properties:

$$\langle f(t)\rangle = 0, \quad \langle f(t)f(t')\rangle = \frac{D}{\lambda}e^{-\lambda|t-t'|}$$

where $\langle ...\rangle$ denotes the ensemble average. D is a constant representing the strength of the stochastic force, and λ denotes the inverse of the stochastic force correlation time, i.e., the temporal correlation rate. Its magnitude relative to the disk oscillation frequency affects the coupling between disk oscillations and turbulent effects [18]. Here, we assume λ is smaller than the disk oscillation frequency.

We assume that the vertical oscillation frequency of the accretion disk fluctuates due to external influences, which can be expressed as [19]:

$$\omega_\perp = \omega_0 + \eta(t)$$

The frequency fluctuation $\eta(t)$ is a Gaussian white noise random variable with statistical properties:

$$\langle \eta(t)\rangle = 0, \quad \langle \eta(t)\eta(t')\rangle = \varepsilon\delta(t-t')$$

where ε represents the noise intensity parameter, which we set within the range $0 < \varepsilon \leq 1$, as larger ε values would broaden the peaks in the power spectral density curve and reduce the quality factor Q of the peaks [18]. ω_0 is the characteristic angular frequency of the vertical oscillation of the accretion disk, expressed as [10]:

$$\omega_0 = \sqrt{\frac{GM}{r_{adj}^3}} \left(\frac{r_{adj}}{r_{out}}\right)^{\gamma-1} \left(1 - \sqrt{\frac{r_{in}}{r_{adj}}}\right)$$

where $m = M/M_{\odot}$, with M and $M_{\odot}$ being the masses of the central object and the Sun, respectively; r_{in} is the inner edge radius, r_s is the Schwarzschild radius; r_{out} and r_{adj} are the outer edge radius and adjustment radius of the disk; and the surface density index γ can take values of 3/5 or 3/4 [10].

For computational simplicity, assuming $\beta = \xi/2$ as the viscous damping coefficient within the accretion disk, equation (1) can be written as:

$$\ddot{z} + 2\beta\dot{z} + [\omega_0 + \eta(t)]^2 z = f(t)$$

Assuming the two random variables are uncorrelated, satisfying $\langle \eta(t) f(t') \rangle = 0$. By applying a Laplace transform to equation (6), we can obtain the displacement and velocity of the particle at any time:

$$z(t) = H(t)z_0 + \int_0^t H(t-t')f(t')dt'$$

$$v(t) = \dot{H}(t)z_0 + \int_0^t \dot{H}(t-t')f(t')dt'$$

where $\dot{H}$ denotes the time derivative of H. Assuming initial displacement and velocity of $z(0) = z_0$ and $v(0) = v_0$, the average values of displacement and velocity are:

$$\langle z(t) \rangle = H(t)z_0, \quad \langle v(t) \rangle = \dot{H}(t)z_0$$

In the case where $\omega_0 > \beta$, the response functions $H(t)$ and $\dot{H}(t)$ of the equation are:

$$H(t) = \frac{1}{\omega_1}e^{-\beta t}\sin(\omega_1 t), \quad \dot{H}(t) = e^{-\beta t}\cos(\omega_1 t) - \frac{\beta}{\omega_1}e^{-\beta t}\sin(\omega_1 t)$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

1.2 Correlation Functions of Oscillation Displacement and Velocity

Following the method for calculating displacement and velocity correlation functions in generalized Langevin equations presented by Mankin et al. [20], we define the displacement variance $\sigma_z^2(t)$ and the time correlation functions for displacement and velocity, $\kappa_{zz}(t,\tau)$ and $\kappa_{zv}(t,\tau)$, as:

$$\sigma_z^2(t) = \langle [z(t) - \langle z(t)\rangle]^2 \rangle$$
$$\kappa_{zz}(t,\tau) = \langle [z(t) - \langle z(t)\rangle][z(t+\tau) - \langle z(t+\tau)\rangle] \rangle$$
$$\kappa_{zv}(t,\tau) = \langle [z(t) - \langle z(t)\rangle][v(t+\tau) - \langle v(t+\tau)\rangle] \rangle$$

Based on the statistical independence of the random variables $\eta(t)$ and $f(t)$, they satisfy the following statistical relations:

$$\langle \eta(t) z(t)\rangle = 0, \quad \langle f(t) z(t)\rangle = 0$$

The correlation functions of displacement and velocity can then be expressed as:

$$\kappa_{zz}(t,\tau) = \int_0^t dt_1 \int_0^{t+\tau} dt_2 H(t-t_1)H(t+\tau-t_2)\langle f(t_1)f(t_2)\rangle + \int_0^t dt_1 \int_0^t dt_2 H(t-t_1)H(t-t_2)\langle \eta(t_1)\eta(t_2)\rangle$$

When $t \to \infty$, the above formulas become:

$$\kappa_{zz}(\tau) = \frac{D}{\lambda}\int_0^\infty dt_1 \int_0^\infty dt_2 H(t_1)H(t_2)e^{-\lambda|\tau+t_1-t_2|} + \varepsilon\omega_0^2 \int_0^\infty dt_1 \int_0^\infty dt_2 H(t_1)H(t_2)\delta(\tau+t_1-t_2)$$

where $\sigma_{z\infty}^2$ is the time-homogeneous part of the displacement variance in the steady state. As $t \to \infty$, we obtain:

$$\sigma_{z\infty}^2 = \frac{D}{2\beta\lambda(\omega_0^2 + \beta\lambda + \lambda^2)} + \frac{\varepsilon\omega_0^2}{4\beta\omega_1^2}$$

1.3 Power Spectral Density of Stochastic Oscillating Luminosity

The total energy of a unit mass particle oscillating in the accretion disk is:

$$E(t) = \frac{1}{2}v^2(t) + \frac{1}{2}\omega_0^2 z^2(t)$$

Due to the effects of external stochastic forces, internal viscous dissipation, and frequency fluctuation noise, the particle' s total energy is partially lost. This lost energy constitutes the oscillation luminosity $L(t)$, which from equations (6) and (20) yields:

$$L(t) = 2\beta v^2(t) + \omega_0 z^2(t)\eta(t)$$

In the limit $t \to \infty$, the steady-state mean autocorrelation function of the luminosity is defined as:

$$C_L(\tau) = \lim_{t\to\infty} \langle [L(t) - \langle L\rangle][L(t+\tau) - \langle L\rangle]\rangle$$

Substituting equations (17), (18), and (21) into equation (22), we obtain:

$$C_L(\tau) = 4\beta^2\kappa_{vv}(\tau) + \varepsilon\omega_0^2\sigma_{z\infty}^2\delta(\tau)$$

Through Fourier transformation, equation (23) converts to the power spectral density of the oscillating luminosity, $PSD(\nu)$. After complex calculations and neglecting constant terms independent of frequency ν, we obtain the expression for $PSD(\nu)$:

$$PSD(\nu) = \frac{a_0 + a_1\nu^2 + a_2\nu^4}{b_0 + b_1\nu^2 + b_2\nu^4 + b_3\nu^6 + b_4\nu^8}$$

where the coefficients a_i and b_i are functions of the system parameters ω_0, β, λ, and ε.

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2 Influence of Model Parameters on LFQPOs in Power Spectral Density Curves

Based on the analytical expression (25), we can calculate the power density spectra of disk oscillating luminosity under different parameter combinations, revealing both single-peaked and double-peaked stochastic resonance phenomena in the curves. These stochastic resonance peaks correspond to LFQPOs. Below, we primarily discuss how system parameters (ε, λ, β, ν_0) influence LFQPOs.

2.1 Variation of LFQPOs with Characteristic Oscillation Frequency ν_0

In our model, calculations reveal that the stochastic force strength D does not alter the spectral profile, so we set $D = 1$ in the following analysis. The characteristic frequency of vertical disk oscillations, $\nu_0 = \omega_0/2\pi$, approximates the fundamental frequency of Type C LFQPOs. From equation (5), ω_0 depends on the accretion disk structure and the central object' s mass. We select different disk parameters to ensure oscillation frequencies fall within the LFQPO range. With other parameters held constant, Figure 1 presents power spectral density curves for various characteristic frequencies, all exhibiting low-frequency truncated power-law spectral profiles consistent with LFQPOs.

[Figure 1: see original paper]

Figure 1. The power spectral density curves of stochastically oscillating luminosity for different characteristic frequencies, with parameters $\varepsilon = 0.38$, $\lambda = 0.01$, $\beta = 0.1$.

As evident from Figure 1, when ν_0 takes different values, the power at low frequencies below the cutoff changes minimally, and the high-frequency spectral slope remains essentially constant. However, the stochastic resonance peaks (LFQPOs) in the curves are highly sensitive to ν_0. Only when ν_0 reaches certain values do both a fundamental peak and an additional second harmonic peak appear. The central frequencies of the double peaks correspond to ν_0 and $2\nu_0$, respectively. As ν_0 decreases, both peaks become lower and broader, with the fundamental peak changing more rapidly. When ν_0 decreases to 0.07 Hz, the fundamental peak nearly disappears, leaving only a relatively broad second harmonic peak in the curve. As ν_0 decreases further, the harmonic peak also weakens progressively until it vanishes completely. At very small ν_0, no LFQPOs exist. This demonstrates that higher accretion disk oscillation frequencies produce more pronounced quasi-periodic oscillation phenomena through stochastic resonance, while at very low ν_0, the power spectral density decreases monotonically with increasing frequency.

3.2 Variation of LFQPOs with Parameters ε, λ, and β

Equation (21) reveals that the oscillating luminosity of the accretion disk primarily arises from the combined effects of viscous dissipation, stochastic forces, and frequency fluctuation noise. In our model, these three effects are controlled by three parameters: the viscous damping coefficient β, the temporal correlation rate λ, and the noise intensity ε. To investigate how each parameter influences LFQPOs, we adopt accretion disk geometry consistent with reference [10]: $r_{in} = 3r_s$, $r_{out} = 10^{4r}_s$, $r_{adj} = 10^{3/5}r_s$ (with $\gamma = 3/5$). For the central object mass, we select the mass of the black hole candidate XTE J1550-564, which exhibits Type C LFQPOs in the 0.1-10 Hz range [7]. Based on the mass estimate of 8.4-10.8 $M_\odot$ from Remillard et al. (2006) [1], we take $m = 9.6$. Substituting these parameters into equation (5) yields a characteristic oscillation frequency of $\nu_0 = 0.289$ Hz. With ν_0 fixed and ε, λ, β taking different values, the power spectral density curves of oscillating luminosity are shown in Figure 2 [Figure 2: see original paper]. All curves exhibit the expected single-peaked or double-peaked LFQPOs, with each parameter affecting the spectral lines differently.

[Figure 2: see original paper]

Figure 2. The power spectral density curves of stochastically oscillating luminosity for different values of λ, ε, and β: (a) $\lambda = 0.01, 0.10, 0.50, 1.00$ with $\varepsilon = 0.38$, $\beta = 0.1$; (b) $\varepsilon = 0.10, 0.25, 0.32, 0.38$ with $\lambda = 0.01$, $\beta = 0.1$; (c) $\beta = 0.10, 0.12, 0.15, 0.30$ with $\varepsilon = 0.38$, $\lambda = 0.01$. Vertical lines mark the central frequencies of LFQPOs.

Figure 2 clearly demonstrates that when the accretion disk oscillation frequency is fixed, the central frequency of the LFQPOs fundamental peak remains unchanged with other parameters. These parameters affect only the height and width of the LFQPO peaks. For instance, the stochastic force correlation rate λ significantly influences the fundamental peak while having almost no effect on the second harmonic peak (Figure 2a). As λ increases, the curve shifts upward, indicating that shorter correlation times of the stochastic force produce larger spectral density, but the peak value corresponding to the fundamental frequency decreases, making the peak broader and flatter until it disappears. At this point, LFQPOs consist only of a single peak with central frequency $2\nu_0$. Conversely, the frequency noise intensity ε and viscous damping coefficient β primarily affect the second harmonic peak while having minimal impact on the fundamental peak (Figures 2b and 2c). When ε increases or β decreases, the power density spectrum curve shifts upward, and the second harmonic peak becomes sharper. When ε takes small values or β takes large values, the second harmonic peak disappears. These results indicate that shorter stochastic force correlation times, larger frequency noise intensity, and smaller damping facilitate resonance between disk oscillations and turbulent effects, internal noise, and viscous effects, making LFQPO phenomena more pronounced.

This study improves the Langevin equation model for vertical stochastic oscillations of accretion disks by incorporating frequency fluctuation noise and exponentially correlated stochastic forces. We find that generalized single-peaked and double-peaked stochastic resonance appears in the power spectral density curves of oscillating luminosity, providing an explanation for LFQPOs containing both a fundamental peak and a second harmonic peak. The LFQPO peaks result from resonance produced by the combined effects of viscous dissipation, frequency noise, and external turbulent stochastic forces. The central frequency of the fundamental peak corresponds to the characteristic oscillation frequency ν_0 of the accretion disk, the correlation time of the stochastic force determines the height and width of the fundamental peak, and the frequency noise intensity and viscous damping affect the second harmonic peak.

Our results confirm that the stochastic oscillation model of accretion disks is approximately effective for describing X-ray variability and can satisfactorily explain LFQPO phenomena. In this model, the central frequencies of LFQPOs are determined solely by the oscillation frequency of the accretion disk. One can imagine that by adjusting the geometric structure of the accretion disk to instantaneously achieve sufficiently large oscillation frequencies, high-frequency quasi-periodic oscillations (HFQPOs) could be produced. Under appropriate dynamical conditions, the observed 3:2 HFQPO pairs could be explained through resonance. This will require establishing a more sophisticated stochastic oscillation model for accretion disks in future work to fit the observed HFQPOs.

References

[1] REMILLARD R A, MCCLINTOCK J E. X-ray properties of black-hole binaries [J]. Annual Review of Astronomy & Astrophysics, 2006, 44(1): 49-92.
[2] INGRAM A, MOTTA S. A review of quasi-periodic oscillations from black hole X-ray binaries: observation and theory [J]. New Astronomy Reviews, 2019, 85:101524.
[3] 张亮, 陈黎, 李兆辉, 等. 黑洞暂现源中的准周期振荡现象 [J], 天文学进展，2014，32(3):313-337.
[4] CASELLA P, BELLONI T, STELLA L. The ABC of low-frequency quasi-periodic oscillations in black hole candidates: Analogies with Z sources [J]. The Astrophysical Journal, 2005, 629(1): 403-407.
[5] ZHANG L, MÉNDEZ M, ALTAMIRANO D, et al. A systematic analysis of the phase lags associated with the type-C quasi-periodic oscillation in GRS 1915+105 [J]. Monthly Notices of the Royal Astronomical Society, 2020, 494(1): 1375-1386.
[6] CASLLA P, BELLONI T, HOMAN J, et al. A study of the low-frequency quasi-periodic oscillations in the X-ray light curves of the black hole candidate XTE J1859+226 [J]. Astronomy and Astrophysics, 2004, 426: 587-600.
[7] HOMAN J, WIJNANDS R, VAN DER KLIS M, et al. Correlated X-Ray spectral and timing behavior of the black hole candidate XTE J1550-564: A new interpretation of black hole states [J]. The Astrophysical Journal Supplement Series, 2001, 132(2): 377-402.
[8] KATO S. Oscillations of disks [M]. Astrophysics and Space Science Library, Vol. 437, Springer Japan, 2016, doi:10.1007/978-4-431-56208-5.
[9] ABRAMOWICZ M A, FRAGILE P C. Foundations of black hole accretion disk theory (J), Living Reviews in Relativity, 2013, 16(1) :88.
[10] TITARCHUK L，OSHEROVICH V. The Global normal disk oscillations and the persistent low-frequency quasi-periodic oscillations in X-ray binaries[J]. The Astrophysical Journal, 2000,542(2): L111-L114.
[11] HARKO T，MOCANU G. Stochastic oscillations of general relativistic discs [J]. Monthly Notices of the Royal Astronomical Society, 2012, 421(4): 3102-3110.
[12] LEUNG C S，WEI J Y，HARKO T，et al. Modelling the IDV emissions of the BL Lac objects with a Langevin type stochastic differential equation[J]. Journal of Astrophysics and Astronomy，2011，32(1-2): 189-192.
[13] LONG G B, OU J W, ZHENG Y G. Power spectrum density of stochastic oscillating accretion disk [J].Journal of Astrophysics and Astronomy, 2016, 37 (2) : 10.

[14] 徐江，郭峰，陈勇．双稳系统中的随机共振乘性信噪作用研究 [J]，天文研究与技术，2020，17(3)：292-298.
[15] WANG Z Y, CHEN P J, WANG D X, et al. Stochastic resonance of accretion disk and the persistent low-frequency quasi-periodic oscillations in black hole X-ray binaries[J]. Journal of Astrophysics and Astronomy, 2013，34(1) :33-40.
[16] WANG Z Y，CHEN P J，ZHANG L Y. Stochastic resonance of a general relativistic accretion disk [J]. Chinese Physics Letters, 2013,30(9):099801-1-4.
[17] WANG Z Y，CHEN P J，Stochastic oscillations of general relativistic disks described by a fractional Langevin equation with fractional Gaussian noise [J]. Journal of Astrophysics and Astronomy,2016, 37(2)：8.
[18] ORTEGA-RODRÍGUEZ M, SOLÍS-SÁNCHEZ H, ÁIVAREZ-GARCÍA L. On twin peak quasi-periodic oscillations resulting from the interaction between discoseismic modes and turbulence in accretion discs around black holes [J]. Monthly Notices of the Royal Astronomical Society, 2020, 492(2):1755-1760.
[19] GITTERMAN M. Classical harmonic oscillator with multiplicative noise [J]. Physica A: Statistical Mechanics and its Applications, 2005, 352(2)：309-334.
[20] MANKIN R, LAAS K, SAUGA A. Generalized Langevin equation with multiplicative noise: Temporal behavior of the autocorrelation functions [J]. Physical Review E, 2011, 83(6)：061131.

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