New ordinal relative fuzzy entropy

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Abstract

In real-world scenarios, events typically occur in a sequential order. Therefore, sequence must be regarded as a crucial factor when managing various entities in fuzzy environments. However, limited related research has been conducted to provide reasonable solutions to this requirement. Therefore, measuring the degree of uncertainty of ordinal fuzzy sets remains an open issue. To address this issue, this paper proposes a novel ordinal relative fuzzy entropy that takes the order of propositions into consideration when measuring uncertainty levels in fuzzy environments. Compared with previously proposed entropies, the proposed method embodies the effects of sequential proposition ordering on the degree of fuzzy uncertainty within its measured values. Moreover, numerical examples are provided to verify the correctness and validity of the proposed entropy.

Full Text

Preamble

New Ordinal Relative Fuzzy Entropy

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Abstract

In real-world scenarios, events and phenomena typically unfold in a sequential order. Consequently, sequence should be regarded as a crucial factor when managing various entities within a fuzzy environment. However, few studies have addressed this requirement with a reasonable solution, leaving the measurement of uncertainty in ordinal fuzzy sets an open research question. To tackle this issue, this paper proposes a novel ordinal relative fuzzy entropy that incorporates the order of propositions when measuring uncertainty levels in fuzzy environments. Unlike previously proposed entropy measures, the proposed method explicitly captures the effects of sequential proposition ordering on fuzzy uncertainty within its measurement values. Numerical examples are provided to verify the correctness and validity of the proposed entropy.

Keywords: Order; Fuzzy environment; Degree of uncertainty; Entropy

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1. Introduction

Measuring the degree of uncertainty in fuzzy environments has attracted considerable attention from researchers worldwide. Numerous meaningful theories have been developed to extract useful information from uncertain data, including Dempster-Shafer evidence theory [?, ?], complex mass functions [?], D-numbers [?], Z-numbers [?], soft theory [?], fuzzy theory [?], and others [?]. Due to their effectiveness in handling uncertainty, these theories have been successfully applied across various domains such as pattern recognition [?], decision making [?], and other fields [?], enabling the extraction of critical information from uncertain situations.

Before processing uncertain information, however, it is essential to measure the uncertainty level of the entire system. Currently, entropy serves as the most efficient tool for quantifying system uncertainty. Numerous relevant works have been proposed, including Deng entropy [?], motion entropy [?], interval entropy [?], and non-additive entropy [?]. Nevertheless, all existing entropy measures neglect the order of propositions within a fuzzy system as a factor affecting uncertainty levels, resulting in a significant research gap in measuring uncertainty for ordinal fuzzy systems.

To address this limitation, this paper proposes a new ordinal relative fuzzy entropy that provides an accurate description of ordinal fuzzy systems, applicable to intuitionistic, Pythagorean, Fermatean, and orthopair environments. Unlike existing entropies, the proposed method reflects the influence of proposition sequences, aligning with real-world operational rules and criteria.

This paper is organized as follows: Section 2 introduces preliminary concepts. Section 3 elaborates on the proposed methodology with detailed explanations. Section 4 provides numerical examples across five different fuzzy environments to verify the rationality and validity of the proposed entropy. Finally, conclusions summarize the contributions of this work.

2. Preliminaries

This section briefly introduces relevant concepts that have been extensively applied to solve problems across various domains [?].

2.1. Fuzzy Sets [?]

Definition 1. Let P be a fuzzy set (FS) in a finite universe of discourse H. The mathematical representation of an FS P in H is defined as:

$$P = \{x, \mu(x)\}$$

where $\mu(x)$ represents the membership degree of $x \in H$.

2.2. Intuitionistic Fuzzy Sets [?]

Definition 2. Let A be an intuitionistic fuzzy set (IFS) in a finite universe of discourse X. The mathematical representation of an IFS A in X is defined as:

$$A = \{\langle x, \mu_A(x), \nu_A(x) \mid x \in X\rangle\}$$

The IFS satisfies the following properties: $\mu_A(x) : X \to [0, 1]$ represents the membership degree of $x \in X$, while $\nu_A(x) : X \to [0, 1]$ represents the non-membership degree of $x \in X$. These parameters satisfy the constraint:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

For an IFS defined in X, the degree of hesitation $\pi(x)$ is defined as:

$$\pi(x) = 1 - \mu_A(x) - \nu_A(x)$$

where $\pi(x)$ reflects the hesitation degree of $x \in X$.

2.3. Pythagorean Fuzzy Sets [?, ?]

Definition 3. Let B be a Pythagorean fuzzy set (PFS) in a finite universe of discourse Z. The mathematical representation of a PFS B in Z is defined as:

$$B = \{\langle x, B_Y(x), B_N(x) \mid x \in Z\rangle\}$$

The PFS satisfies the following properties: $B_Y(x) : Z \to [0, 1]$ represents the membership degree of $x \in Z$, while $B_N(x) : Z \to [0, 1]$ represents the non-membership degree of $x \in Z$. These parameters satisfy the constraint:

$$0 \leq B_Y^2(x) + B_N^2(x) \leq 1$$

For a PFS defined in Z, given $K^2(x) = B_Y^2(x) + B_N^2(x)$, the degree of hesitation $B_H(x)$ is defined as:

$$B_H(x) = \sqrt{1 - K^2(x)}$$

where $B_H(x)$ reflects the hesitation degree of $x \in Z$.

2.4. Fermatean Fuzzy Sets [?]

Definition 4. Let C be a Fermatean fuzzy set (FFS) in a finite universe of discourse Q. The mathematical representation of an FFS C in Q is defined as:

$$C = \{\langle x, \alpha_F(x), \beta_F(x)\rangle \mid x \in Q\}$$

The FFS satisfies the following properties: $\alpha_F(x) : Q \to [0, 1]$ represents the membership degree of $x \in Q$, while $\beta_F(x) : Q \to [0, 1]$ represents the non-membership degree of $x \in Q$. These parameters satisfy the constraint:

$$0 \leq \alpha_F(x)^3 + \beta_F(x)^3 \leq 1$$

For an FFS defined in Q, the degree of hesitation $\pi_F(x)$ is defined as:

$$\pi_F(x) = \sqrt[3]{1 - \alpha_F(x)^3 - \beta_F(x)^3}$$

where $\pi_F(x)$ reflects the hesitation degree of $x \in Q$.

2.5. Orthopair Fuzzy Sets [?]

Definition 5. Let D be an orthopair fuzzy set (OFS) in a finite universe of discourse R. The mathematical representation of an OFS D in R is defined as:

$$D = \{\langle x, D^+(x), D^-(x) \mid x \in R\rangle\}$$

The OFS satisfies the following properties: $D^+(x) : R \to [0, 1]$ represents the membership degree of $x \in R$, while $D^-(x) : R \to [0, 1]$ represents the non-membership degree of $x \in R$. These parameters satisfy the constraint:

$$0 \leq (D^+(x))^n + (D^-(x))^n \leq 1$$

For an OFS defined in R, the degree of hesitation $\pi_O(x)$ is defined as:

$$\pi_O(x) = \sqrt[n]{1 - (D^{+}(x))^n - (D^{-}(x))^n}$$

where $\pi_O(x)$ reflects the hesitation degree of $x \in R$.

2.6. Shannon Entropy [?]

Shannon entropy, denoted as Sh, is defined as:

$$Sh = -\sum_{i=1}^{n} p_i \log_b p_i$$

where n is the number of targets in a system and p_i represents the probability of a particular event occurring. When the logarithm base b equals 2, the unit of Shannon entropy is bits.

2.7. De Luca and Termini' s Fuzzy Set Entropy [?]

Definition 6. Given a fuzzy set $E = \{x, \mu(x)\}$ where $\mu(x)$ represents the membership of certain propositions, the corresponding entropy is defined as:

$$DT(E) = \sum_{i=1}^{n} [\mu_E(x_i) \log \mu_E(x_i) + (1 - \mu_E(x_i)) \log(1 - \mu_E(x_i))]$$

2.8. Pal and Pal' s Fuzzy Exponential Entropy [?]

Definition 7. Given a fuzzy set $F = \{x, \mu(x)\}$ where $\mu(x)$ represents the membership of certain propositions, the corresponding entropy is defined as:

$$PP(F) = \frac{1}{e-1} \sum_{i=1}^{n} [\mu_F(x_i) e^{(1-\mu_F(x_i))} + (1 - \mu_F(x_i)) e^{\mu_F(x_i)} - 1]$$

2.9. Zhang and Jiang' s Intuitionistic Fuzzy Entropy [?]

Definition 8. Given an intuitionistic fuzzy set $G = \{x, \mu_G(x), \nu_G(x)\}$ where $\mu_G(x)$ represents membership and $\nu_G(x)$ represents non-membership of propositions, the corresponding entropy is defined as:

$$ZJ(G) = \sum_{i=1}^{n} \left[-\frac{\mu_G(x) + 1 - \nu_G(x)}{\mu_G(x) + 1 - \nu_G(x)} \log \frac{\mu_G(x)}{\mu_G(x) + 1 - \nu_G(x)} - \frac{\nu_G(x) + 1 - \mu_G(x)}{\nu_G(x) + 1 - \mu_G(x)} \log \frac{\nu_G(x)}{\nu_G(x) + 1 - \mu_G(x)} \right]$$

2.10. Hung and Yang' s Intuitionistic Fuzzy Entropy [?]

Definition 9. Given an intuitionistic fuzzy set $H = \{x, \mu_H(x), \nu_H(x)\}$ where $\mu_H(x)$ represents membership and $\nu_H(x)$ represents non-membership of propositions, the corresponding entropy is defined as:

$$HY(x) = -(\mu_H(x)\log\mu_H(x) + \nu_H(x)\log\nu_H(x) + \pi_H(x)\log\pi_H(x))$$

2.11. Xu' s Pythagorean Fuzzy Entropy [?]

Definition 10. Given a Pythagorean fuzzy set $I = \{x, I_Y(x), I_N(x)\}$ where $I_Y(x)$ represents membership and $I_N(x)$ represents non-membership of propositions, the corresponding entropy is defined as:

$$X(I) = \sum_{i=1}^{n}[1 - (1 - I_H(x_i))|I_Y(x_i) - I_N(x_i)|]$$

2.12. Yang' s Pythagorean Fuzzy Entropy [?]

Definition 11. Given a Pythagorean fuzzy set $J = \{x, J_Y(x), J_N(x)\}$ where $J_Y(x)$ represents membership and $J_N(x)$ represents non-membership of propositions, the corresponding entropy is defined as:

$$Y(J) = -\sum_{i=1}^{n}(J_Y(x)^2\log J_Y(x)^2 + J_N(x)^2\log J_N(x)^2 + J_H(x)^2\log J_H(x)^2)$$

3. Generalized Entropies for Fermatean Fuzzy Sets and Orthopair Fuzzy Sets

Due to the lack of proper entropies for measuring uncertainty in Fermatean fuzzy sets and orthopair fuzzy sets, existing entropies are generalized to address this gap. This section adapts Zhang and Jiang' s intuitionistic fuzzy entropy [?] and Yang' s Pythagorean fuzzy entropy [?] for Fermatean fuzzy sets. Additionally, Hung and Yang' s intuitionistic fuzzy entropy [?] and Xu' s Pythagorean fuzzy entropy [?] are adapted for orthopair fuzzy sets.

3.1. Fermatean Fuzzy Entropies

Definition 11. Given a Fermatean fuzzy set $L = \{\langle x, \alpha_L(x), \beta_L(x)\rangle\}$ where $\alpha_L(x)$ represents membership and $\beta_L(x)$ represents non-membership of propositions, the corresponding entropies are defined as:

$$ZJ_{Fermatean}(L) = \sum_{i=1}^{n}\left[-\frac{\alpha_L(x) + 1 - \beta_L(x)}{\alpha_L(x) + 1 - \beta_L(x)}\log\frac{\alpha_L(x)}{\alpha_L(x) + 1 - \beta_L(x)} - \frac{\beta_L(x) + 1 - \alpha_L(x)}{\beta_L(x) + 1 - \alpha_L(x)}\log\frac{\beta_L(x)}{\beta_L(x) + 1 - \alpha_L(x\right.$$

$$Y(L) = -\sum_{i=1}^{n}(\alpha_L(x)^2 \log \alpha_L(x)^2 + \beta_L(x)^2 \log \beta_L(x)^2 + \pi_L(x)^2 \log \pi_L(x)^2)$$

3.2. Orthopair Fuzzy Entropies

Definition 12. Given an orthopair fuzzy set $M = \{\langle x, M^+(x), M^-(x)\rangle\}$ where $M^+(x)$ represents membership and $M^-(x)$ represents non-membership of propositions, the corresponding entropies are defined as:

$$HY_{Orthopair}(M) = -\sum_{i=1}^{n}(M^+(x)\log M^+(x)+M^-(x)\log M^-(x)+\pi_O(x)\log \pi_O(x))$$

$$X(M) = \sum_{i=1}^{n}[1-(1-\pi_O(x))|M^+(x)-M^-(x)|]$$

4. Proposed New Ordinal Relative Fuzzy Entropy

Although entropies have been widely used to measure uncertainty in systems containing sequences of fuzzy sets [?], existing methods fail to consider proposition order—a critical limitation for real-world applications where sequential relationships are inherent. Since events occur in order, with one event enabling subsequent ones, a new entropy measure is needed that accounts for the effects of proposition sequences on uncertainty levels.

4.1. Sequential Fuzzy Sets

In ordinal fuzzy systems, all elements appear in a specific order, and their relationships are determined by their sequences to some extent. For an ordinal system $\Theta_{Ordinal} = \{P_1, P_2, P_3, P_4\}$, proposition P_1 must occur before P_2, P_3, P_4, and cannot be replaced by them. The properties of ordinal fuzzy systems are defined as:

- The sequence of each proposition in an ordinal fuzzy system is fixed and cannot be altered; changing the order creates a new system.
- The only relationship among propositions in an ordinal fuzzy system is their order; no other relationships exist among designated propositions.
- As more propositions are confirmed, the uncertainty of the entire ordinal fuzzy system becomes further determined.

Case 1: Consider an ordinal fuzzy system $\Theta_{Ordinal} = \{P_1, P_2, P_3\}$ containing three distinct propositions. Proposition P_1 occurs first, followed by P_2. After confirming P_1 and P_2, only P_3 can be added to the system. If the sequence is disturbed, the conditions of the propositions change accordingly.

4.2. Distributed Weights for Fuzzy Sets in Proposed Entropy

Since every proposition in a fuzzy system appears in order, the system' s uncertainty level becomes progressively determined. It is necessary to recognize that each confirmation step plays a different role, as confirming a proposition has both direct and indirect effects on other proposition values. Let a be the number of propositions and b the sequence of a particular proposition. The weight determination process is defined as:

1. The weight of a specific proposition is $a - b + 1$, calculated as:

$$\text{Weight}_{P_b} = a - b + 1$$

2. Original proposition values are denoted as $\langle x, \text{Pre}, \text{Aft} \rangle$. In fuzzy sets, Pre represents $\mu(x)$ and Aft equals 0. In intuitionistic environments, Pre represents $\mu_A(x)$ and Aft represents $\nu_A(x)$. For Pythagorean fuzzy sets, Pre represents $\alpha_B(x)$ and Aft represents $\beta_B(x)$. Finally, in orthopair environments, Pre represents $D^+(x)$ and Aft represents $D^-(x)$. Intermediate values are calculated as:

$$\text{Mass}_{\text{Pre}} = \text{Weight}_{P_b} \times \text{Pre}_{P_b}$$

$$\text{Mass}_{\text{Aft}} = \text{Weight}_{P_b} \times \text{Aft}_{P_b}$$

3. A normalization step is designed for elements Pre and Aft:

$$\text{Value}_{\text{PreFinal}} = \frac{\text{Mass}_{\text{Pre}_{P_b}}}{2 \times \sum_{t=1}^{n} \text{Mass}_{\text{Pre}_{P_t}}}$$

$$\text{Value}_{\text{AftFinal}} = \frac{\text{Mass}_{\text{Aft}_{P_b}}}{2 \times \sum_{t=1}^{n} \text{Mass}_{\text{Aft}_{P_t}}}$$

Note: Normalization is necessary because a proposition' s importance decreases as its sequence number increases, so its hesitation index is exaggerated to reduce its impact on the overall system' s uncertainty. To ensure the modified values still satisfy fuzzy set properties, both values are divided by 2.

4.3. Relative Fuzzy Entropy

Existing entropies cannot reflect influences caused by proposition order because they treat each fuzzy set as an individual rather than examining underlying relationships and their cumulative effects on the system. To maintain consistency with the weight assignment operation, a variable k is restricted between 1 and $a - 1$, and relative fuzzy entropy is defined as:

$$RFE(P_b, P_c) = \text{PreFinal}_{P_b} \times \log \frac{\text{PreFinal}_{P_b}}{\text{PreFinal}_{P_c}} + \text{AftFinal}_{P_b} \times \log \frac{\text{AftFinal}_{P_b}}{\text{AftFinal}_{P_c}}, \quad b > c$$

If the denominator of a logarithmic index and an addition operation is canceled, the proposed entropy degenerates to Shannon entropy form [**?**], demonstrating consistency with traditional information entropy.

Case 2: Consider an ordinal fuzzy system with three propositions P_1, P_2, P_3 satisfying sequential fuzzy set properties. The corresponding fuzzy set details are given in Table 1 .

Using Table 1 values, the relative fuzzy entropy mass for proposition P_1 is calculated as:

$$RFE(P_1, P_2) = \text{PreFinal}_{P_1} \times \log \frac{\text{PreFinal}_{P_1}}{\text{PreFinal}_{P_2}} + \text{AftFinal}_{P_1} \times \log \frac{\text{AftFinal}_{P_1}}{\text{AftFinal}_{P_2}}$$

$$RFE(P_1, P_3) = \text{PreFinal}_{P_1} \times \log \frac{\text{PreFinal}_{P_1}}{\text{PreFinal}_{P_3}} + \text{AftFinal}_{P_1} \times \log \frac{\text{AftFinal}_{P_1}}{\text{AftFinal}_{P_3}}$$

4.4. Individual Ordinal Relative Fuzzy Entropy

Previously proposed fuzzy entropies, including relative fuzzy entropy, calculate uncertainty non-directionally, which does not reflect real-world situations or manifest influences from proposition sequences. To capture ordinal system features, an individual ordinal relative fuzzy entropy is proposed.

For three propositions P_1, P_2, P_3, the individual ordinal relative fuzzy entropy for P_1 is calculated by summing $RFE(P_1, P_2)$ and $RFE(P_1, P_3)$. For P_2, only the $P_2 \to P_3$ process is considered. For P_3, the value is 0 because, in an ordinal system, the entire system is already determined before the final proposition appears. The calculation is defined as:

$$IORFE(P_b, P_c) = \sum_{c=b+1}^{n} RFE(P_b, P_c)$$

Case 3: Consider an ordinal fuzzy system with three propositions P_1, P_2, P_3 satisfying sequential fuzzy set properties. The fuzzy set details are given in Table 2 .

Using Table 2 values, the individual relative fuzzy entropy mass is calculated as:

$$IORFE(P_1, P_c) = \sum_{c=1+1}^{n} RFE(P_1, P_c) = RFE(P_1, P_2) + RFE(P_1, P_3) = 1.5681 + 1.5257 = 3.0938$$

$$IORFE(P_2, P_c) = \sum_{c=2+1}^{n} RFE(P_2, P_c) = RFE(P_2, P_3) = 1.1981$$

$$IORFE(P_3, P_c) = 0$$

Since the fuzzy system is ordinal, the calculation is directional. Each individual ordinal relative fuzzy entropy value represents the uncertainty level at that system stage. As more propositions are confirmed, the system' s uncertainty becomes further determined, as reflected by the obtained values. The proposed entropy appropriately measures each component' s condition within the fuzzy system.

4.5. Complete Ordinal Relative Fuzzy Entropy

Complete ordinal relative fuzzy entropy synthesizes individual ordinal relative fuzzy entropy to measure conditions at every system stage, accounting for all phases. The calculation is defined as:

$$CORFE(P_b, P_c) = \sum_{b=1}^{n} IORFE(P_b, P_c)$$

Case 4: Consider an ordinal fuzzy system with three propositions P_1, P_2, P_3 satisfying sequential fuzzy set properties. The fuzzy set details are given in Table 3 .

Using Table 3 values, the calculation proceeds as:

$$CORFE(P_b, P_c) = \sum_{b=1}^{n} IORFE(P_b, P_c) = RFE(P_1, P_2) + RFE(P_1, P_3) + RFE(P_2, P_3) + RFE(P_3, P_c) = 1.3984$$

This represents the final measurement of the given ordinal fuzzy system.

4.6. Measuring Traditional Fuzzy Systems Using Ordinal Relative Fuzzy Entropy

The previous sections address ordinal fuzzy system measurement. However, the proposed ordinal entropy can also measure traditional fuzzy systems by considering all possible proposition sequences. In other words, by examining all sequential combinations, the uncertainty of an unordered fuzzy system can be calculated as a synthesis of all possible situations. The detailed process is:

- List all possible proposition sequences in the ordinal fuzzy system.
- Calculate the sum of complete ordinal relative fuzzy entropy values for each ordinal system.
- Compute the average based on the sum and the number of propositions, yielding the final evaluation of the classic unordered fuzzy system.

Case 5: Consider an intuitionistic fuzzy system with three propositions P_1, P_2, P_3 satisfying sequential fuzzy set properties. The fuzzy set details are given in Table 4 .

All possible combinations are: $\{P_1, P_2, P_3\}, \{P_1, P_3, P_2\}, \{P_2, P_1, P_3\}, \{P_2, P_3, P_1\}, \{P_3, P_1, P_2\}, \{P_3, P_2, P_1\}$.

Each sequence yields different proposition values. Based on the complete ordinal relative fuzzy entropy definition, the values are:

$CORFE_{System1}(P_b, P_c) = 5.1671$, $CORFE_{System2}(P_b, P_c) = 3.6493$, $CORFE_{System3}(P_b, P_c) = 4.9177$, $CORFE_{System4}(P_b, P_c) = 3.8328$, $CORFE_{System5}(P_b, P_c) = 3.7105$, $CORFE_{System6}(P_b, P_c) = 4.0399$.

With 6 different systems, the average uncertainty level is:

$$CORFE_{Unordered} = \frac{\sum_{i=1}^{6} CORFE_{System_i}(P_b, P_c)}{6} = \frac{5.1671 + 3.6493 + 4.9177 + 3.8328 + 3.7105 + 4.0399}{6}$$

Since the system is intuitionistic, its uncertainty can also be measured using Zhang and Jiang' s intuitionistic fuzzy entropy [**?**] and Hung and Yang' s intuitionistic entropy [**?**]. Table 5 compares these three entropy measures.

The results demonstrate that the proposed ordinal relative fuzzy entropy can measure traditional fuzzy system uncertainty when considering all sequential combinations, validating its effectiveness across different scenarios.

5. Numerical Examples

This section provides five examples across different fuzzy environments to verify the superior validity and correctness of the proposed ordinal relative fuzzy entropy compared to existing methods.

Example 1: Consider an ordinal fuzzy system with three propositions P_1, P_2, P_3. Their original values are listed in Table 6 .

First, obtain proposition weights:

$$\text{Weight}_{P_1} = a - b_{P_1} + 1 = 3, \quad \text{Weight}_{P_2} = a - b_{P_2} + 1 = 2, \quad \text{Weight}_{P_3} = a - b_{P_3} + 1 = 1$$

Second, calculate intermediate values:

$$\text{Mass}_{\text{Pre}_{P_1}} = \text{Weight}_{P_1} \times \text{Pre}_{P_1} = 1.44$$

$$\text{Mass}_{\text{Pre}_{P_2}} = \text{Weight}_{P_2} \times \text{Pre}_{P_2} = 1.12$$
$$\text{Mass}_{\text{Pre}_{P_3}} = \text{Weight}_{P_3} \times \text{Pre}_{P_3} = 0.66$$

Third, perform normalization:

$$\text{Value}_{\text{PreFinal}_{P_1}} = \frac{\text{Mass}_{\text{Pre}_{P_1}}}{2 \times \sum_{t=1}^{n} \text{Mass}_{\text{Pre}_{P_t}}} = 0.2236$$

$$\text{Value}_{\text{PreFinal}_{P_2}} = \frac{\text{Mass}_{\text{Pre}_{P_2}}}{2 \times \sum_{t=1}^{n} \text{Mass}_{\text{Pre}_{P_t}}} = 0.1024$$

Fourth, enumerate all proposition combinations: $\{P_1, P_2, P_3\}, \{P_1, P_3, P_2\}, \{P_2, P_1, P_3\}, \{P_2, P_3, P_1\}, \{P_3, P_1, P_2$

Fifth, calculate uncertainty values using the ordinal relative fuzzy entropy definition. Table 7 presents measurements from three different entropy methods.

The results clearly show that existing entropies cannot reflect proposition order influences, while the proposed entropy values fluctuate according to proposition sequence, conforming to real-world situations.

Example 2: Consider an intuitionistic fuzzy system with three propositions P_1, P_2, P_3. Original values are in Table 8 and results in Table 9 .

The proposed entropy successfully manifests sequence effects, with values fluctuating as order changes. In contrast, the two existing entropies cannot reflect such influences.

Example 3: Consider a Pythagorean fuzzy system with three propositions P_1, P_2, P_3. Original values are in Table 10 and results in Table 11 .

This example demonstrates sequence influence on uncertainty levels in Pythagorean environments, proving the proposed method' s effectiveness. The other two entropies fail to capture order effects.

Example 4: Consider a Fermatean fuzzy system with three propositions P_1, P_2, P_3. Original values are in Table 12 and results in Table 13 .

The proposed method appropriately reflects sequence influences, while the generalized existing entropies cannot properly capture order effects.

Example 5: Consider an orthopair fuzzy system with three propositions P_1, P_2, P_3. Original values are in Table 14 and results in Table 15 .

This example confirms the proposed method' s effectiveness in considering sequence influences, while the generalized existing entropies cannot reflect this phenomenon.

6. Conclusion

In reality, all events occur in an underlying sequence with specific relationships and mutual influences. The main contribution of this paper is proposing an entropy that treats proposition order in ordinal fuzzy systems as a crucial factor in uncertainty measurement. Existing entropies cannot properly reflect real-world operational rules. The proposed entropy effectively measures ordinal fuzzy system uncertainty in accordance with actual situations, with numerical examples providing strong evidence for its validity and correctness.

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