Quantum entanglement measurement based on belief entropy

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Abstract

Partial entropy entanglement is a very popular method to measure the entanglement of quantum systems, which is based on the classic von Neumann entropy. However, because of the problem of classical von Neumann entropy in measuring the uncertainty of quantum systems, the partial entropy entanglement is not efficient enough to measure the entanglement of quantum systems. The new entropy measure of quantum uncertainty is a model for measuring the uncertainty of a quantum system based on the classic von Neumann entropy and belief entropy, which has higher performance than the classic von Neumann entropy in measuring the uncertainty entropy of a quantum system.Based on the new entropy measure of quantum uncertainty and the classic partial entropy entanglement, this paper proposes a new model to measure the quantum entanglement measurement, named quantum entanglement measurement based on belief entropy. When the new entropy measure of quantum uncertainty degenerates to classical von Neumann entropy, the quantum entanglement measurement based on belief entropy will degenerate to classical partial entropy entanglement. Numerical examples are used to prove that quantum entanglement measurement based on belief entropy is more efficient and reliable in measuring the entanglement of quantum systems than the classic partial entropy entanglement. The experimental results show that the quantum entanglement measurement based on belief entropy can measure the uncertainty of quantum systems more efficiently and reliably than the classical classic partial entropy entanglement.

Full Text

Preamble

Quantum Entanglement Measurement Based on Belief Entropy

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Abstract

Partial entropy entanglement is a widely adopted method for measuring quantum system entanglement, grounded in the classical von Neumann entropy. However, due to inherent limitations of classical von Neumann entropy in quantifying quantum uncertainty, partial entropy entanglement proves insufficiently efficient for entanglement measurement. The new entropy measure of quantum uncertainty, which integrates classical von Neumann entropy with belief entropy, demonstrates superior performance over classical von Neumann entropy in assessing quantum system uncertainty. Building upon this new entropy measure and classical partial entropy entanglement, this paper proposes a novel quantum entanglement measurement model called quantum entanglement measurement based on belief entropy. When the new entropy measure reduces to classical von Neumann entropy, the proposed model naturally degenerates to classical partial entropy entanglement. Numerical examples demonstrate that the quantum entanglement measurement based on belief entropy is more efficient and reliable than classical partial entropy entanglement for measuring quantum system entanglement. Experimental results confirm that the proposed model can measure quantum system uncertainty more efficiently and reliably than classical partial entropy entanglement.

Keywords: Quantum system, Belief entropy, Von Neumann entropy, Uncertainty, Entanglement, Partial entropy entanglement

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1. Introduction

The unknown world contains vast amounts of unknown information [1, 2, 3, 4]. To process such information, we must first represent it appropriately, prompting scholars to propose numerous mathematical methods and theories [5, 6, 7, 8]. For instance, Song et al. [9] introduced a novel combination method for temporal evidence, while Pan et al. [10] proposed new Pythagorean fuzzy sets and their similarity measures. Following reasonable representation of unknown information, experts have developed various models and methods to handle it [11, 12, 13]. Deng and Jiang [14] applied maximum uncertainty allocation to improve the Dempster–Shafer belief structure. Garg et al. [15, 16] employed entropy theory to enhance measurement methods in Pythagorean fuzzy environments.

Prajapati and Saha [17] utilized entropy theory for next-word prediction in text using language models. Abellan [18] analyzed belief entropy properties in evidential environments. Zhu [19] investigated the maximum value dimension and power law of belief distribution for maximum belief entropy. Kang and Deng [20] proposed the maximum belief entropy, while Gao and Deng [21] introduced the Pseudo-Pascal Triangle form for maximum belief entropy.

Quantum theory has emerged as a prominent field in recent years [22, 23, 24, 25], attracting extensive research from numerous experts [26, 27, 28]. A quantum system, based on quantum theory [29, 30, 31], differs fundamentally from classical systems and exhibits unique characteristics [32, 33, 34], including quantum uncertainty, quantum state superposition, and quantum state entanglement [35, 36, 37]. Consequently, evaluating quantum system entanglement remains a critical academic challenge [38, 39, 40]. In response, researchers have proposed diverse methods and models for measuring quantum entanglement [41, 42, 43]. Xavier and Rajabpour [44] investigated entanglement and boundary entropy in quantum spin chains. Karabali [45] introduced a new entanglement entropy for the integer quantum Hall effect in two and higher dimensions. Alimuddin et al. [46] studied the independence of entropy and work in equal-energetic finite quantum systems. Moitra and Sensarma [47] proposed entanglement entropy of fermions from Wigner functions. Grimmett et al. [48] examined bounded entanglement entropy under the quantum Ising model. Verga and Elas [49] explored thermal state entanglement entropy in quantum graphs. Hirano [50] developed a new entanglement entropy for all orders in 1/N expansion.

Among these approaches, partial entropy entanglement stands as one of the most popular models [51], with extensive research conducted by many experts [52]. Han and Kye [53] studied convex cones under classifications of partial entanglement in three-qubit systems. Huber et al. [54] investigated high-dimensional entanglement in states under positive partial transposition. Dwivedi et al. [55] examined multi-boundary entanglement in Chern-Simons theory based on finite gauge groups. Shapourian and Ryu [56] analyzed finite-temperature entanglement negativity of free fermions. Bauml et al. [57] studied fundamental limits on the capacities of bipartite quantum interactions.

Recently, Xue and Deng [58] proposed a new entropy measure for quantum system uncertainty that combines von Neumann entropy with belief entropy. This measure outperforms classical von Neumann entropy in quantifying quantum system uncertainty. Leveraging these advantages, this paper introduces quantum entanglement measurement based on belief entropy, which integrates the new entropy measure with partial entropy entanglement. The proposed model derives quantum system entanglement from the uncertainty of the quantum system and its subsystems. Specifically, entanglement equals half the difference between the sum of uncertainties of all subsystems and the uncertainty of the entire quantum system. The primary advantage over partial entropy entanglement lies in its foundation on the new entropy measure of quantum uncertainty, which proves superior to classical von Neumann entropy. When the new entropy measure reduces to classical von Neumann entropy, the proposed model naturally degenerates to classical partial entropy entanglement.

The remainder of this paper is organized as follows. Section 2 introduces preliminary concepts. Section 3 presents the quantum entanglement measurement based on belief entropy. Section 4 demonstrates the flexibility and accuracy of the proposed model through numerical examples. Section 5 concludes the paper.

2. Preliminaries

Quantum systems are inherently uncertain [59, 60, 61], prompting the development of numerous models and theories to address these uncertainties [62, 63, 64].

2.1. Belief Entropy

Given a mass function m on frame of discernment $X = \{x1, \cdots, xn\}$, belief entropy is defined as follows:

Definition 2.1. (Belief entropy) [65]
Ed $= -$ (cid:88) $m(A) \log 2|A| - 1$

Belief entropy generalizes Shannon entropy [66].

2.2. Von Neumann Entropy

Quantum theory represents a significant field [67, 68, 69] that has attracted considerable scholarly attention [70, 71, 72, 73]. Von Neumann entropy serves as a classical method for evaluating quantum system uncertainty, drawing extensive research interest [74]. Given a density matrix , von Neumann entropy is defined as:

Definition 2.2. (Von Neumann Entropy) [75]
$S( ) = -\mathrm{Tr}[\ \ln\ ]$

Von Neumann entropy can also be defined as:

$S( ) = -$ (cid:88) λi ln λi

where $0 * \ln 0 = 0$, and λi, i $\{1, 2, \cdots, m\}$ are the eigenvalues of .

2.3. A New Entropy Measure of Quantum Uncertainty

Definition 2.3. (A new entropy measure of quantum system uncertainty) [58]
Given a density matrix   of a quantum system, the new entropy measure of   is defined as:

$D( ) = -\ \lambda$i ln $2|\lambda$i$| - 1$ (cid:88) where λi is the eigenvalue of , and $|\lambda i|$ represents the cardinality of , indicating how many orthogonal ground states compose the eigenvector corresponding to λi.

Example 2.1. Consider a pure state quantum system:

| > = |10> = |10> + |01>, |01> =

The resulting density matrix is:

= | >< | =

The eigenvalues and corresponding eigenvectors of are shown in Table 1 :

Table 1: The eigenvalues and the corresponding eigenvectors of

Eigenvalues	Eigenvectors

Using Eq.(4), we obtain:

$D(\ ) = -\ \lambda i\ \ln 2|\lambda i| - 1$ (cid:88) = ln3

2.4. Partial Entropy Entanglement

Definition 2.4. (Partial entropy entanglement) [51]
Given a density matrix of a quantum system, partial entropy entanglement is defined as:

El(AB) = $\{S($ A) + $S($ B) − $S($ AB)$\}$

where $S($ A), $S($ B), and $S($ AB) represent the von Neumann entropy of subsystem A, subsystem B, and the two-body quantum system AB, respectively.

3. The Proposed Method

The world is inherently uncertain [76, 77, 78], creating numerous challenges [79, 80, 81]. Quantum systems exhibit flexibility, necessitating the development of more effective models [82, 83, 84].

Definition 3.1. (Quantum entanglement measurement based on belief entropy)
Given a density matrix of a quantum system, the quantum entanglement measurement based on belief entropy is defined as:

ED(AB) = $\{D($ A) + $D($ B) − $D($ AB)$\}$

where $D($ A), $D($ B), and $D($ AB) represent the new entropy measure of quantum uncertainty for subsystem A, subsystem B, and the two-body quantum system AB, respectively.

The calculation process of the proposed model is illustrated in Fig. 1 [Figure 1: see original paper].

Figure 1: The calculation process of the proposed model

Theorem 3.1. When the new entropy measure of quantum uncertainty degenerates to classical von Neumann entropy, the quantum entanglement measurement based on belief entropy reduces to classical partial entropy entanglement.

Proof 3.1. From Eq.(6), we have:

ED(AB) = {D(A) + D(B) − D(AB)}

When the new entropy measure of quantum uncertainty degenerates to classical von Neumann entropy, we obtain:

ED(AB) = {D(A) + D(B) − D(AB)}
= {S(A) + S(B) − S(AB)}
= El(AB)

Thus, the quantum entanglement measurement based on belief entropy degenerates to classical partial entropy entanglement.

4. Numerical Examples

Example 4.1. Consider a pure state quantum system:

| > = |00> + |11>

where |00> = , |11> =

The resulting density matrix is:

 AB = | >< | =

The reduced density matrices of AB are:

 A = (cid:18) 1/2 (cid:18) 1/2 , B = (cid:19) (cid:19)

The eigenvalues and corresponding eigenvectors of AB are shown in Table 2:

Table 2 : The eigenvalues and the corresponding eigenvectors of

Eigenvalues	Eigenvectors
−0.7071	

Using Eq.(4), we obtain:

D(AB) = ln 3, D(A) = ln 2, D(B) = ln 2

From Eq.(6), the entanglement degree of quantum system AB is:

ED(AB) = {D(A) + D(B) − D(AB)}
= {ln 2 + ln 2 − ln 3}

The comparison between partial entropy entanglement and quantum entanglement measurement based on belief entropy is shown in Table 3:

Table 3 : The comparison of two models

Partial entropy entanglement	The proposed model
2 ln 4	

The results demonstrate that the proposed model differs from partial entropy entanglement, which is attributable to the limitations of von Neumann entropy in measuring quantum uncertainty.

Example 4.2. Consider a pure state quantum system:

| > = |00> + |01> + |10>

where |00> = , |01> = , |10> =

The resulting density matrix is:

AB = | >< | =

The reduced density matrices of AB are:

A = (cid:18) 2/3 (cid:18) 2/3 , B = (cid:19) (cid:19)

The eigenvalues and corresponding eigenvectors of AB are shown in Table 4:

Table 4 : The eigenvalues and the corresponding eigenvectors of

Eigenvalues	Eigenvectors	
−0.8026	−0.1498	−0.6202

Using Eq.(4), we obtain:

D(AB) = ln 7

The eigenvalues and corresponding eigenvectors of A and B are shown in Table 5 :

Table 5: The eigenvalues and the corresponding eigenvectors of A

Eigenvalues	Eigenvectors	
−0.8507	−0.8507	−0.5257

Using Eq.(4), we obtain:

D(A) = D(B) = 1.48

From Eq.(6), the entanglement degree of quantum system AB is:

ED(AB) = {D(A) + D(B) − D(AB)}
= {1.48 + 1.48 − ln 7}

The comparison between partial entropy entanglement and quantum entanglement measurement based on belief entropy is shown in Table 6:

Table 6 : The comparison of two models

Partial entropy entanglement	The proposed model

The results again show that the proposed model differs from partial entropy entanglement, reflecting the limitations of von Neumann entropy in measuring quantum uncertainty.

5. Conclusion

Partial entropy entanglement is a classical tool for measuring quantum system entanglement. However, since it is based on von Neumann entropy, which has inherent problems in measuring quantum uncertainty, partial entropy entanglement is not sufficiently efficient for entanglement measurement. This article proposes quantum entanglement measurement based on belief entropy, which combines partial entropy entanglement with the new entropy measure of quantum system uncertainty. Because the new entropy measure is founded on classical von Neumann entropy yet outperforms it in measuring quantum uncertainty, the proposed model is more efficient and reasonable than classical partial entropy entanglement. Numerical examples demonstrate that quantum entanglement measurement based on belief entropy is more efficient and reliable for measuring quantum system entanglement. Experimental results confirm that the proposed model can measure quantum systems more efficiently and reliably than classical partial entropy entanglement.

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Note: Figure translations are in progress. See original paper for figures.