A new entropy measure of quantum system uncertainty

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Abstract

Quantum theory is currently the most important research field. Before processing information in a quantum system, we must first understand how to measure quantum system uncertainty. Von Neumann entropy is a classical method for measuring uncertainty in quantum systems. However, due to the unique characteristics of quantum systems, measuring uncertainty is very difficult, resulting in low measurement efficiency of classical von Neumann entropy in some cases. Based on classical von Neumann entropy and belief entropy, this paper proposes a new entropy model to measure quantum system uncertainty, which can fully utilize the eigenvalues and eigenvectors of the quantum system' s density matrix to quantify uncertainty. Numerical examples are used to demonstrate that the proposed entropy is more efficient and reliable for measuring quantum systems than classical von Neumann entropy. Experimental results show that the proposed entropy can measure quantum system uncertainty more efficiently and reliably than classical von Neumann entropy.

Full Text

Preamble

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A New Entropy Measure of Quantum System Uncertainty

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Abstract Quantum theory represents one of the most important frontiers in contemporary research. Before processing information in a quantum system, we must first understand how to measure its uncertainty. Von Neumann entropy provides a classic method for quantifying quantum system uncertainty. However, due to the unique characteristics of quantum systems, measuring their uncertainty is exceptionally challenging, and the classical von Neumann entropy’ s measurement efficiency proves inadequate in certain scenarios. This paper proposes a novel entropy model, grounded in classical von Neumann entropy and belief entropy, to measure quantum system uncertainty. The proposed approach fully utilizes both the eigenvalues and eigenvectors of the density matrix, thereby providing a comprehensive assessment of quantum system uncertainty. Numerical examples demonstrate that the proposed entropy measure offers greater efficiency and reliability than classical von Neumann entropy for quantum system analysis. Experimental results confirm that the proposed method outperforms classical von Neumann entropy in measuring quantum system uncertainty.

Keywords Quantum system · Belief entropy · Von Neumann entropy · Uncertainty

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1 Introduction

Human beings inhabit an uncertain world [?, ?, ?]. To address these uncertainties, numerous models and methods have been proposed [?, ?, ?], including quantum theory [?, ?, ?], complex networks, Dempster-Shafer evidence theory [?], Information Volume [?, ?], information entropy [?, ?, ?], belief entropy [?, ?], game theory [?, ?], and von Neumann entropy [?, ?, ?]. Deng and Jiang [?] applied maximum uncertainty allocation to improve the Dempster–Shafer belief structure. Jaunzemis et al. [?] employed judicial evidential reasoning to gather evidence information for hypothesis resolution. Pedrycz and Bargiela [?] utilized fuzzy membership functions to enhance the fuzzy modeling method for fuzzy fractal dimension. Khan and Anwar [?] applied weighted evidence and the Dempster–Shafer combination rule to improve time-domain data fusion for object classification. Among these theories and models, belief entropy [?], known as Deng entropy, extends information entropy and can evaluate uncertainties more flexibly than traditional information entropy [?, ?].

Leveraging its advantages in representing uncertainty, belief entropy has been widely studied by scholars [?, ?]. Zhu et al. [?] proposed the maximum value dimension and power law for the belief distribution of maximum belief entropy. Gao and Deng [?] introduced the Pseudo-Pascal Triangle form for maximum belief entropy. Liu et al. [?] applied generalized belief entropy to identify conflict evidence.

Quantum systems represent complex and fascinating systems [?, ?]. Due to phenomena such as quantum entanglement, they are more complicated than general systems [?, ?]. In other words, quantum systems exhibit significant uncertainty arising from characteristics like the principle of state superposition and the uncertainty principle [?, ?]. The principle of state superposition entangles multiple quantum states, making the system' s state unpredictable before collapse [?, ?, ?]. Entropy theory also serves to measure quantum system uncertainty [?, ?]. Zozor et al. [?] proposed general entropy-like uncertainty relations under finite dimensions. Vladimir Majernik [?] expressed uncertainty relations using a new entropic function. Von Neumann [?] introduced von Neumann entropy for measuring quantum system uncertainty. Specifically, measuring uncertainty via von Neumann entropy requires first determining the density matrix [?], then calculating its eigenvalues and eigenvectors [?, ?, ?]. Based on these eigenvalues, the von Neumann entropy can be determined [?, ?].

However, our research reveals that von Neumann entropy is less effective than expected for certain quantum systems. We believe this limitation stems from its reliance solely on density matrix eigenvalues, while eigenvectors also contain internal information about the matrix that the von Neumann entropy calculation neglects. Furthermore, since quantum states involve complex numbers, measuring quantum system uncertainty proves more difficult than measuring uncertainty in general systems.

This paper proposes a new entropy measure for quantum system uncertainty, based on von Neumann entropy and belief entropy. The proposed method fully utilizes both eigenvalues and eigenvectors of the density matrix, thereby extracting complete information from the density matrix representation and efficiently evaluating quantum system uncertainty. When the cardinalities of all eigenvectors equal 1, the proposed entropy degenerates to classical von Neumann entropy. Numerical examples verify the efficiency and reliability of the proposed entropy, demonstrating its superior performance and reliability compared to classical von Neumann entropy for measuring quantum system uncertainty.

The remainder of this paper is structured as follows. Section 2 introduces preliminary concepts. Section 3 presents the new entropy measure for quantum system uncertainty. Section 4 illustrates the flexibility and accuracy of the proposed measure. Section 5 concludes the paper.

2 Preliminaries

Quantum systems are inherently uncertain [?, ?, ?]. To address these uncertainties, numerous models and methods have been proposed [?, ?, ?].

2.1 Density Matrix

The density matrix, ρ, of a quantum system in a pure state $|\phi\rangle$ is defined as follows:

Definition 1 (Density Matrix of Pure State) [?]

$$\rho = |\phi\rangle\langle\phi|$$

The density matrix, ρ, of a quantum system in a mixed state is defined as follows:

Definition 2 (Density Matrix of Mixed State) [?]

$$\rho = \sum_i P_i |\phi_i\rangle\langle\phi_i|$$

where $\{P_i, |\phi_i\rangle\}$ represents the state and probability of the system.

2.2 Von Neumann Entropy

Von Neumann entropy serves as a classic method for evaluating quantum system uncertainty [?, ?] and has attracted considerable scholarly attention [?]. Given a density matrix ρ, the von Neumann entropy is defined as:

Definition 3 (Von Neumann Entropy) [?]

$$S(\rho) = -\mathrm{Tr}[\rho \ln \rho]$$

This can be expressed as:

$$S(\rho) = -\sum_i \lambda_i \ln \lambda_i$$

where $0 \cdot \ln 0 = 0$ and λ_i, $i \in \{1, 2, \dots, m\}$ are the eigenvalues of ρ.

2.3 Belief Entropy

Given a mass function m under frame of discernment Y, belief entropy is defined as:

Definition 4 (Belief Entropy) [?]

$$E_d = -\sum_{C \subseteq Y} \frac{m(C) \log_2 m(C)}{2^{|C|} - 1}$$

where $|C|$ is the cardinality of C.

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3 A New Entropy Measure of Quantum System Uncertainty

The world is inherently uncertain [?, ?], which creates numerous challenges [?, ?, ?]. Quantum systems are flexible, necessitating more effective models [?, ?, ?]. While von Neumann entropy can effectively evaluate quantum system uncertainty based on the density matrix, the density matrix itself is difficult to predict, limiting von Neumann entropy' s applicability. Belief entropy, however, demonstrates high performance in indicating uncertainty.

According to the principle of quantum state superposition, a quantum system' s state comprises a superposition of orthogonal basis states with complex weights [?, ?, ?]. Complex numbers inherently possess high uncertainty, making classical von Neumann entropy inadequate for correctly measuring uncertainty in some quantum systems [?, ?]. To address these limitations, this paper proposes a new entropy measure for quantum system uncertainty based on belief entropy. The proposed method can efficiently measure both quantum systems that are measurable by classical von Neumann entropy and those that are not. Since belief entropy effectively measures uncertainty in unknown systems, this section adapts its principles to propose a quantum system uncertainty measure.

Definition 5 (New Entropy Measure of Quantum System Uncertainty)
Given a density matrix ρ of a quantum system, the new entropy measure is defined as:

$$D(\rho) = -\sum_i \frac{\lambda_i \ln \lambda_i}{2^{|\lambda_i|} - 1}$$

where λ_i are the eigenvalues of ρ, and $|\lambda_i|$ represents the cardinality—the number of orthogonal basis states composing the eigenvector corresponding to λ_i.

Example 1
Assume the density matrix ρ has an eigenvector $|\psi\rangle = \alpha|10\rangle + \beta|01\rangle$ with eigenvalue λ_i, consisting of two orthogonal basis states $|10\rangle$ and $|01\rangle$ as follows:

$$|10\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then $|\lambda_i| = 2$.

Theorem 1
When all eigenvectors of a quantum system' s density matrix consist of only one orthogonal basis state, the proposed entropy degenerates to classical von Neumann entropy.

Proof
From Eq. (6), we have:

$$D(\rho) = -\sum_i \frac{\lambda_i \ln \lambda_i}{2^{|\lambda_i|} - 1}$$

Since all eigenvectors consist of only one orthogonal basis state, $|\lambda_i| = 1$ for all $i \in \{1, 2, \ldots, m\}$. Therefore:

$$D(\rho) = -\sum_i \frac{\lambda_i \ln \lambda_i}{2^1 - 1} = -\sum_i \lambda_i \ln \lambda_i$$

Thus, the proposed entropy degenerates to classical von Neumann entropy.

For a quantum system with four eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and corresponding eigenvectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, the calculation process of the proposed entropy is illustrated in Fig. 1 [Figure 1: see original paper].

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4 Numerical Examples

Numerical examples were designed to verify the proposed entropy' s advantages over classical von Neumann entropy in measuring uncertainty across various quantum systems.

Example 2
Consider a pure-state quantum system:

$$|\psi\rangle = \alpha|10\rangle + \beta|01\rangle$$

where $|10\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|01\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The corresponding density matrix is:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$$

The eigenvalues and eigenvectors of ρ are shown in Table 1 .

Using Eq. (4) and Eq. (6), we obtain:

$$S(\rho) = -\sum_i \lambda_i \ln \lambda_i = 0$$

$$D(\rho) = -\sum_i \frac{\lambda_i \ln \lambda_i}{2^{|\lambda_i|} - 1} = \ln 3$$

The comparison between the proposed entropy and von Neumann entropy is presented in Table 2 .

The von Neumann entropy of ρ is 0, suggesting zero uncertainty—a problematic result. Although this quantum system is in a pure state, it is not completely certain; for instance, the phase angle of the complex number α remains uncertain. This demonstrates that classical von Neumann entropy may be inaccurate for certain quantum systems. In contrast, the proposed entropy yields $\ln 3 > 0$, effectively measuring the system' s uncertainty.

Example 3
Consider a mixed-state quantum system:

$$|\psi\rangle = \sqrt{0.64}|10\rangle + \sqrt{0.36}|01\rangle$$

where $|10\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|01\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The density matrix is:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} 0.64 & 0 \\ 0 & 0.36 \end{pmatrix}$$

The eigenvalues and eigenvectors are shown in Table 3 .

Using Eq. (4) and Eq. (6):

$$S(\rho) = -\sum_i \lambda_i \ln \lambda_i = -0.64 \ln 0.64 - 0.36 \ln 0.36 = 0.6534$$

$$D(\rho) = -\sum_i \frac{\lambda_i \ln \lambda_i}{2^{|\lambda_i|} - 1} = 0.6534$$

The comparison is shown in Table 4 .

Both entropies equal 0.6534, demonstrating that the proposed entropy degenerates to classical von Neumann entropy in certain cases, highlighting its flexibility.

Example 4
Consider another pure-state quantum system:

$$|\psi\rangle = \alpha|10\rangle + \beta|01\rangle$$

where $|10\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|01\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The density matrix is:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$$

The eigenvalues and eigenvectors are shown in Table 5 .

Using Eq. (4) and Eq. (6):

$$S(\rho) = -\sum_i \lambda_i \ln \lambda_i = 0$$

$$D(\rho) = -\sum_i \frac{\lambda_i \ln \lambda_i}{2^{|\lambda_i|} - 1} = \ln 3$$

The comparison appears in Table 6 .

When the quantum system ρ still contains significant unknowns and uncertainties, classical von Neumann entropy incorrectly calculates zero uncertainty, while the proposed entropy measures it as $\ln 3$, again demonstrating superior reliability.

Example 5
Given the density matrix:

$$\rho = \begin{pmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 \end{pmatrix}$$

This quantum system is in a fully mixed state. The eigenvalues and eigenvectors are shown in Table 7 .

Using Eq. (4) and Eq. (6):

$$S(\rho) = -\sum_i \lambda_i \ln \lambda_i = -4 \times 0.25 \ln 0.25 = 2 \ln 2$$

$$D(\rho) = -\sum_i \frac{\lambda_i \ln \lambda_i}{2^{|\lambda_i|} - 1} = 2 \ln 2$$

The comparison is presented in Table 8 .

In this special case of a completely mixed state where classical von Neumann entropy reaches its maximum, the proposed entropy equals the classical value, confirming its effectiveness even in extreme quantum scenarios.

Example 6
Consider a quantum state:

$$|\eta\rangle = \cos(\psi)|a_1\rangle + \sin(\psi)|a_2\rangle = \cos(\psi - \varepsilon)|b_1\rangle + \sin(\psi - \varepsilon)|b_2\rangle$$

where ψ ranges between 0 and 2π.

Based on this form, Portesi and Plastino [?] proposed a generalized entropic uncertainty measure:

$$U_q(\hat{A},\hat{B};\eta)=\frac{1-[\cos^{2q}\psi+\sin^{2q}\psi][\cos^{2q}(\psi-\varepsilon)+\sin^{2q}(\psi-\varepsilon)]}{q-1}$$

The MU-like [?] expression is:

$$\text{where } \xi=\frac{1-(1/c)^{2(1-q)}}{q-1},\quad c=\max\{|\cos\varepsilon|,|\sin\varepsilon|\}$$

A comparison of the three models is shown in Fig. 2 [Figure 2: see original paper].

In the figure, ε varies from 0 to 2π on the x-axis, while the y-axis shows uncertainty measurements from the three models (with $\psi=\pi/2$ for computational convenience). The proposed entropy exceeds Portesi and Plastino' s model, indicating higher efficiency in measuring quantum system uncertainty as it captures more information. The Maassen and Uffink model produces insufficiently smooth curves with sharp points, suggesting limited universality. The proposed entropy is both larger and smoother, demonstrating superior efficiency and adaptability.

5 Conclusion

This paper proposes a novel entropy model for measuring quantum system uncertainty, combining classical von Neumann entropy with belief entropy. The proposed method fully exploits both eigenvalues and eigenvectors of the density matrix to quantify quantum system uncertainty. When all eigenvector cardinalities equal 1, the proposed entropy degenerates to classical von Neumann entropy. Numerical examples prove the proposed entropy' s superior efficiency and reliability compared to classical von Neumann entropy. Experimental results demonstrate that the proposed method measures quantum system uncertainty more effectively and reliably than classical von Neumann entropy.

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Ethical approval
This article contains no studies involving human participants or animals performed by any authors.

References

Abdel-Khalek S, Abo-Dahab S, Ragab M, Ahmad H (2020a) Engineering entanglement, geometric phase, and quantum fisher information of a three-level system with energy dissipation. *Mathematical Methods in the Applied Sciences* DOI 10.1002/mma.6781

Abdel-Khalek S, Alhag A, Ragab M, Abo-Dahab S, Algarni A, Ahmad H (2020b) Atomic fisher information and entanglement forecasting for quantum system based on artificial neural network and time series model. *International Journal of Quantum Chemistry* p 26446, DOI 10.1002/qua.26446

Alkassar Y, Agarwal VK, Pandey R, Behera N (2020) Experimental study and shannon entropy analysis of pressure fluctuations and flow mode transition in fluidized dense phase pneumatic conveying of fly ash. *Particuology* 49:169–178, DOI 10.1016/j.partic.2019.03.003

Alkhateeb S, Abdel-Khalek S (2020) Some features of the nonlocal correlation and geometric phase of the quantum system in two-mode nondegenerate entangled states. *Journal of Mathematical Chemistry* 58, DOI 10.1007/s10910-019-01057-6

Anaya Contreras J, Moya-Cessa H, Ziga-Segundo A (2019) The von neumann entropy for mixed states. *Entropy* 21:49, DOI 10.3390/e21010049

Anjaria K (2020) Negation and entropy: Effectual knowledge management equipment for learning organizations. *Expert Systems with Applications* p 113497, DOI 10.1016/j.eswa.2020.113497

Balazadeh L, Najarbashi G, Tavana A (2020) Quantum renormalization of the l(1)-norm and relative entropy of coherence in various spin chains. *Quantum Information Processing* 19:181, DOI 10.1007/s11128-020-02677-7

Bannwarth C, Ehlert S, Grimme S (2019) Gfn2-xtb—an accurate and broadly parametrized self-consistent tight-binding quantum chemical method with multipole electrostatics and density-dependent dispersion contributions. *Journal of Chemical Theory and Computation* 15(3):1652–1671, DOI 10.1021/acs.jctc.8b01176

Benabdallah F, Slaoui A, Daoud M (2020) Quantum discord based on linear entropy and thermal negativity of qutrit-qubit mixed spin chain under the influence of external magnetic field. *Quantum Information Processing* 19, DOI 10.1007/s11128-020-02754-x

Berrada K, Al-Rajhi M (2017) Information quantifiers, entropy squeezing and entanglement properties of superconducting qubit-deformed bosonic field system under dephasing effect. *Quantum Information Processing* 16:239, DOI 10.1007/s11128-017-1701-0

Berrada K, Eleuch H (2021) Einstein-podolsky-rosen steering and nonlocality in quantum dot systems. *Physica E: Low-dimensional Systems and Nanostructures* 126:114,412, DOI 10.1016/j.physe.2020.114412

Boes P, Eisert J, Gallego R, Mller M, Wilming H (2019) Von neumann entropy from unitarity. *Physical Review Letters* 122, DOI 10.1103/PhysRevLett.122.210402

Boutabba N, Eleuch H (2020) Quantum control of an optically dense atomic medium: Pulse shaping in a v-type three-level system. *Results in Physics* 19:103,421, DOI 10.1016/j.rinp.2020.103421

Carrijo T, Avelar A (2019) Weak quantum correlation quantifiers with generalized entropies. *Quantum Information Processing* 18:308, DOI 10.1007/s11128-019-2416-

Chakraborty S, Shaikh SH, Chakrabarti A, Ghosh R (2020) A hybrid quantum feature selection algorithm using a quantum inspired graph theoretic approach. *Applied Intelligence* 50:1775–1793, DOI 10.1007/s10489-019-01604-3

Dai J, Deng Y (2020) A new method to predict the interference effect in quantum-like bayesian networks. *Soft Computing* pp 1–8, DOI 10.1007/s00500-020-04693-2

Deng J, Deng Y (2021) Information volume of fuzzy membership function. *International Journal of Computers Communications & Control* 16(1):4106, DOI https://doi.org/10.15837/ijccc.2021.1.4106

Deng S, Diao P, Li F, Yu Q, Yu S, Wu H (2018) Observation of dynamical super-efimovian expansion in a unitary fermi gas. *Physical Review Letters* 120(12):125,301, DOI 10.1103/PhysRevLett.120.125301

Deng X, Jiang W (2020) On the negation of a dempster–shafer belief structure based on maximum uncertainty allocation. *Information Sciences* 516:346–352, DOI 10.1016/j.ins.2019.12.080

Deng Y (2020a) Information volume of mass function. *International Journal of Computers Communications & Control* 15(6):3983, DOI https://doi.org/10.15837/ijccc.2020.6.3983

Deng Y (2020b) Uncertainty measure in evidence theory. *SCIENCE CHINA Information Sciences* 63(11):210,201

Duboscq R, Pinaud O (2020) Constrained minimizers of the von neumann entropy and their characterization. *Calculus of Variations and Partial Differential Equations* 59:105, DOI 10.1007/s00526-020-01753-1

Farhadinia B, Xu Z (2018) Novel hesitant fuzzy linguistic entropy and cross-entropy measures in multiple criteria decision making. *Applied Intelligence* 48(11):3915–

Feng X, Wei W, Zhang R, Wang J, Shi Y, Zheng Z (2019) Exploring the heterogeneity for node importance by von neumann entropy. *Physica A: Statistical Mechanics and its Applications* 517, DOI 10.1016/j.physa.2018.11.019

Gal M (2019) Maps on quantum states in c-*algebras preserving von neumann entropy or schatten p-norm of convex combinations.* Canadian Mathematical Bulletin* 62:1–6, DOI 10.4153/CMB-2018-011-3

Gao X, Deng Y (2020) Quantum Model of Mass Function. *International Journal of Intelligent Systems* 35(2):267–282

Gao X, Deng Y (2020) The pseudo-pascal triangle of maximum deng entropy. *International Journal of Computers Communications & Control* 15(1):1006, DOI 10.15837/3735/ijccc.2020.1.3735

Gaonkar B, Beckett J, Attiah M, Ahn C, Edwards M, Wilson B, Laiwalla A, Salehi B, Yoo B, Bui A, Macyszyn L (2021) Eigenrank by committee: Von-neumann entropy based data subset selection and failure prediction for deep learning based medical image segmentation. *Medical Image Analysis* 67:101,834, DOI 10.1016/j.media.2020.101834

Guha A, Das P (2018) An extensive study of bose-einstein condensation in liquid helium using tsallis statistics. *Physica A: Statistical Mechanics and its Applications* 497:272–284, DOI 10.1016/j.physa.2018.01.020

Gupta D, Richhariya B (2018) Entropy based fuzzy least squares twin support vector machine for class imbalance learning. *Applied Intelligence* 48(11):4212–4231, DOI 10.1007/s10489-018-1204-4

Huang Z, Yang L, Jiang W (2019) Uncertainty measurement with belief entropy on the interference effect in the quantum-like bayesian networks. *Applied Mathematics and Computation* 347:417–428, DOI 10.1016/j.amc.2018.11.036

Ion D, Ion M (2001) New nonextensive quantum entropy and strong evidences for equilibrium of quantum hadronic states. *Physics Letters B* 519:63–70, DOI 10.1016/S0370-2693(01)01066-8

Jaunzemis AD, Holzinger MJ, Chan MW, Shenoy PP (2019) Evidence gathering for hypothesis resolution using judicial evidential reasoning. *Information Fusion* 49:26–45, DOI 10.1016/j.inffus.2018.09.010

Jauregui M, Zunino L, Lenzi E, Mendes R, Valentin Ribeiro H (2018) Characterization of time series via rényi complexity-entropy curves. *Physica A: Statistical Mechanics and its Applications* 498:74–85, DOI 10.1016/j.physa.2018.01.026

Johnson CR, Horn RA (1985) *Matrix Analysis*. Cambridge University Press Cambridge

Karci A (2016) Fractional order entropy: New perspectives. *Optik* 127(20):9172–9177, DOI 10.1016/j.ijleo.2016.06.119

Khan MN, Anwar S (2019) Time-domain data fusion using weighted evidence and dempster–shafer combination rule: Application in object classification. *Sensors* 19(23):5187, DOI 10.3390/s19235187

Koh JM, Cheong KH (2019) New doubly-anomalous parrondos games suggest emergent sustainability and inequality. *Nonlinear Dynamics* 96(1):257–266, DOI 10.1007/s11071-019-04788-y

Kontopoulou EM, Dexter GP, Szpankowski W, Grama A, Drineas P (2020) Randomized linear algebra approaches to estimate the von neumann entropy of density matrices. *IEEE Transactions on Information Theory* 66(8):5003–5021, DOI 10.1109/TIT.2020.2971991

Kurzyk D, Pawela u, Puchaa Z (2018) Conditional entropic uncertainty relations for tsallis entropies. *Quantum Information Processing* 17, DOI 10.1007/s11128-018-

Li D, Deng Y, Cheong KH (2021) Multi-source basic probability assignment fusion based on information quality. *International Journal of Intelligent Systems* 36:10.1002/int.22,363

de Lima Bernardo B (2020) Proposal for a direct measurement of the von neumann entropy and the relative entropy of coherence. *Physica Scripta* 95(4):045,104, DOI 10.1088/1402-4896/ab6359

Liu F, Gao X, Zhao J, Deng Y (2019) Generalized belief entropy and its application in identifying conflict evidence. *IEEE Access* 7(1):126,625–126,633

Longo R, Xu F (2020) Von neumann entropy in qft. *Communications in Mathematical Physics* pp 1–24, DOI 10.1007/s00220-020-03702-7

Luis A, Rodil A (2013) Alternative measures of uncertainty in quantum metrology: Contradictions and limits. *Physical Review A* 87, DOI 10.1103/PhysRevA.87.034101

Luis A, Bosyk G, Portesi M (2016) Entropic measures of joint uncertainty: Effects of lack of majorization. *Physica A: Statistical Mechanics and its Applications* 444:905–913, DOI 10.1016/j.physa.2015.10.097

Maassen H, Uffink J (1988) Generalized entropic uncertainty relations. *Physical Review Letters* 60:1103–1106, DOI 10.1103/PhysRevLett.60.1103

Majernik V (2008) The uncertainty relation expressed by means of a new entropic function. *Open Physics* 6:363–371, DOI 10.2478/s11534-008-0057-6

Majernik V, Majernkov E, Shpyrko S (2003) Uncertainty relations expressed by shannon-like entropies. *Central European Journal of Physics* 1:393–420, DOI

10.2478/BF02475852

Mukhamedov F, Watanabe N (2018) On s-mixing entropy of quantum channels. *Quantum Information Processing* 17, DOI 10.1007/s11128-018-1916-8

Mukhamedov F, Ohmura K, Watanabe N (2020) A formulation of renyi entropy on c-*algebras.* Quantum Information Processing* 18, DOI 10.1007/s11128-019-2430-

Özkan K (2018) Comparing shannon entropy with deng entropy and improved deng entropy for measuring biodiversity when a priori data is not clear. *Forestist* 68(2):136–140, DOI 10.26650/forestist.2018.340634

Pan L, Deng Y (2020) Probability transform based on the ordered weighted averaging and entropy difference. *International Journal of Computers Communications & Control* 15(4):3743, DOI 10.15837/ijccc.2020.4.3743

Pan Y, Zhang L, Wu X, Skibniewski MJ (2020) Multi-classifier information fusion in risk analysis. *Information Fusion* 60:121–136, DOI 10.1016/j.inffus.2020.02.003

Panigrahi K, Howli P, Chattopadhyay KK (2020) 3d network of v2o5 for flexible symmetric supercapacitor. *Electrochimica Acta* 337:135,701, DOI 10.1016/j.electacta.2020.135701

Pedrycz W, Bargiela A (2003) Fuzzy fractal dimensions and fuzzy modeling. *Information Sciences* 153:199–216

Perez-Bernal F, Torres-Herrera E, Schiulaz M, Santos L (2020) Self-averaging in many-body quantum systems out of equilibrium: Approach to the localized phase. *Physical Review B* 102:094,310, DOI 10.1103/PhysRevB.102.094310

Portesi M, Plastino A (1996) Generalized entropy as a measure of quantum uncertainty. *Physica A: Statistical Mechanics and its Applications* 225:412–430, DOI 10.1016/0378-4371(95)00475-0

Prunkl C (2020) On the equivalence of von neumann and thermodynamic entropy. *Philosophy of Science* 87, DOI 10.1086/707565

Rastegin AE (2012) Number-phase uncertainty relations in terms of generalized entropies. *Quantum Information and Computation* 12:743–762

Rastegin AE (2015) Uncertainty relations for general canonically conjugate observables in terms of unified entropies. *Foundations of Physics* 45: 23–942

ur Rehman J, Shin H (2019) purity-based continuity bounds for von neumann entropy. *Scientific Reports* 9(1):1–9, DOI 10.1038/s41598-019-50309-7

Romagnoli S (2019) A vague multidimensional dependency structure: Conditional versus unconditional fuzzy copula models. *Information Sciences* 512:1202–1213, DOI 10.1016/j.ins.2019.10.052

Romagnoli S (2020) A vague multidimensional dependency structure: Conditional versus unconditional fuzzy copula models. *Information Sciences* 512:1202–1213, DOI 10.1016/j.ins.2019.10.052

Sakamoto S, Tanimura Y (2020) Numerically exact simulations of entropy production in the fully quantum regime: Boltzmann entropy vs von neumann entropy. *The Journal of Chemical Physics* 153:234,107, DOI 10.1063/5.0033664

Sarantoglou G, Skontranis M, Mesaritakis C (2020) All optical integrate and fire neuromorphic node based on single section quantum dot laser. *IEEE Journal of Selected Topics in Quantum Electronics* 26:1–10, DOI 10.1109/JSTQE.2019.2945549

Turhan H, Demirekler M (2017) A novel combination methodology for dempster-shafer theory. In: *2017 25th Signal Processing and Communications Applications Conference (SIU)*, pp 1–4, DOI 10.1109/SIU.2017.7960625

Ur Rehman J, Shin H (2019) Purity-based continuity bounds for von neumann entropy. *Scientific Reports* 9:1–9, DOI 10.1038/s41598-019-50309-7

Valero-Toranzo I, Zozor S, Brossier J (2018) Generalization of the de bruijn identity to general ϕ-entropies and ϕ-fisher informations. *IEEE Transactions on Information Theory* 64(10):6743–6758, DOI 10.1109/TIT.2017.2771209

Von Neumann J (1955) *Mathematical Foundations of Quantum Mechanics*

Wang D, Gao J, Wei D (2019) A new belief entropy based on deng entropy. *Entropy* 21(10):987, DOI 10.3390/e21100987

Wang L, Ye SQ, Cheong KH, Bao W, Xie Ng (2018) The role of emotions in spatial prisoners dilemma game with voluntary participation. *Physica A: Statistical Mechanics and its Applications* 490:1396–1407, DOI 10.1016/j.physa.2017.08.033

Yurischev M (2018) Bimodal behavior of post-measured entropy and one-way quantum deficit for two-qubit x states. *Quantum Information Processing* 17, DOI 10.1007/s11128-017-1776-7

Yurischev MA (2017) Extremal properties of conditional entropy and quantum discord for xxz, symmetric quantum states. *Quantum Information Processing* 16, DOI 10.1007/s11128-017-1701-0

Zhu JM (2020) Von neumann entropy distribution with influence factors in matrix product states. *International Journal of Theoretical Physics* 59:1–15, DOI 10.1007/s10773-020-04517-x

Zhu R, Chen J, Kang B (2020) Power law and dimension of the maximum value for belief distribution with the maximum deng entropy. *IEEE Access* 8:47,713–47,719, DOI 10.1109/ACCESS.2020.2979060

Zozor S, Portesi M, Vignat C (2008) Some extensions of the uncertainty principle. *Physica A: Statistical Mechanics and its Applications* 387:4800–4808, DOI 10.1016/j.physa.2008.04.010

Zozor S, Bosyk G, Portesi M (2013) On a generalized entropic uncertainty relation in the case of the qubit. *Journal of Physics A Mathematical and Theoretical* 46, DOI 10.1088/1751-8113/46/46/465301

Zozor S, Bosyk G, Portesi M (2014) General entropy-like uncertainty relations in finite dimensions. *Journal of Physics A: Mathematical and Theoretical* 47, DOI 10.1088/1751-8113/47/49/495302

Note: Figure translations are in progress. See original paper for figures.