Postprint: The Origin of Soft Photons in the Inverse Compton Mechanism of Flat-Spectrum Radio Quasars

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Abstract

To verify the dominant process in the inverse Compton mechanism and the origin of soft photons therein, we collected multi-band (quasi-)simultaneous data for 75 flat-spectrum radio quasars from the literature, fitted the spectral energy distributions using a log-parabolic function, analyzed the relationship between the ratio of synchrotron peak flux to inverse Compton peak flux and the Doppler factor, and employed the bootstrap method to calculate the seed factor distribution of the sample to investigate the origin of soft photons. The analysis results are as follows: (1) Under the leptonic model framework, the significant correlation between log(〖U_B (F__)〗 *{EC}*/ 〖*U*{ext} (F__)〗 _{syn}) and logδ for the entire sample and subsamples indicates that the high-energy radiation of flat-spectrum radio quasars is preferentially dominated by the external Compton process. (2) For populations with relatively low synchrotron peak frequencies, the intersection between the observed seed factor distribution and the theoretical seed factor range of the dusty torus suggests that soft photons in their inverse Compton process are likely dominated by the dusty torus, whereas for populations with relatively high synchrotron peak frequencies, the observed seed factor exceeds the range of seed factors from the broad-line region or dusty torus, and therefore we cannot yet determine the origin of their soft photons.

Full Text

Study on the Origin of Soft Photons for the Inverse Compton Mechanism in Flat-Spectrum Radio Quasars

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Abstract

To verify the dominant process in the inverse Compton mechanism and the origin of soft photons in this process, we collected multi-wavelength (quasi-)simultaneous data for 75 flat-spectrum radio quasars from the literature. We fitted their spectral energy distributions using logarithmic parabolic functions, analyzed the relationship between the ratio of synchrotron peak flux to inverse Compton peak flux and the Doppler factor, and applied the bootstrap method to calculate the seed factor distribution of the sample to investigate the origin of soft photons. Our analysis yields the following results: (1) Under the leptonic model framework, the significant correlation between ($L_{\rm IC}/L_{\rm syn}$) and δ for both the entire sample and subsamples indicates that the high-energy radiation of flat-spectrum radio quasars is preferentially dominated by the external Compton process. (2) For populations with relatively low synchrotron peak frequencies, the intersection between the observed seed factor distribution and the theoretical dust torus seed factor range suggests that soft photons in the inverse Compton process are likely dominated by the dust torus. However, for populations with relatively high synchrotron peak frequencies, the observed seed factor exceeds the range of seed factors from either the broad-line region or dust torus, leaving the origin of their soft photons undetermined.

Keywords: radiation mechanisms; active galactic nuclei; flat-spectrum radio quasars; inverse Compton scattering
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0 Introduction

Blazars are a subclass of radio-loud active galactic nuclei [1] observed along the relativistic jet axis, comprising two categories: flat-spectrum radio quasars (FSRQs) with strong emission lines and BL Lacertae objects (BL Lacs) with weak or no emission lines [2]. Observationally, the broadband radiation of blazars, spanning from radio to gamma-ray wavelengths, forms a spectral energy distribution (SED) that exhibits a characteristic double-peaked structure in νF_ν space. Extensive research on this double-peaked structure within the widely adopted leptonic model framework identifies the low-energy peak as synchrotron radiation from relativistic electrons and the high-energy peak as inverse Compton scattering produced by interactions between relativistic electrons and soft photons [3]. When soft photons originate from synchrotron radiation, the model is called synchrotron self-Compton (SSC) [4]; when they originate from outside the jet, it is called external Compton (EC). In the EC scenario, soft photons may come from the accretion disk [5], broad-line region [6], or dust torus [7]. To date, the origin of soft photons in inverse Compton scattering remains uncertain [8].

Some researchers argue that short-timescale variability in blazars implies that the emission region lies within the BLR [9-10]. Under the standard one-zone leptonic model, the observed very-high-energy gamma-ray emission ($ $100 GeV) from FSRQs such as 3C 279 [11], 3C 454.3 [12], and 4C 21.35 [13] cannot be explained. Very-long-baseline interferometry [14] observations of radio emission have revealed bright regions in the radio band, known as radio cores, located far from the central black hole, even beyond the dust torus [15]. Tan et al. [16] compiled 60 FSRQs with multi-wavelength quasi-simultaneous data and, using a one-zone model that considered seed photon contributions from both the BLR and DT, found that the high-energy radiation of FSRQs originates between the BLR and DT. Harvey et al. [17] collected 62 FSRQs with multi-wavelength quasi-simultaneous data and, within the leptonic model framework, introduced a parameter to calculate the peak frequencies and fluxes of the double-peaked structure based on observed multi-band SEDs to identify the dominant soft photon field in the EC process of blazars. This parameter is called the seed factor (SF). Using bootstrapping, they found that the observed seed factor distribution statistically matches the expected distribution from the DT but not from the BLR, indicating that the soft photon field in the IC mechanism is primarily contributed by the DT.

In summary, the origin of soft photons in IC radiation and the location of the emission region remain uncertain. In this paper, we analyze the origin of soft photons in the IC mechanism from a statistical perspective using a large sample with simultaneous data.

1 Sample

We collected 69 flat-spectrum radio quasars with multi-wavelength quasi-simultaneous data from the Fermi 4-LAC catalog of active galactic nuclei [18] and an additional 6 FSRQs with multi-wavelength quasi-simultaneous data from the literature [19], totaling 75 FSRQs. Our sample satisfies the following criteria: (1) known redshift; (2) coverage from ultraviolet to gamma-ray bands (excluding radio data) with observation times across bands intersecting within 2 months, enabling construction of double-peaked SEDs; (3) available bulk Lorentz factors [20] or Doppler factors [19]. In this work, we assume the bulk Lorentz factor is approximately equal to the Doppler factor [21], and the magnetic field (B) parameters are obtained from model fitting in [16].

Our sample includes one source with a relatively high peak frequency (PKS 2141+175, $\nu_{\rm syn} = 10^{14.11}$ Hz).

We used the ASDC SED Builder tool[1] to fit SEDs from the Fermi sample [22] and Python to fit SEDs constructed from data extracted from [19]. We fitted both the low-energy and high-energy peaks of the blazar SEDs using a quadratic function of the form [23]:

$$\log(\nu F_\nu) = a(\log \nu)^2 + b(\log \nu) + c$$

where a, b, and c are constants. The peak frequencies and integrated luminosities obtained from fitting the observational data are in the observer' s frame. To remove redshift effects, we transform these quantities to the AGN rest frame using:

$$\nu_{\rm rest} = \nu_{\rm obs}(1+z)$$

$$L_{\rm rest} = 4\pi D_L^2 (\nu F_\nu)_{\rm obs}(1+z)^{-1}$$

In equation (2), $\nu_{\rm obs}$ is the observed peak frequency of either the synchrotron or inverse Compton component, and z is the redshift. In equation (3), D_L is the luminosity distance, and $(\nu F_\nu)_{\rm obs}$ is the observed integrated flux of the synchrotron or inverse Compton peak.

presents our sample of 75 sources, with columns for: (1) source name; (2) redshift; (3) (quasi-)simultaneous observation period (data from [19] lack time information); (4) logarithm of synchrotron peak frequency (in Hz); (5) logarithm of synchrotron peak flux (in $\mathrm{erg\,cm^{-2}\,s^{-1}}$); (6) logarithm of synchrotron peak luminosity (in $\mathrm{erg\,s^{-1}}$); (7) logarithm of IC peak frequency (in Hz); (8) logarithm of inverse Compton peak flux (in $\mathrm{erg\,cm^{-2}\,s^{-1}}$); (9) logarithm of inverse Compton peak luminosity (in $\mathrm{erg\,s^{-1}}$); (10) magnetic field energy density (in Gauss); (11) bulk Lorentz factor (Doppler factor indicated by superscript a); (12) observed seed factor.

2.1 Ratio of Synchrotron to Inverse Compton Peak Flux and Doppler Factor

Dermer et al. [24] analyzed a homogeneous one-zone leptonic model, assuming electrons are injected into a relativistically moving plasma blob with a truncated power-law spectrum that evolves over time. All multi-wavelength radiation originates from this compact emission region moving at relativistic speed. They found that if the IC process is dominated by EC, the relationship between EC flux and synchrotron peak flux approximates:

$$\frac{L_{\rm IC}}{L_{\rm syn}} \propto \frac{U_{\rm ext}}{U_B}\delta^{2+2\alpha}$$

where $U_B = B^2/8\pi$ is the magnetic field energy density, $U_{\rm ext}$ is the external photon energy density, δ is the Doppler factor, and α is the photon spectral index. Assuming $U_{\rm ext}$ is constant [17], the constant term typically disperses data but does not affect correlation analysis. Therefore, equation (4) predicts a correlation between the ratio of IC peak flux to synchrotron peak flux and the Doppler factor.

If the IC process is dominated by SSC, the ratio of SSC-produced radiation to synchrotron radiation can be expressed as [24]:

$$\frac{L_{\rm SSC}}{L_{\rm syn}} \propto \frac{\sigma_T u_e R}{3c} \delta^{3+2\alpha} \left(\frac{\nu_{\rm syn}^{\rm max}}{\nu_{\rm syn}}\right)^{\alpha} \left(\frac{\nu_{\rm IC}^{\rm max}}{\nu_{\rm IC}}\right)^{-\alpha} \log\left(\frac{\nu_{\rm IC}}{\nu_{\rm syn}}\right)$$

where σ_T is the Thomson scattering cross-section, u_e is the normalization factor of the non-thermal radiating electron number density, R is the radius of the emission region, $\nu_{\rm syn}^{\rm max}$ is the upper limit defining the synchrotron component, $\nu_{\rm IC}^{\rm max}$ is the upper limit defining the high-energy Compton component, and $\log(\nu_{\rm IC}/\nu_{\rm syn})$ is the Compton-to-synchrotron logarithmic conversion factor. In this case, we expect no correlation between the ratio of IC peak flux to synchrotron peak flux and the Doppler factor.

[Figure 1: see original paper] shows the correlation between $\log(L_{\rm IC}/L_{\rm syn})$ and $\log\delta$ for the entire sample. Spearman' s test yields a confidence level of $p = 0.0003$ (threshold $p < 0.01$) and correlation coefficient $R = 0.470$. Linear regression using the least squares method gives $\log(L_{\rm IC}/L_{\rm syn}) = (1.12 \pm 0.28)\log\delta + (0.18 \pm 0.08)$. These results indicate that the IC mechanism in FSRQs is dominated by the EC process.

According to [25], the synchrotron peak frequency may be related to the maximum energy that particles can be accelerated to. For the commonly used LSP/ISP/HSP classification [26], FSRQs statistically belong mostly to LSPs, with synchrotron peak frequencies concentrated around $10^{12.5} - 10^{14.5}$ Hz, and inverse Compton scattering occurs in the Thomson regime. In this work, we investigate whether different synchrotron peak frequencies in our sample correspond to physical differences, dividing our sample into two subsamples based on the synchrotron peak frequency distribution around $10^{13.05}$ Hz: a relatively low synchrotron peak frequency population ($12.10 < \log\nu_{\rm syn} < 13.05$, containing 38 sources, hereafter "low-frequency population") and a relatively high synchrotron peak frequency population ($13.05 < \log\nu_{\rm syn} < 14.11$, containing 37 sources, hereafter "high-frequency population").

[Figure 2: see original paper] shows the correlation between $\log(L_{\rm IC}/L_{\rm syn})$ and $\log\delta$ for both populations. For the low-frequency population, Spearman' s test yields $p = 0.013$ and $R = 0.520$, with linear regression giving $\log(L_{\rm IC}/L_{\rm syn}) = (1.31 \pm 0.48)\log\delta + (0.03 \pm 0.14)$. For the high-frequency population, Spearman' s test yields $p = 0.008$ and $R = 0.433$, with linear regression giving $\log(L_{\rm IC}/L_{\rm syn}) = (0.93 \pm 0.33)\log\delta + (0.30 \pm 0.10)$. These results show no significant difference from the entire sample, confirming that the IC mechanism in FSRQs is dominated by the EC process.

2.2 Seed Factor

Our previous analysis revealed that IC radiation in FSRQs is dominated by the EC process, which involves multiple possible origins for soft photons, including the accretion disk (AD), broad-line region (BLR), and dust torus (DT). To further investigate the origin of these soft photons, we apply the bootstrapping method proposed by Harvey et al. [17] to calculate the observed seed factor distributions for the low- and high-frequency populations (values listed in Table 1). The seed factor is calculated as [17]:

$$\log(\mathrm{SF}) = \log\left(\frac{U_{\mathrm{ext}}}{10^{-3}}\right) + \log\left(\frac{\epsilon_{\mathrm{ext}}}{10^{-5}}\right) + \log(\mathrm{CD}) - \log\left(\frac{\nu_{\mathrm{syn}}}{10^{13}}\right) - \log\left(\frac{\nu_{\mathrm{IC}}}{10^{21}}\right)$$

where U_{ext} is the external photon energy density of the jet (in CGS units), ϵ_{ext} is the characteristic energy of external photons (in units of electron rest energy), CD is the Compton dominance ($L_{\mathrm{IC}}/L_{\mathrm{syn}}$, i.e., the ratio of IC peak luminosity to synchrotron peak luminosity, in units of 10), ν_{syn} is the synchrotron peak frequency (in Hz), and ν_{IC} is the IC peak frequency (in Hz). The theoretically expected seed factor distributions for the BLR and DT are estimated using external photon fields and characteristic energies in equation (6). We adopt the ranges provided by Harvey et al. [17]: $-0.5 < \log(\mathrm{SF}_{\mathrm{BLR}}) < 0.5$ and $0.5 < \log(\mathrm{SF}_{\mathrm{DT}}) < 1.5$.

We performed normality tests on the seed factor distributions of each sample using the two-sided Kolmogorov-Smirnov test. We found that the normality hypothesis cannot be rejected for either the low- or high-frequency populations. In the low-frequency population, the significance for rejecting normality is 0.37σ ($p = 0.71$), while in the high-frequency population it is 0.01σ ($p = 0.99$), both below the 2σ confidence level. Therefore, these seed factor distributions follow a normal distribution, consistent with Harvey et al. [17].

Using bootstrapping, we constructed large samples of observed seed factors from the low- and high-frequency populations (38×10^4 for the low-frequency population and 37×10^4 for the high-frequency population). From these large samples, we generated 10^4 surrogate samples containing the same number of sources. Using these surrogate samples, we estimated the median ($\tilde{x}$) and 2σ confidence intervals for the observed seed factor distributions in different populations:

Low-frequency population: $\tilde{x} = 0.68^{+0.32}_{-0.31}$
High-frequency population: $\tilde{x} = 1.78^{+0.33}_{-0.32}$

[Figure 3: see original paper] shows histograms of the seed factor distributions for both populations, with theoretically expected seed factor ranges for the BLR and DT marked. The histogram peaks appear near the dust torus location.

Histograms do not incorporate estimation uncertainties. To include these uncertainties, we performed kernel density estimation (KDE) for the low- and high-frequency populations using bootstrapping. We applied Silverman' s Rule

[27] for automatic bandwidth smoothing in the KDE [17]. [Figure 4: see original paper] presents the KDE results for both populations. In the left panel (low-frequency population), the observed seed factor distribution intersects with the DT distribution, suggesting that soft photons are likely dominated by the dust torus. In the right panel (high-frequency population), the observed seed factor distribution does not intersect with the theoretical seed factor distributions, indicating that soft photons may not be dominated solely by either the BLR or DT.

To further test whether the observed seed factor distributions correspond to the BLR or DT locations, we calculated the significance of rejecting median equality between the observed seed factor distributions and theoretically expected distributions for both populations. In terms of standard deviations, for the low-frequency population, the rejection significance is:

BLR: $(2.87\sigma, p = 0.004)$
DT: $(0.32\sigma, p = 0.75)$

For the high-frequency population, the rejection significance is:

BLR: $(6.12\sigma, p < 0.001)$
DT: $(2.14\sigma, p = 0.03)$

Thus, the low-frequency population shows the smallest rejection significance when compared to the theoretical DT seed factor distribution, indicating that the origin of soft photons in this population is most likely dominated by the dust torus. In the high-frequency population, although the rejection significance with the DT distribution is smaller than with the BLR distribution, the KDE shows clear differences, suggesting that the dust torus may not be the sole source of soft photons and that both BLR and DT may contribute.

3 Discussion and Summary

Our study finds no significant difference between the entire sample and the two equally divided subsamples regarding the correlation between $\log(L_{\text{IC}}/L_{\text{syn}})$ and $\log \delta$, as shown in [Figure 1: see original paper] and [Figure 2: see original paper]. This indicates that under the leptonic model framework, the IC mechanism in FSRQs is likely dominated by the EC process. [Figure 1: see original paper] includes 75 FSRQs, while the left and right panels of [Figure 2: see original paper] include 38 and 37 FSRQs, respectively. Both figures show some dispersion in the data distributions, which may weaken our conclusions to some extent. With the addition of more observational facilities in the future, larger samples can be collected to further verify the dominant high-energy radiation process under the leptonic model framework.

Seed factor calculation is based on multi-band SED fitting, requiring that broadband radiation shows no significant variability [17]. We applied the bootstrapping method proposed by Harvey et al. [17] to estimate the observed seed factor distribution for our sample. The histogram in [Figure 3: see original paper] shows the observed seed factor distribution without incorporating estimation uncertainties, while the KDE in [Figure 4: see original paper] includes these uncertainties. The histogram reveals that the observed seed factor distribution is concentrated near the peak frequency, a result also shown by the KDE curves in [Figure 4: see original paper], which additionally illustrate differences between our sample' s observed seed factor distribution and Harvey et al.' s theoretically expected distributions.

As shown in [Figure 4: see original paper] (left panel), the observed seed factor distribution statistically intersects with the DT seed factor distribution range, indicating that soft photons in the low-frequency population are likely dominated by the dust torus. In [Figure 4: see original paper] (right panel), the observed seed factor distribution exceeds the theoretically expected ranges for both the BLR and DT seed factor distributions, suggesting that in the high-frequency population, the DT is likely not the sole dominant source of soft photons and that both BLR and DT may contribute. Harvey et al. found that the emission region of FSRQs statistically coincides with the DT location, indicating that soft photons are primarily contributed by the DT. In contrast, our sample shows different results across different populations. The histograms in [Figure 3: see original paper] contain only 38 and 37 seed factors, respectively, which is statistically limited and may weaken our conclusions. We hope that future studies will increase the sample of simultaneous data and use observational quantities obtained through different methods for further analysis and verification.

This paper statistically investigates the origin of soft photons in 75 flat-spectrum radio quasars. Correlation analysis indicates that the IC mechanism in FSRQs is likely dominated by the EC process. Using seed factor distributions to further examine the origin of soft photons in the EC process reveals statistically different results across different populations.

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Note: Figure translations are in progress. See original paper for figures.