# A Necessary and Sufficient Condition of Positive Definiteness for 4th Order Symmetric Tensors Defined in Particle Physics

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**Date:** 2024-09-04T00:00:00+00:00

## Abstract

In this paper, we mainly discuss analytical expressions of positive definiteness for a special 4th order 3-dimensional symmetric tensor defined by the constructed model for a physical phenomenon. Firstly, an analytically necessary and sufficient conditions of 4th order 2-dimensional symmetric tensors are given to test its positive definiteness. Furthermore, by means of such a result, a necessary and sufficient condition of positive definiteness is obtained for a special 4th order 3-dimensional symmetric tensor. Such an analytical conditions can be used for verifying the vacuum stability of general scalar potentials of two real singlet scalar fields and the Higgs boson. The positive semi-definiteness conclusions are presented too.

## Full Text

## Preamble

**Boundedness from below conditions for a general scalar potential of two real scalar fields and the Higgs boson**

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## Abstract

The most general scalar potential of two real scalar fields and a Higgs boson is a quartic homogeneous polynomial in three variables, which defines a 4th-order 3-dimensional symmetric tensor. Hence, the boundedness from below of such a scalar potential involves the positive (semi-)definiteness of the corresponding tensor. In this paper, we mainly discuss analytic expressions of positive (semi-)definiteness for such a special tensor. First, an analytically necessary and sufficient condition is given to test the positive (semi-)definiteness of a 4th-order 2-dimensional symmetric tensor. Furthermore, by means of such a result, the analytic necessary and sufficient conditions of the boundedness from below are obtained for a general scalar potential of two real scalar fields and the Higgs boson.


**Keywords:** scalar potentials; boundedness from below; 4th order tensors; positive definiteness; homogeneous polynomial; analytical expression.

## Introduction

The boundedness from below of a scalar potential makes physical sense, which simply implies that such a scalar potential is positive (or non-negative). The polynomial degree of the potential is 4 when one keeps the scalar interactions renormalizable [**?**, **?**]. Then the condition for the potential of $n$ real scalar fields $\phi_i$ $(i = 1, 2, \cdots, n)$ to be bounded from below in the strong sense is equivalent to the requirement that $V(\phi) = \sum_{i,j,k,l=1}^{n} v_{ijkl}\phi_i\phi_j\phi_k\phi_l > 0$.

Let $V = (v_{ijkl})$. Then $V$ is a 4th-order symmetric tensor, and hence, the above requirement is the positive definiteness of the tensor $V$. Qi [**?**, **?**] first used and introduced the positive definiteness and copositivity of tensors. An $m$th-order $n$-dimensional real tensor $V = (v_{i_1 i_2 \cdots i_m})$ is said to be (i) positive semi-definite if $Vx^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} v_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \geq 0$ for all $x \in \mathbb{R}^n$ and an even number $m$; (ii) positive definite if $Vx^m > 0$ for all $x \in \mathbb{R}^n \ \{0\}$ and an even number $m$; (iii) copositive if $Vx^m \geq 0$ for all $x \geq 0$; (iv) strictly copositive if $Vx^m > 0$ for all $x \geq 0$ and $x \neq 0$.

Kannike [**?**, **?**, **?**] presented the vacuum stability conditions of general scalar potentials of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs boson $H$, and studied the boundedness from below condition for scalar potential of the Standard Model (for short, SM) Higgs $H_1$, an inert doublet $H_2$ and a complex singlet $S$. In fact, such two problems were solved by Kannike [**?**], where the first problem involves the positive definiteness of the corresponding symmetric tensor and the second problem requires the copositivity of the corresponding symmetric tensor. Chauhan [**?**] gave an analytical vacuum stability condition of the left-right symmetric model for successful symmetry breaking. Ivanov [**?**] presented the stability conditions in multi-Higgs potentials. Bahl et al. [**?**] provided the analytically sufficient conditions of the vacuum stability for the two-Higgs-doublet potential with CP conservation, and showed a vacuum stability condition for the two-Higgs-doublet potential with CP violation depends on the Lagrange multiplier $\zeta$. Recently, Song [?] established the boundedness from below conditions of scalar potential for the two-Higgs-doublet with explicit CP conservation. Song [?] obtained the boundedness from below conditions of scalar potential for a general two-Higgs-doublet, which includes necessary conditions and sufficient conditions. Also see Faro-Ivanov [?], Belanger-Kannike-Pukhov-Raidal [?, ?], Ivanov-Köpke-Mühlleitner [?] for more details. In Refs. [?, ?], one can construct only one quadratic term and five quartic terms for the Higgs potential with the help of three Higgs doublets with equal electroweak quantum numbers, which is a quartic polynomial with real coefficients defined on complex field. Toorop-Bazzocchi-Merlo-Paris [?, ?] and Degee-Ivanov-Keus [?] turned such a polynomial from complex field to real field. In fact, they were trying to look for the analytical condition of such a polynomial to be positive.

Recently, Song-Qi [?] and Liu-Song [?] gave a different sufficient condition of copositivity for 4th-order 3-dimensional symmetric tensors to find the boundedness from below conditions of scalar potential of the SM Higgs $H_1$, an inert doublet $H_2$ and a complex singlet $S$. Very recently, Qi-Song-Zhang [?] presented a necessary and sufficient condition of copositivity for such a tensor given by the above particle physical model. Song-Li [?] provided an analytically necessary and sufficient condition of the boundedness from below for such a scalar potential model.

In the past decades, many numerical algorithms were established to find some H-(Z-)eigenvalues of a tensor [?, ?], and were applied to test the positive definiteness of such an even order tensor by means of the sign of the smallest H-(Z-)eigenvalue. On the other hand, some classes of tensors with special structure may be determined directly their positive definiteness such as Hilbert tensor [?], diagonal dominant tensor [?], B-tensor [?] and others. However, the practical matters such as the vacuum stability of general scalar potentials of a few fields require analytical expressions. The most general scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$ (Kannike [?, ?, ?]) is

$$V(\phi_1,\phi_2,|H|)=\lambda_H|H|^4+\lambda_{H20}|H|^2\phi_2+\lambda_{40}\phi_1^4+\lambda_{31}\phi_1^3+\lambda_{H11}|H|^2\phi_1\phi_2+\lambda_{H02}|H|^2\phi_2^2+\lambda_{13}\phi_1\phi_2^3+\lambda_{04}\phi_4$$

Clearly, such a quartic homogeneous polynomial defines a 4th-order 3-dimensional symmetric tensor $V=(v_{ijkl})$, with $v_{1111}=\lambda_{40}$, $v_{2222}=\lambda_{04}$, $v_{3333}=\lambda_H$, $v_{1112}=v_{1133}=v_{1233}=\lambda_{H20}$, $v_{1122}=\lambda_{22}$, $v_{2233}=\lambda_{H02}$, $\lambda_{H11}$, and $v_{ijkl}=0$ for the others. Hence, the boundedness from below of such a scalar potential is equivalent to the positive (semi-)definiteness of such a tensor $V$. So this requires an analytic condition of positive (semi-)definiteness.

For a 4th-order 2-dimensional symmetric tensor, the analytical condition of the positive definiteness traced back to ones of Refs. Gadem-Li [?], Ku [?] and Jury-Mansour [?]. Wang-Qi [?] improved their proof and conclusions. However, the above result depends on the discriminant of such a polynomial. Recently, Guo

[?] showed a new necessary and sufficient condition without the discriminant. Very recently, Qi-Song-Zhang [?] gave a new necessary and sufficient condition other than the above results. Hasan-Hasan [?] claimed that a necessary and sufficient condition of positive definiteness was proved without the discriminant. However, there is a problem in their argumentations. In 1998, Fu [?] pointed out that Hasan-Hasan's results are sufficient only. Song [?] gave several analytically sufficient conditions of the positive definiteness of 4th-order 3-dimensional symmetric tensor. Until now, people have not found an analytic necessary and sufficient condition of positive (semi-)definiteness for a 4th-order 3-dimensional symmetric tensor.

In this paper, we mainly concentrate on the analytic expressions of positive (semi-)definiteness for a special 4th-order tensor given by the scalar potential. More precisely, by means of Qi-Song-Zhang's result, we first show an analytic necessary and sufficient condition of positive (semi-)definiteness of 4th-order 2-dimensional symmetric tensors. Secondly, with the help of this conclusion, we discuss the analytic expressions of positive (semi-)definiteness of a 4th-order 3-dimensional symmetric tensor defined by the potential. Then these analytic conditions are the necessary and sufficient conditions of the boundedness from below for a scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$.

## 3 Non-negativeness of Quadratic and Quartic Polynomials

Let $P_2(t)$ be a quadratic polynomial, $P_2(t) = at^2 + bt + c$, with $a > 0$. Then the following results should be well-known, which was shown hundreds of years ago: (1) $P_2(t) > 0$ for all $t \geq 0$ if and only if $b \geq 0$ and $c > 0$, or $b < 0$ and $4ac - b^2 > 0$. (2) $P_2(t) \geq 0$ for all $t \geq 0$ if and only if $b \geq 0$ and $c \geq 0$, or $b < 0$ and $4ac - b^2 \geq 0$.

Let $P_4(t)$ be a quartic polynomial, $P_4(t) = at^4 + bt^3 + ct^2 + dt + e$, where $a > 0$ and $e > 0$. Then the positivity (non-negativeness) of $P_4(t)$ were proved by Qi-Song-Zhang [?], recently: (3) $P_4(t) \geq 0$ for all $t \in \mathbb{R}$ if and only if $e - d\sqrt{a} \leq 4\sqrt{ace} + 2ae$ and $\Delta \geq 0$, $|b\sqrt{e}| \leq c \leq 6\sqrt{ae}$ and $|b\sqrt{c} > 6e + d\sqrt{a}| \leq 4\sqrt{ace} - 2ae$, where $\Delta = 4(12ae - 3bd + c^2)^3 - (72ace + 9bcd - 2c^3 - 27ad^2 - 27b^2e)^2$. (4) $P_4(t) > 0$ for all $t \in \mathbb{R}$ if and only if $\Delta = 0$, $b\sqrt{e} = d\sqrt{a}$, $e - d\sqrt{a} \leq 4\sqrt{ace} + 2ae$ and $|b\sqrt{a}| \leq c \leq 6\sqrt{ae}$, or $\Delta > 0$, $|b\sqrt{a}| \leq c \leq 6\sqrt{ae}$ and $|b\sqrt{c} > 6\sqrt{a}e + d\sqrt{a}| \leq 4\sqrt{ace} - 2ae$.

## 4 Boundedness from Below of Scalar Potential of Two Real Scalar Fields and a Higgs Boson

The most general scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$ (Kannike [?, ?]) is

$$V(\phi_1, \phi_2, |H|) = \lambda_H |H|^4 + \lambda_{H20} |H|^2 \phi_2 + \lambda_{40} \phi_1^4 + \lambda_{31} \phi_1^3 + \lambda_{H11} |H|^2 \phi_1 \phi_2 + \lambda_{H02} |H|^2 \phi_2^2 + \lambda_{13} \phi_1 \phi_2^3 + \lambda_{04} \phi_4$$

This scalar potential defines a 4th-order 3-dimensional symmetric tensor $V = (v_{ijkl})$ with its entries $v_{1111} = \lambda_{40}$, $v_{2222} = \lambda_{04}$, $v_{3333} = \lambda_H$, $v_{1112} = \lambda_{31}$, $v_{1222} = \lambda_{H20}$, $v_{1122} = \lambda_{22}$, $v_{2233} = \lambda_{H02}$, $\lambda_{H11}$, $v_{1133} = v_{1233} = \lambda_{H20}$, and $v_{ijkl} = 0$ for the others. In this section, we mainly discuss analytical expressions of positive definiteness of the 4th-order tensor $V = (v_{ijkl})$ given by the potential. Furthermore, we present a necessary and sufficient condition of the boundedness from below of scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$.

### 4.1 Positive Definiteness of 4th-Order 2-Dimensional Symmetric Tensors

Let $V = (v_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor with $v_{1111} > 0$ and $v_{2222} > 0$. For a vector $x = (x_1, x_2)^\top$ such that $\|x\| = \sqrt{x_1^2 + x_2^2} = 1$, we may assume $x_2 \neq 0$ without loss of generality. We have

$$Vx^4 = \sum_{i,j,k,l=1}^{2} v_{ijkl}x_ix_jx_kx_l = v_{1111}x_1^4+4v_{1112}x_1^3x_2+6v_{1122}x_1^2x_2^2+4v_{1222}x_1x_2^3+v_{2222}x_2^4$$

and hence,

$$Vx^4 = v_{1111}\left(\frac{x_1}{x_2}\right)^4 + 4v_{1112}\left(\frac{x_1}{x_2}\right)^3 + 6v_{1122}\left(\frac{x_1}{x_2}\right)^2 + 4v_{1222}\left(\frac{x_1}{x_2}\right) + v_{2222}$$

Clearly, $Vx^4 > 0$ if and only if $P(t) = at^4 + bt^3 + ct^2 + dt + e > 0$ for all $t \in \mathbb{R}$, where $a = v_{1111}$, $b = 4v_{1112}$, $c = 6v_{1122}$, $d = 4v_{1222}$, $e = v_{2222}$. The discriminant is

$$\Delta = 4(12ae - 3bd + c^2)^3 - (72ace + 9bcd - 2c^3 - 27ad^2 - 27b^2e)^2$$

$$= 4(12v_{1111}v_{2222}-48v_{1112}v_{1222}+36v_{1122}^2)^3-(72\times6v_{1111}v_{1122}v_{2222}+72\times12v_{1112}v_{1122}v_{1222}-72\times6v_{1122}^3-72\times6v_{1111}$$

$$= 4\times12^3(I^3 - 27J^2)$$

where

$$I = v_{1111}v_{2222} - 4v_{1112}v_{1222} + 3v_{1122}^2$$

$$J = v_{1111}v_{1122}v_{2222} + 2v_{1112}v_{1122}v_{1222} - v_{1122}^3 - v_{1111}v_{1222}^2 - v_{1112}^2 v_{2222}$$

and hence, the sign of $\Delta$ is the same as that of $(I^3 - 27J^2)$. So, it follows from the conditions for quartic polynomials that, by simply calculating, $V$ is positive definite, i.e., $Vx^4 > 0$ for all $x \in \mathbb{R}^2$ if and only if

$$\begin{cases} v_{1111} > 0, v_{2222} > 0 \\ \text{(i) } v_{1111}v_{2222} \leq 3v_{1122}^2 \text{ and } |v_{1112}\sqrt{v_{2222}} + v_{1222}\sqrt{v_{1111}}| \leq \sqrt{6v_{1111}v_{1122}v_{2222} + 2\sqrt{(v_{1111}v_{2222})^3}} \\ \text{(ii) } v_{1111}v_{2222} > 3v_{1122}^2 \text{ and } |v_{1112}\sqrt{v_{2222}} - v_{1222}\sqrt{v_{1111}}| \leq \sqrt{6v_{1111}v_{1122}v_{2222} - 2\sqrt{(v_{1111}v_{2222})^3}} \\ I^3 - 27J^2 = 0, v_{1112}^2 + v_{1222}^2 + 8v_{1111}v_{1122} = 4v_{1111}v_{22} < 24v_{1111}v_{1122}v_{2222} \\ \text{or } I^3 - 27J^2 > 0 \end{cases}$$

Similarly, it follows that $V = (v_{ijkl})$ is positive semi-definite, i.e., $Vx^4 \geq 0$ for all $x \in \mathbb{R}^2$ if and only if

$$\begin{cases} v_{1111} > 0, v_{2222} > 0 \\ \text{(i) } v_{1111}v_{2222} \leq 3v_{1122}^2 \text{ and } |v_{1112}\sqrt{v_{2222}} + v_{1222}\sqrt{v_{1111}}| \leq \sqrt{6v_{1111}v_{1122}v_{2222} + 2\sqrt{(v_{1111}v_{2222})^3}} \\ \text{(ii) } v_{1111}v_{2222} > 3v_{1122}^2 \text{ and } |v_{1112}\sqrt{v_{2222}} - v_{1222}\sqrt{v_{1111}}| \leq \sqrt{6v_{1111}v_{1122}v_{2222} - 2\sqrt{(v_{1111}v_{2222})^3}} \\ I^3 - 27J^2 \geq 0 \end{cases}$$

Next we give an analytically necessary and sufficient condition of the boundedness from below of scalar potential of two real scalar fields $\phi_1$ and $\phi_2$. The most general scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ may be written as (Kannike [**?**, **?**, **?**])

$$\bar{V}(\phi_1, \phi_2) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4$$

Let $V = (v_{ijkl})$ be the coupling tensor with its entries $v_{1111} = \lambda_{40}$, $v_{2222} = \lambda_{04}$, $v_{1112} = \lambda_{31}$, $v_{1122} = \lambda_{22}$, $v_{1222} = \lambda_{13}$. In fact, the boundedness from below of two real scalar fields $\phi_1$ and $\phi_2$ is equivalent to the positive definiteness of the coupling tensor $V = (v_{ijkl})$. Then we define

$$\Delta' = 4(12\lambda_{40}\lambda_{04} - 3\lambda_{31}\lambda_{13} + \lambda_{22}^2)^3 - (72\lambda_{40}\lambda_{22}\lambda_{04} + 9\lambda_{31}\lambda_{22}\lambda_{31} - 2\lambda_{22}^3 - 27\lambda_{40}\lambda_{13}^2 - 27\lambda_{31}^2\lambda_{04})^2$$

From the conditions above, the following results are easy to obtain. Let $\lambda_{40} > 0$, $\lambda_{04} > 0$. Then $\bar{V}(\phi_1, \phi_2) > 0$ if and only if

$$\begin{cases}\Delta' = 0, \lambda_{31}^2 + 8\lambda_{40}\lambda_{22} = 4\lambda_{40}\lambda_{22} < 24\lambda_{40}\lambda_{22}\lambda_{04} \\ \text{or } \Delta' > 0 \\ \text{(i) } \lambda_{22} > 6\sqrt{\lambda_{40}\lambda_{04}} \text{ and } |\lambda_{31}\sqrt{\lambda_{04}} + \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}} \\ \text{(ii) } \lambda_{22} \leq 6\sqrt{\lambda_{40}\lambda_{04}} \text{ and } |\lambda_{31}\sqrt{\lambda_{04}} - \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} - 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}\end{cases}$$

Similarly, $\bar{V}(\phi_1, \phi_2) \geq 0$ if and only if

$$\begin{cases}\Delta' \geq 0 \\ \text{(i) } \lambda_{22} > 6\sqrt{\lambda_{40}\lambda_{04}} \text{ and } |\lambda_{31}\sqrt{\lambda_{04}} + \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}} \\ \text{(ii) } \lambda_{22} \leq 6\sqrt{\lambda_{40}\lambda_{04}} \text{ and } |\lambda_{31}\sqrt{\lambda_{04}} - \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} - 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}\end{cases}$$

In fact, the analytical condition above gives the boundedness from below in the stronger sense for the scalar potential of two real scalar fields $\phi_1$ and $\phi_2$.

### 4.2 Boundedness from Below of Two Real Scalar Fields and a Higgs Boson

The most general scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$ (Kannike [**?**, **?**]) is

$$V(\phi_1, \phi_2, |H|) = \lambda_H|H|^4 + \lambda_{H20}|H|^2\phi_2 + \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3 + \lambda_{H11}|H|^2\phi_1\phi_2 + \lambda_{H02}|H|^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_4$$

which can be rewritten as

$$V(\phi_1, \phi_2, |H|) = \lambda_H|H|^4 + M(\phi_1, \phi_2)|H|^2 + \bar{V}(\phi_1, \phi_2)$$

where

$$M(\phi_1, \phi_2) = \lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2$$

$$\bar{V}(\phi_1, \phi_2) = V(\phi_1, \phi_2, 0) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4$$

Recently, Kannike [**?**, **?**] studied the boundedness from below of $V(\phi_1, \phi_2, |H|)$, and gave an analytical condition for $V(\phi_1, \phi_2, |H|) > 0$.

In this subsection, we will present the analytic conditions of positive (semi-)definiteness for this special 4th-order 3-dimensional symmetric tensor, and moreover, show the analytic necessary and sufficient conditions for the boundedness from below of scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$.

Let $x = (\phi_1, \phi_2, |H|)^\top$. Then $V(\phi_1, \phi_2, |H|) = Vx^4$, where $V = (v_{ijkl})$ is a 4th-order 3-dimensional symmetric tensor given by the potential. Clearly, the tensor corresponding to $\bar{V}(\phi_1, \phi_2)$ is a 4th-order 2-dimensional principal sub-tensor of $V$.

Let $\lambda_H > 0$. It follows from the equation above that

$$Vx^4 = \lambda_H |H|^4 + M(\phi_1, \phi_2)|H|^2 + \bar{V}(\phi_1, \phi_2)$$

which may be regarded as a quadratic polynomial with respect to $t = |H|^2$:

$$P_2(t) = at^2 + bt + c$$

where $a = \lambda_H$, $b = M(\phi_1, \phi_2)$, $c = \bar{V}(\phi_1, \phi_2)$.

From the quadratic polynomial conditions, it yields the following conclusion: $V(\phi_1, \phi_2, |H|) = Vx^4 > 0$ for all $\phi_1, \phi_2, H$ if and only if for all $\phi_1, \phi_2$,

$$\begin{cases} M(\phi_1, \phi_2) \geq 0 \text{ and } \bar{V}(\phi_1, \phi_2) > 0 \\ \text{or } M(\phi_1, \phi_2) < 0 \text{ and } 4\lambda_H \bar{V}(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2 > 0 \end{cases}$$

Similarly, $V(\phi_1, \phi_2, |H|) = Vx^4 \geq 0$ for all $\phi_1, \phi_2, H$ if and only if for all $\phi_1, \phi_2$,

$$\begin{cases} M(\phi_1, \phi_2) \geq 0 \text{ and } \bar{V}(\phi_1, \phi_2) \geq 0 \\ \text{or } M(\phi_1, \phi_2) < 0 \text{ and } 4\lambda_H \bar{V}(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2 \geq 0 \end{cases}$$

It is obvious that $M(\phi_1, \phi_2) = \lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2$ is a quadratic form with respect to two variables $\phi_1, \phi_2$, and hence, the inequality $M(\phi_1, \phi_2) \geq 0$ is equivalent to positive semi-definiteness of its coefficient matrix, which is equivalent to

$$\lambda_{H20} \geq 0, \quad \lambda_{H02} \geq 0, \quad 4\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 \geq 0$$

Similarly, the inequality $M(\phi_1, \phi_2) < 0$ is equivalent to negative definiteness of its coefficient matrix, i.e., the matrix $-\begin{pmatrix} \lambda_{H20} & \lambda_{H11} \\ \lambda_{H11} & \lambda_{H02} \end{pmatrix}$ is positive definite if and only if

$$\lambda_{H20} < 0, \quad \lambda_{H02} < 0, \quad 4\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 > 0$$

At the same time, the inequality $\bar{V}(\phi_1, \phi_2) > 0$ can be obtained by the condition for positive definiteness, and $\bar{V}(\phi_1, \phi_2) \geq 0$ can be obtained by the condition for positive semi-definiteness. Next we only need to show $4\lambda_H \bar{V}(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2 \geq 0$ (or $> 0$) for all $\phi_1, \phi_2$.

In order to prove this inequality, we define transformed coefficients:

$$\lambda'_{40} = 4\lambda_{40}\lambda_H - \lambda_{H20}^2, \quad \lambda'_{04} = 4\lambda_{04}\lambda_H - \lambda_{H02}^2$$

$$\lambda'_{31} = 4\lambda_H\lambda_{31} - 2\lambda_{H20}\lambda_{H11}, \quad \lambda'_{13} = 4\lambda_H\lambda_{13} - 2\lambda_{H02}\lambda_{H11}$$

$$\lambda'_{22} = 4\lambda_H\lambda_{22} - 2\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2$$

and

$$\Delta'' = 4(12\lambda'_{40}\lambda'_{04} - 3\lambda'_{31}\lambda'_{13} + \lambda'^2_{22})^3 - (72\lambda'_{40}\lambda'_{22}\lambda'_{04} + 9\lambda'_{31}\lambda'_{22}\lambda'_{31} - 2\lambda'^3_{22} - 27\lambda'_{40}\lambda'^2_{13} - 27\lambda'^2_{31}\lambda'_{04})^2$$

Let $V'(\phi_1, \phi_2) = 4\lambda_H \bar{V}(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2$. We may expand the polynomial $V'(\phi_1, \phi_2)$ as follows:

$$V'(\phi_1, \phi_2) = 4\lambda_H \bar{V}(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2$$

$$= (4\lambda_{40}\lambda_H - \lambda_{H20}^2)\phi_1^4 + (4\lambda_H\lambda_{31} - 2\lambda_{H20}\lambda_{H11})\phi_1^3\phi_2 + (4\lambda_H\lambda_{22} - 2\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2)\phi_1^2\phi_2^2$$

$$+(4\lambda_H\lambda_{13} - 2\lambda_{H02}\lambda_{H11})\phi_1\phi_2^3 + (4\lambda_{04}\lambda_H - \lambda_{H02}^2)\phi_2^4$$

$$= \lambda'_{40}\phi_1^4 + \lambda'_{31}\phi_1^3\phi_2 + \lambda'_{22}\phi_1^2\phi_2^2 + \lambda'_{13}\phi_1\phi_2^3 + \lambda'_{04}\phi_2^4$$

So this defines a 4th-order 2-dimensional symmetric tensor $V = (v_{ijkl})$ with its entries $v_{1111} = \lambda'_{40}$, $v_{2222} = \lambda'_{04}$, $v_{1112} = \lambda'_{31}$, $v_{1122} = \lambda'_{22}$, $v_{1222} = \lambda'_{13}$.

Let $\lambda'_{40} > 0$, $\lambda'_{04} > 0$. From the conditions for 2D tensors, we easily obtain the following conclusions: $V'(\phi_1, \phi_2) = 4\lambda_H \bar{V}(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2 > 0$ if and only if

$$\begin{cases}\Delta''=0,\lambda_{31}'^2+8\lambda_{40}'\lambda_{22}'=4\lambda_{40}'\lambda_{22}'<24\lambda_{40}'\lambda_{22}'\lambda_{04}'\\ \text{or } \Delta''>0\\ \text{(i) } \lambda_{22}'>6\sqrt{\lambda_{40}'\lambda_{04}'} \text{ and } |\lambda_{31}'\sqrt{\lambda_{04}'}+\lambda_{13}'\sqrt{\lambda_{40}'}|\leq 4\sqrt{\lambda_{40}'\lambda_{22}'\lambda_{04}'+2\lambda_{40}'\lambda_{04}'\sqrt{\lambda_{40}'\lambda_{04}'}}\\ \text{(ii) } \lambda_{22}'\leq 6\sqrt{\lambda_{40}'\lambda_{04}'} \text{ and } |\lambda_{31}'\sqrt{\lambda_{04}'}-\lambda_{13}'\sqrt{\lambda_{40}'}|\leq 4\sqrt{\lambda_{40}'\lambda_{22}'\lambda_{04}'-2\lambda_{40}'\lambda_{04}'\sqrt{\lambda_{40}'\lambda_{04}'}}\end{cases}$$

If $\lambda_{40}'>0$ and $\lambda_{04}'>0$, then from the semi-definiteness condition we easily yield: $V'(\phi_1,\phi_2)=4\lambda_H\bar{V}(\phi_1,\phi_2)-(M(\phi_1,\phi_2))^2\geq 0$ if and only if

$$\begin{cases}\Delta''\geq 0\\ \text{(i) } \lambda_{22}'>6\sqrt{\lambda_{40}'\lambda_{04}'} \text{ and } |\lambda_{31}'\sqrt{\lambda_{04}'}+\lambda_{13}'\sqrt{\lambda_{40}'}|\leq 4\sqrt{\lambda_{40}'\lambda_{22}'\lambda_{04}'+2\lambda_{40}'\lambda_{04}'\sqrt{\lambda_{40}'\lambda_{04}'}}\\ \text{(ii) } \lambda_{22}'\leq 6\sqrt{\lambda_{40}'\lambda_{04}'} \text{ and } |\lambda_{31}'\sqrt{\lambda_{04}'}-\lambda_{13}'\sqrt{\lambda_{40}'}|\leq 4\sqrt{\lambda_{40}'\lambda_{22}'\lambda_{04}'-2\lambda_{40}'\lambda_{04}'\sqrt{\lambda_{40}'\lambda_{04}'}}\end{cases}$$

If $\lambda_{40}'=0$, then $V'(\phi_1,\phi_2)=\phi_2(\lambda_{31}'\phi_1^3+\lambda_{22}'\phi_1^2\phi_2+\lambda_{13}'\phi_1\phi_2^2+\lambda_{04}'\phi_2^3)$, and hence, $V'(\phi_1,\phi_2)\geq 0$ for all $\phi_1,\phi_2\geq 0$ is equivalent to $V''(\phi_1,\phi_2)=\lambda_{31}'\phi_1^3+\lambda_{22}'\phi_1^2\phi_2+\lambda_{13}'\phi_1\phi_2^2+\lambda_{04}'\phi_2^3\geq 0$. So an application of Schmidt-Heß [**?**, **?**] or Liu-Song [**?**, **?**] yields $V''(\phi_1,\phi_2)\geq 0$ for all $\phi_1,\phi_2\geq 0$ if and only if

$$\begin{cases}\lambda_{31}'\geq 0,\lambda_{31}'\lambda_{13}'^3+4\lambda_{04}'\geq 0 \text{ and } \lambda_{04}'\lambda_{22}'^3+27\lambda_{22}'^2\geq 0,\lambda_{31}'\lambda_{13}'^2\geq 0\\ \text{or } \lambda_{04}'-18\lambda_{31}'\lambda_{13}'\lambda_{22}'-\lambda_{22}'^2\lambda_{13}'^2\geq 0\end{cases}$$

Similarly, if $\lambda_{04}'=0$, then $V'(\phi_1,\phi_2)\geq 0$ for all $\phi_1,\phi_2\geq 0$ if and only if

$$\begin{cases}\lambda_{13}'\geq 0,\lambda_{13}'\lambda_{31}'^3+4\lambda_{40}'\geq 0,\lambda_{40}'\lambda_{22}'^3+27\lambda_{13}'^2\lambda_{22}'^2\geq 0,\lambda_{40}'-18\lambda_{31}'\lambda_{13}'\lambda_{22}'\geq 0\\ \text{or } \lambda_{40}'-\lambda_{22}'^2\lambda_{31}'^2\geq 0\end{cases}$$

However, for all $\phi_1,\phi_2\in\mathbb{R}$, both $V'(\phi_1,\phi_2)\geq 0$ and $V''(\phi_1,\phi_2)\geq 0$ may not hold.

Altogether, combining the conditions for $\bar{V}$, $M$, $V'$, and the inequalities relating them, the analytical necessary and sufficient condition is established for the boundedness from below in the stronger sense of scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$, which also gives the analytic expressions of positive definiteness of a 4th-order 3-dimensional symmetric tensor $V$ defined by the potential.

Let $\lambda_H>0$, $\lambda_{40}>0$ and $\lambda_{04}>0$. Then $V(\phi_1,\phi_2,|H|)>0$ for all $\phi_1,\phi_2,H$ if and only if

$$\begin{cases} \lambda_{40} > 0, \lambda_{04} > 0 \\ \Delta' = 0, \lambda_{31}^2 + 8\lambda_{40}\lambda_{22} = 4\lambda_{40}\lambda_{22} < 24\lambda_{40}\lambda_{22}\lambda_{04} \\ \text{or } \Delta' > 0 \\ \text{(i) } \lambda_{22} > 6\sqrt{\lambda_{40}\lambda_{04}} \text{ and } |\lambda_{31}\sqrt{\lambda_{04}} + \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}} \\ \text{(ii) } \lambda_{22} \leq 6\sqrt{\lambda_{40}\lambda_{04}} \text{ and } |\lambda_{31}\sqrt{\lambda_{04}} - \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} - 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}} \\ 4\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 \geq 0 \\ \lambda'_{40} > 0, \lambda'_{04} > 0 \\ \Delta'' = 0, \lambda_{31}'^2 + 8\lambda'_{40}\lambda'_{22} = 4\lambda'_{40}\lambda'_{22} < 24\lambda'_{40}\lambda'_{22}\lambda'_{04} \\ \text{or } \Delta'' > 0 \\ \text{(i) } \lambda'_{22} > 6\sqrt{\lambda'_{40}\lambda'_{04}} \text{ and } |\lambda'_{31}\sqrt{\lambda'_{04}} + \lambda'_{13}\sqrt{\lambda'_{40}}| \leq 4\sqrt{\lambda'_{40}\lambda'_{22}\lambda'_{04} + 2\lambda'_{40}\lambda'_{04}\sqrt{\lambda'_{40}\lambda'_{04}}} \\ \text{(ii) } \lambda'_{22} \leq 6\sqrt{\lambda'_{40}\lambda'_{04}} \text{ and } |\lambda'_{31}\sqrt{\lambda'_{04}} - \lambda'_{13}\sqrt{\lambda'_{40}}| \leq 4\sqrt{\lambda'_{40}\lambda'_{22}\lambda'_{04} - 2\lambda'_{40}\lambda'_{04}\sqrt{\lambda'_{40}\lambda'_{04}}} \end{cases}$$

Combining the conditions for semi-definiteness, the analytical necessary and sufficient condition is built for the boundedness from below of such a scalar potential, which also gives the analytic condition of positive semi-definiteness of a 4th-order 3-dimensional symmetric tensor $V$ defined by the potential.

Let $\lambda_H > 0$, $\lambda_{40} > 0$ and $\lambda_{04} > 0$. Then $V(\phi_1, \phi_2, |H|) \geq 0$ for all $\phi_1, \phi_2, H$ if and only if

$$\begin{cases} \Delta' \geq 0 \\ \text{(i) } \lambda_{22} > 6\sqrt{\lambda_{40}\lambda_{04}} \text{ and } |\lambda_{31}\sqrt{\lambda_{04}} + \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}} \\ \text{(ii) } \lambda_{22} \leq 6\sqrt{\lambda_{40}\lambda_{04}} \text{ and } |\lambda_{31}\sqrt{\lambda_{04}} - \lambda_{13}\sqrt{\lambda_{40}}| \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} - 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}} \\ 4\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 \geq 0 \\ \text{(i) } \lambda'_{40} > 0, \lambda'_{04} > 0, \Delta'' \geq 0, \text{ and the corresponding conditions for } \lambda'_{22}, \lambda'_{31}, \lambda'_{13} \\ \text{(ii) } \lambda'_{31} = 0, \lambda'_{40} = \lambda'_{13} = 0, \lambda'_{22} \geq 0 \text{ and } \lambda'_{04} - 4\lambda'_{22} \geq 0 \\ \text{(iii) } \lambda'_{13} = 0, \lambda'_{04} = \lambda'_{31} = 0, \lambda'_{22} \geq 0 \text{ and } \lambda'_{40} - 4\lambda'_{22} \geq 0 \\ \text{(iv) } \lambda'_{31} - 4\lambda'_{22} \geq 0 \text{ and } \lambda'_{04} - 4\lambda'_{22} \geq 0 \end{cases}$$

## 5 Conclusions

In this paper, for a scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$, the analytically necessary and sufficient conditions of the boundedness from below are achieved with the help of the analytical expressions of positive definiteness for 4th-order 2-dimensional symmetric tensors. More precisely:

- For a 4th-order 2-dimensional symmetric tensor, the condition for positive definiteness is an analytically necessary and sufficient condition, and the condition for positive semi-definiteness is an analytically necessary and sufficient condition.

- For a scalar potential of two real scalar fields $\phi_1$ and $\phi_2$, the condition for boundedness from below in the stronger sense is an analytically necessary and sufficient condition, and the condition for boundedness from below is an analytically necessary and sufficient condition.
- For a scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$, the condition for boundedness from below is an analytically necessary and sufficient condition, and the condition for boundedness from below in the stronger sense is an analytically necessary and sufficient condition.

**Conflict of Interest:** The authors of this work declare that they have no conflicts of interest.

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*Note: Figure translations are in progress. See original paper for figures.*