Shadows of rotating Hayward-de Sitter black holes with astrometric observables

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Abstract

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Full Text

Shadows of Rotating Hayward-de Sitter Black Holes with Astrometric Observables

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Abstract

Motivated by recent work on rotating black hole shadows [Phys. Rev. D101, 084029 (2020)], we investigate the shadow behaviors of rotating Hayward-de Sitter black holes for static observers at a finite distance in terms of astronomical observables. This paper employs the newly introduced distortion parameter from [arXiv:2006.00685] to quantitatively describe the shadow's shape. We demonstrate that the spin parameter distorts shadows, while the magnetic monopole charge increases the degree of deformation. Simultaneously, this distortion can be alleviated by the cosmological constant, though it increases with distance from the black hole. Additionally, increases in the spin parameter, magnetic monopole charge, and cosmological constant all cause the shadow to shrink.

Keywords: Rotating Hayward-de Sitter black holes, Shadows of black holes, Observable

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Introduction

According to General Relativity (GR), black holes represent among the most fascinating predicted celestial bodies. The strong gravitational field near a black hole can bend light rays, and due to this extreme gravitational lensing, black hole shadows typically appear in an observer's sky [1]. The received light originates from the black hole's unstable photon orbits, or the photon region [2, 3]. The first image of a black hole captured by the Event Horizon Telescope (EHT) [4] confirmed their existence and has attracted increasing research attention toward observable black hole effects, such as shadows, gravitational deflection of light, and massive particle trajectories.

For the simplest case of a spherical black hole, the shadow boundary forms a perfect circle. In the 1960s, Synge calculated the angular radius of the Schwarzschild black hole shadow for a static observer in his seminal work [5]. For rotating black holes, the shadow shape is no longer circular but becomes flattened on one side due to the "dragging" of null geodesics by the black hole. Bardeen first determined the shadow shape of the Kerr black hole for a distant observer; these results can be found in Chandrasekhar's book [6] and in [7]. Since these pioneering studies, black hole shadows have been extensively investigated (see Refs. [8–27]).

Very recently, the authors of Refs. [28, 29] proposed a new method for calculating shadow size and shape using astrometric observables for finite-distance observers and introduced a novel distortion parameter to characterize deviations from circularity. The shadows of Kerr-de Sitter black holes for static observers were revisited using this approach without introducing tetrads [28]. Furthermore, the appearance of shadows for static spherical and Kerr black holes was discussed within a unified framework [29].

This paper applies this method to study the shadows of rotating Hayward-de Sitter black holes and examine how various parameters affect the shadow's size and distortion. We organize this article as follows. In Sec. II, we briefly review the method for calculating black hole shadows using astronomical observables. In Sec. III, we apply this approach to rotating Hayward-de Sitter black holes to analyze parameter influences on shadow shape and size. We conclude in Sec.

IV. Throughout this paper, we set $G = c = 1$.

II. Shadows of Rotating Black Holes

To make this article self-contained, we briefly introduce the fundamentals in this section; readers may consult Refs. [28, 29] for details.

In astrometry, the angle ϵ between two incident light rays can be expressed as [31]:

$$\cos\epsilon \equiv \frac{\gamma_* w \cdot \gamma_* k}{|\gamma_* w||\gamma_* k|} = \frac{w \cdot k}{(u \cdot w)(u \cdot k)}$$

Here, k and w are the tangent vectors of the two light rays, respectively, and γ_* is the projection operator, $\gamma^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + u^{\mu} u_{\nu}$, for an observer with 4-velocity u^{μ}.

Generally, the metric of a rotating black hole can be written as:

$$ds^2 = g_{00} dt^2 + g_{11} dr^2 + g_{22} d\theta^2 + g_{33} d\phi^2 + 2 g_{03} dt d\phi$$

The 4-velocity of a static observer is $u = \frac{1}{\sqrt{-g_{00}}} \partial_t$. For asymptotically de Sitter spacetime, a cosmological horizon exists, and the observer is fixed in the so-called domain of outer communication—the region between the event horizon and the cosmological horizon. When the observer is located at $\theta = 0$, they will find the shadow as a disk with angular radius:

$$\cot\psi = \mathrm{sgn}\left(\frac{g_{33}}{g_{11}}\right)\sqrt{-\frac{g_{33}}{g_{11}}}$$

Here, we have chosen a light ray $l = (l^0, l^1, l^2, l^3)$ originating from the photon region, and "sgn" represents the sign function.

For an observer located at $\theta > 0$, the shadow silhouette is not a perfect circle due to frame-dragging effects. As an example, assume the observer is at $\theta = \pi/2$. Let $k = (k^0, k^1, 0, k^3)$ represent a light ray from a prograde orbit moving in the same direction as the black hole's rotation, and $w = (w^0, w^1, 0, w^3)$ represent a light ray from a retrograde orbit moving against the rotation.

The angle between these two light rays is given by:

$$\cot\gamma = \mathrm{sgn}(k, w)\sqrt{\frac{(K - W)^2}{g_{11}} + \left(g_{33} - \frac{g_{03}^2}{g_{00}}\right) KW}$$

where $K \equiv k^3/k^1$, $W \equiv w^3/w^1$, and $\mathrm{sgn}(k, w) = \mathrm{sgn}(\cos\gamma) = \mathrm{sgn}\left(g_{11} + \left(g_{33} - \frac{g_{03}^2}{g_{00}}\right) KW\right)$.

Similarly, the angle α between a light ray $l = (l^0, l^1, l^2, l^3)$ from the photon region and k is:

$$\cot\alpha = \mathrm{sgn}(k, l)\sqrt{\frac{(1 - L_3)^2}{g_{11}} + \left(g_{33} - \frac{g_{03}^2}{g_{00}}\right) L_3}$$

and the angle β between l and w is:

$$\cot\beta = \mathrm{sgn}(w,l)\sqrt{\frac{(1-L_3)^2}{g_{11}} + \left(g_{33} - \frac{g_{03}^2}{g_{00}}\right) L_3}$$

In these equations, $L_2 \equiv l^2/l^1$, $L_3 \equiv l^3/l^1$, $\mathrm{sgn}(k,l) = \mathrm{sgn}(\cos\alpha) = \mathrm{sgn}\left(g_{11} + \left(g_{33} - \frac{g_{03}^2}{g_{00}}\right) KL_3\right)$, and $\mathrm{sgn}(w,l) = \mathrm{sgn}(\cos\beta) = \mathrm{sgn}\left(g_{11} + \left(g_{33} - \frac{g_{03}^2}{g_{00}}\right) WL_3\right)$.

The angles γ, α, and β provide the black hole shadow on the celestial sphere. For convenience in studying the shadow, we can use the following stereographic projection for celestial coordinates to describe the shadow shape in a 2D plane [28]:

$$Y_{sh} = \frac{2\sin\Phi\sin\Psi}{1+\cos\Phi\sin\Psi}\frac{2\cos\beta\sin\gamma - 2\cot\gamma}{1+\cos\beta\cos\gamma+\sin^2\gamma\sin^2\beta+(\cos(\beta+\gamma)-\cos\alpha)(\cos(\beta-\gamma)-\cos\alpha)}$$

$$Z_{sh} = \frac{2\cos\Psi}{1+\cos\Phi\sin\Psi}\frac{2\csc\gamma(\cos\alpha-\cos(\beta+\gamma))(\cos(\beta-\gamma)-\cos\alpha)}{1+\cos\beta\cos\gamma+\sin^2\gamma\sin^2\beta+(\cos(\beta+\gamma)-\cos\alpha)(\cos(\beta-\gamma)-\cos\alpha)}$$

Here, Φ and Ψ are the azimuthal and polar angles in the celestial coordinate system.

To quantitatively describe the shadow's shape, a distortion parameter Ξ in terms of α, β, and γ is introduced:

$$\cos\Xi \equiv \frac{1+\cos\gamma-\cos\alpha-\cos\beta}{\sqrt{(1-\cos\alpha)(1-\cos\beta)}}$$

where Ξ ranges from 0 to π. For a circular shadow on the celestial sphere, $\cos\Xi = 0$. For non-vanishing $\cos\Xi$, it quantifies the deviation from circularity. The authors of Ref. [29] first proposed this quantity for shadow analysis. Now we can use γ and Ξ to represent shadow sizes and shapes unambiguously.

III. Application to Rotating Hayward-de Sitter Black Holes

In this section, we apply the method described above to obtain the shadows of rotating Hayward-de Sitter black holes without introducing tetrads.

The metric of rotating Hayward-de Sitter black holes in Boyer-Lindquist coordinates (t,r,θ,ϕ) is [30]:

$$ds^2 = -\frac{\Delta_r}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \Sigma d\theta^2 + \frac{\Delta_\theta\sin^2\theta}{\Sigma}\left(adt - (r^2+a^2)d\phi\right)^2$$

where

$$\Sigma = r^2+a^2\cos^2\theta, \quad \rho = 1+\frac{\Lambda}{3}(r^2+a^2), \quad \Delta_r = 1-\frac{2\tilde{m}(r)}{r}-\frac{\Lambda}{3}(r^2+a^2), \quad \Delta_\theta = 1+\frac{\Lambda}{3}a^2\cos^2\theta,$$

and

$$\tilde{m}(r) = \frac{Mr^3}{r^3 + g^3}$$

Here, M represents the black hole mass, a is the spin parameter, Λ is the cosmological constant, and g is the magnetic monopole charge arising from nonlinear electrodynamics.

A. Null Geodesic Equations and Photon Regions

The photon motion equations in this spacetime, determined by metric (10), can be derived from the Lagrangian $\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$, where an overdot denotes differentiation with respect to an affine parameter. For metric (10), the momenta ($p_\mu = g_{\mu\lambda} \dot{x}^\lambda$) are:

$$p_t = -\frac{\Delta_r}{\Sigma} + \frac{a^2 \Delta_\theta \sin^2\theta}{\Sigma}, \quad p_r = \frac{\Sigma}{\Delta_r}\dot{r}, \quad p_\theta = \Sigma\dot{\theta}, \quad p_\phi = \frac{a\Delta_r \sin^2\theta}{\Sigma} - \frac{a(r^2+a^2)\Delta_\theta \sin^2\theta}{\Sigma}$$

where $p_t = -E$ and $p_\phi = L_\phi$ denote energy and angular momentum, respectively. Combining the momenta with the Hamilton-Jacobi equation yields the null geodesic equations.

The Hamilton-Jacobi equation takes the general form:

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0$$

where λ is an affine parameter and S is the Jacobi action, which can be decomposed as:

$$S = -\frac{1}{2} m^2 \lambda - Et + L_\phi \phi + S_\theta(\theta) + S_r(r)$$

if S is separable. Here m is the particle mass (zero for photons). From (19) and (20), we obtain:

$$\left(\frac{\partial S}{\partial r}\right)^2 = \frac{R}{\Delta_r}, \quad \left(\frac{\partial S}{\partial \theta}\right)^2 = \frac{\Theta}{\Delta_\theta}$$

where Q is a separation constant called the Carter constant, and $\partial S/\partial x^\mu = p_\mu$. Using the Hamilton-Jacobi equation, the null geodesic equations are:

$$(\Sigma\dot{r})^2 = R, \quad (\Sigma\dot{\theta})^2 = \Theta, \quad \Sigma\dot{t} = \frac{E(r^2+a^2)^2}{\Delta_r} - \frac{aL_\phi(r^2+a^2-a\lambda\rho)}{\Delta_r} + \frac{a^2 E \sin^2\theta}{\Delta_\theta} - \frac{aL_\phi \lambda\rho \sin^2\theta}{\Delta_\theta}$$

$$\Sigma\dot{\phi} = \frac{aE(r^2+a^2)}{\Delta_r} - \frac{L_\phi(r^2+a^2-a\lambda\rho)}{\Delta_r} + \frac{aE}{\Delta_\theta} - \frac{L_\phi \lambda\rho}{\Delta_\theta \sin^2\theta}$$

where

$$R = E^2\left[(r^2+a^2-a\lambda\rho)^2 - \eta\Delta_r\right], \quad \Theta = E^2\left[\eta\Delta_\theta - (\lambda\rho\csc\theta - a\sin\theta)^2\right]$$

and we have defined $\lambda \equiv L_\phi/E$, $\eta \equiv Q/E^2$.

For spherical orbits, the conditions $R(r_c) = 0$ and $R'(r_c) = 0$ must be satisfied, which lead to:

$$\lambda = \frac{(a^2 + r_c^2)\Delta_r'(r_c) - 4r_c\Delta_r(r_c)}{a\Delta_r'(r_c)}$$

$$\eta = \frac{16r_c^2\Delta_r(r_c) - (a^2 + r_c^2)^2\Delta_{r'}^2(r_c)}{a^2\Delta_{r'}^2(r_c)}$$

where a prime denotes derivative with respect to r, and r_c is the photon sphere radius. Furthermore, we can rewrite $R''(r_c)$ as:

$$R''(r_c) = 8E^2\left[2r_c\Delta_r'(r_c) + \left(\frac{a^2 + r_c^2}{2}\right)^2\Delta_r''(r_c)\right]$$

A spherical null geodesic at $r = r_c$ is unstable with respect to radial perturbations if $R''(r_c) > 0$, and stable if $R''(r_c) < 0$. Unstable photon orbits determine the shadow contour. The range of r_c (photon region) can be determined by $\Theta \geq 0$ from (24) and (28), which yields:

$$(4r_c\Delta_r(r_c) - \Sigma\Delta_r'(r_c))^2 - 16a^2r_c^2\Delta_r(r_c)\Delta_\theta\sin^2\theta \geq 0$$

From (34) and (35), we obtain $r_{c-} \leq r_c \leq r_{c+}$, where r_{c-} and r_{c+} are the minimum and maximum radial positions of the photon region.

By limiting light rays to those from the photon region, we can regard $p^\mu = \dot{x}^\mu$ as functions of x^μ, E, and r_c.

B. Shadow Sizes

For $\theta = 0$, we can rewrite (35) as:

$$(4r_c\Delta_r(r_c) - (r_c^2 + a^2)\Delta_r'(r_c))^2 = 0$$

This means the photon region becomes a photon sphere, and $r_c = r_{c-} = r_{c+}$. Substituting metric (10) and the geodesic equations into (3), we calculate the angular radius of the shadow as:

$$\cot\psi = \frac{(a^2 + r_c^2 - a\lambda\rho)^2 - \eta\Delta_r(r_c)}{\Delta_r(r_c)\eta + a\lambda\rho(2a^2 + 2r_c^2 - a\lambda\rho)}$$

where λ and η are functions of r_c. We only consider shadows as viewed by observers located outside the photon region.

[Figure 1: see original paper] shows the angular radius ψ of the shadow as a function of distance from the rotating Hayward-de Sitter black hole for selected parameters, with observers at inclination angle $\theta = 0$. The vertical dotted lines indicate the outer boundaries and cosmological horizons (we set $M = 1$). The figures demonstrate that the Schwarzschild black hole has the largest photon sphere radius and consequently the largest shadow size among all cases observed at the same position. Furthermore, regardless of which parameter a, g, or Λ increases, the shadow size decreases.

The situation becomes more complex when the observer is located in the equatorial plane ($\theta = \pi/2$). In this case, (35) can be rewritten as:

$$(4r_c\Delta_r(r_c) - r_c^2\Delta_r'(r_c))^2 - 16a^2r_c^2\Delta_r(r_c) \geq 0$$

yielding $r_{c-} \leq r_c \leq r_{c+}$. From (4), we obtain the angular diameter γ:

$$\cot\gamma = \mathrm{sgn}\left(\frac{\rho^2(\Delta_r - a^2)}{\Delta_r - a^2}\right)\sqrt{\frac{(\Delta_r - a^2)\rho^2}{\Delta_r - a^2}}$$

where the specific forms are determined by evaluating at $r_c = r_{c-}$ and $r_c = r_{c+}$. Notably, since λ and η are functions of r_c, equation (42) is a function of both r and r_c.

[Figure 2: see original paper] illustrates how the angular parameter γ changes with increasing distance from the black hole. It is evident that γ decreases with increasing a, Λ, and g, and that the outer boundary of the photon region r_{c+} is larger than the photon sphere radius in Schwarzschild spacetime. Consequently, the black hole shadow size decreases with increasing a, Λ, and g, with the Schwarzschild black hole shadow having the largest size.

C. Shadow Shape

We now examine the shadow shape under different conditions. Observers at inclination angle $\theta = 0$ see circular shadows, while those at $\theta = \pi/2$ observe distorted silhouettes. According to (5) and (6), the angular distances α and β can be expressed as:

$$\cot\alpha = \mathrm{sgn}(\Delta_r - a^2)\frac{\rho^2}{\Delta_r - a^2}\sqrt{\frac{(\Delta_r - a^2)(1 - L_3)^2}{\Delta_r + (\Delta_r - a^2)\rho^2}}$$

$$\cot\beta = \mathrm{sgn}(\Delta_r - a^2)\frac{\rho^2}{\Delta_r - a^2}\sqrt{\frac{(\Delta_r - a^2)(1 - L_3)^2}{\Delta_r + (\Delta_r - a^2)\rho^2}}$$

where K and W are given by previous equations, and $L_2 \equiv p_\theta/p_r$, $L_3 \equiv p_\phi/p_r$.

[Figure 3: see original paper] shows results for $g = 0$, reproducing the Kerr(-de Sitter) black hole shadows from Ref. [28]. In Figs. 4 and 5, we scale the shadows appropriately to enable qualitative comparison of distortion degrees. The upper panels display scaled shadows for different parameters, while the lower panels show the corresponding distortion parameters as functions of Φ/γ, providing quantitative descriptions of shadow distortion. In [Figure 4: see original paper], observers are relatively close to the black hole's photon regions, while in [Figure 5: see original paper], they are far away. The distortion decreases with increasing cosmological constant but increases with increasing g or a.

[Figure 6: see original paper] plots shadow shapes and distortion parameters for observers at various distances from the black hole center. The distortion parameter increases with distance. From the above discussion, we conclude that when the rotating Hayward-de Sitter black hole parameters g and a are maximal and the cosmological constant is zero, the shadow distortion is largest, and the distortion parameter increases with observer distance.

IV. Conclusions and Discussions

In this article, we calculated the size and shape of rotating Hayward-de Sitter black hole shadows for static observers at finite distances using astronomical observables. For $\theta = 0$, the shadow boundary is a perfect circle, while for $\theta = \pi/2$, it becomes distorted. To quantitatively describe this distortion, we plotted the distortion parameter as a function of black hole parameters, quantifying shape deviations from circularity. We found that regardless of which parameter increases, the shadow size shrinks. At any given distance, the Schwarzschild black hole produces the largest shadow.

Furthermore, when the rotating Hayward-de Sitter black hole parameters g and a are maximal and the cosmological constant vanishes, the shadow distortion is maximal, and the distortion parameter increases with observer distance. We only considered static observers at inclination angles $\theta = 0$ and $\theta = \pi/2$, but this method is applicable to arbitrary observers. Studying black hole shadows provides an important avenue for investigating black hole properties and obtaining rich information about spacetime geometry.

Conflicts of Interest: The authors declare no conflicts of interest regarding this publication.

Figure Captions:

[Figure 1: see original paper] The angular radius ψ of the shadow as a function of distance from rotating Hayward-de Sitter black holes for selected parameters, with observers at inclination angle $\theta = 0$. Vertical dotted lines indicate outer boundaries and cosmological horizons. We set $M = 1$.

[Figure 2: see original paper] The angular diameter γ of the shadow as a function of distance from rotating Hayward-de Sitter black holes for selected parameters, with observers at inclination angle $\theta = \pi/2$. Vertical dotted lines indicate outer boundaries and cosmological horizons. We set $M = 1$.

[Figure 3: see original paper] Shadows of rotating Hayward-de Sitter black holes with $g = 0$ on the projective plane (Y, Z) for selected parameters. r is the distance from observer to black hole. We set $M = 1$. (a) Shadows of rotating Kerr(-de Sitter) black holes for selected spin parameters for distant observers. (b) Shadows of rotating Kerr-de Sitter black holes for observers at $r = 4$.

[Figure 4: see original paper] Shadow shapes and corresponding distortion parameters Ξ as functions of Φ/γ for selected parameters, with observers at $r = 4$. We set $M = 1$. (a) Shadows and distortion parameters for selected cosmological constants. (b) Shadows and distortion parameters for selected magnetic monopole charges. (c) Shadows and distortion parameters for selected spin parameters.

[Figure 5: see original paper] Shadow shapes and corresponding distortion parameters Ξ as functions of Φ/γ for selected parameters, for distant observers. We set $M = 1$. (a) Shadows and distortion parameters for selected spin parameters at $r = 40$. (b) Shadows and distortion parameters for selected magnetic monopole charges at $r = 40$. (c) Shadows and distortion parameters for selected spin parameters at $r = 6.31$. (d) Shadows and distortion parameters for selected magnetic monopole charges at $r = 6.31$.

[Figure 6: see original paper] Shadow shapes and corresponding distortion parameters Ξ as functions of Φ/γ for observers at selected positions r.

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Note: Figure translations are in progress. See original paper for figures.