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Reduced-Degree Sum-of-Squares Representation of Univariate Positive Semidefinite Polynomials and Its L-Algorithm

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Abstract

This paper presents a degree-reduced sum-of-squares representation method for univariate positive semidefinite polynomials, and provides an algorithm for obtaining the degree-reduced sum-of-squares representation from a known positive semidefinite polynomial. In Section 4, we successfully apply this ‘degree-reduced sum-of-squares’ representation idea and algorithm to multivariate polynomials as well.

Full Text

Preamble

This work addresses the problem of determining when a polynomial can be expressed as a sum of squares of polynomials (SOS). For a polynomial $p(x)$, we say $p(x) \geq 0$ for all $x \in \mathbb{R}$ if $p(x)$ is non-negative, and $p(x) > 0$ for all $x \in \mathbb{R}$ if $p(x)$ is positive definite. The SOS decomposition, rooted in Hilbert’s 17th problem, provides a computational framework for certifying non-negativity.

A fundamental result states that a univariate polynomial $p(x)$ is SOS if and only if it can be represented using a Gram matrix B as $p(x) = X(HH^T)X^T$, where $X = (x^n, x^{n-1}, \cdots, x, 1)$ and $B = HH^T$ is positive semidefinite. This establishes the equivalence between the SOS property and the existence of a positive semidefinite Gram matrix.

For a polynomial $p(x) = \sum_{k=0}^{2n} c_k x^k$, the SOS condition can be verified through linear matrix inequalities. Specifically, the set $\{A002(D, V) > 0, D > 0\}$ must be non-empty, where $A002(D, V)$ encodes the Gram matrix constraints. This formulation reduces the problem to checking feasibility of a semidefinite program.

3 Examples

We demonstrate the practical application of SOS decomposition through representative examples, illustrating both the theoretical foundations and computational aspects.

Example 1: Consider the inequality $x^6-x^5-2x^4+x^3+x^2+1 > 0$ for all $x \in \mathbb{R}$. This can be verified by expressing the polynomial as a sum of squares. The decomposition reveals the polynomial's positive definiteness through explicit square terms.

Example 2: The polynomial $x^4 + 2x^3 - 18x^2 - 12x + 117$ admits an SOS representation. Following the method in \cite{example_{ref}}, we obtain:

$$x^4 + 2x^3 - 18x^2 - 12x + 117 = (x^2 + x - 10)^2 + (x + 4)^2 + 1$$

This decomposition clearly demonstrates non-negativity, as it is expressed as a sum of perfect squares plus a positive constant.

Example 3: For the polynomial $x^6 - 2x^5 + 4x^4 - 6x^3 + 6x^2 - 4x + 2$, we find the SOS decomposition:

$$x^6 - 2x^5 + 4x^4 - 6x^3 + 6x^2 - 4x + 2 = (x^3 - x^2 + x - 1)^2 + (x^2 - x + 1)^2 + 1$$

The structure reveals how higher-degree polynomials can be systematically decomposed.

Algorithm 2 (SOS Test for Univariate Polynomials): *Input:* Polynomial $p(x) = \sum_{k=0}^{2n} c_k x^k$
Output: Whether $p(x)$ is SOS

1. Factor $p(x)$ as $p(x) = (f(x))^2 g(x)$ where $g(x)$ is square-free
2. Apply Algorithm 1 to test if $g(x)$ is SOS
3. If $g(x)$ is SOS, then $p(x)$ is SOS
4. Return the SOS certificate

Algorithm 1 checks the feasibility of $\{A002(D, V) > 0, D > 0\}$, where $A002(D, V)$ represents the Gram matrix constraints. The variables D and V parameterize the positive semidefinite matrix $B = HH^T$.

Example 4: The polynomial $x^6 + 1$ is SOS, with decomposition:

$$x^6 + 1 = \left(\frac{x^3}{\sqrt{2}}\right)^2 + \left(\frac{x^3}{\sqrt{2}}\right)^2 + 1$$

Example 5: Consider $p(x) = x^6-6x^5+14x^4-18x^3+17x^2-12x+4$. This polynomial factors as $(x^2+1)(x-1)^2(x-2)^2$, yet also admits an SOS decomposition without obvious factorization:

$$x^6 - 6x^5 + 14x^4 - 18x^3 + 17x^2 - 12x + 4 = (x^3 - 3x^2 + 2x)^2 + (x^2 - 3x + 2)^2$$

This demonstrates that factorization is not prerequisite for SOS decomposition.

Example 6: The polynomial $x^{10} - x + 1$ is positive definite and SOS. Using Algorithm 2 with $V_0 = 0$, we obtain a decomposition involving multiple square terms:

$$x^{10} - x + 1 = \left(x^5 - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}x^5\right)^2 + \cdots$$

The complete decomposition requires solving for coefficients $a_{i,j}$ that satisfy the Gram matrix positivity conditions.

Example 7: The polynomial $x^6 - 2x^5 + 5x^2 - 4x + 1$ is **not** SOS. Analysis via Algorithm 1 yields:

$$A002 = \frac{4s^{10} + (t^2 + 16)s^8 + (28t^2 + 88)s^6 + (8t^4 + 38t^2 + 144)s^4 + (48t^4 - 228t^2 + 324)s^2 + (16t^6 - 120t^4 + 2}{-64s^{2t^2}}$$

where $s = a_{2,2}$ and $t = a_{1,1}$. For $s > 0, t > 0$, the denominator is negative while the numerator is positive, making $A002 < 0$. This violates the necessary condition for SOS representation, proving the polynomial cannot be expressed as a sum of squares despite being non-negative.

Example 8: For $p(x) = 2x^{16} - 4x^{15} - 2x^{14} + 4x^{13} + 2x^{12} - x^5 + 7x^4 - 9x^3 - 7x^2 + 9x + 6$, we first verify $p(x) - 1 = (x^2 - x + 1)^2(2x^{12} - x + 5)$. Since $2x^{12} - x + 5$ is positive definite, we can apply Algorithm 2 recursively. The final SOS decomposition involves 16 square terms and requires solving a system with approximately 1.3×10^{36} possible configurations, demonstrating the computational complexity of the general problem.

Multivariate Extension: For multivariate polynomials, we can apply substitution methods. Consider $f(x, y, z) = x^6 + 4x^{3y^{2z}} + y^6 + 2y^{4z^2} + y^{2z^4} + 4z^6$. Substituting $(x, y, z) \mapsto (t, t^7, t^{43})$ yields $g(t) = 4t^{258} + t^{186} + 2t^{114} + 4t^{60} + t^{42} + t^6$. The univariate SOS decomposition:

$$g(t) = (2t^{129})^2 + (t^{93} - t^{21})^2 + (2t^{57} + t^3)^2$$

can be lifted back to the multivariate case through the substitution mapping.

Motzkin Polynomial: The classic Motzkin polynomial $f(x, y) = x^{4y^2} + x^{2y^4} - 3x^{2y^2} + 1$ is non-negative but **not** SOS. Substituting $g(t) = f(t, t^5) = t^{22} + t^{14} - 3t^{12} + 1$ and attempting to find an SOS representation leads to contradictions in the coefficient matching equations. The resulting system:

$$g_1 - g = (2a_{4,2}a_{4,4} + 2a_{5,1})t^7 + \cdots + a_{1,1}^2 + a_{2,1}^2 + a_{3,1}^2 + a_{4,1}^2 + a_{5,1}^2 - 1$$

cannot be made identically zero, proving the non-existence of an SOS decomposition. This exemplifies the gap between non-negativity and SOS representability.

Computational Aspects: The SDPTools package in Maple provides efficient semidefinite programming solvers for these problems [**?**]. For large polynomials, the Cholesky factorization $B = HH^T$ can be computed to extract explicit SOS decompositions [**?**]. The complexity grows rapidly with degree and number of variables, making symbolic preprocessing essential for practical applications.

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