

Note on anomalies in field theories

Authors: Liu Changyong

Date: 2020-11-02T00:00:00+00:00

Abstract

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Full Text

Preamble

A Note on Anomalies in Field Theories

Chang-Yong Liu

College of Science, Northwest A&F University, Yangling, Shaanxi 712100, China

E-mail: liuchangyong@nwsuaf.edu.cn

November 1, 2020

Abstract

In this note, we investigate anomalies in field theories. The results of anomaly calculations through Feynman diagrams yield multi-valued functions. The single-valued branches of these multi-valued functions are related to the bound states of neutral pseudoscalar mesons. By adding these bound state contributions, we obtain a new anomaly-free condition requiring that all external particles be on-shell, and we find the non-perturbative mass spectrum of neutral pseudoscalar mesons. This yields a mass for the η' meson of $m_{\eta'} = 961$ MeV, which is almost identical to the experimental value $m_{\eta'}^{\text{ex}} \approx 958$ MeV. We also discuss anomalies in 1+1 dimensional quantum field theory. Generalizing this formula, we obtain the mass relation $m_{\eta'}^2(n) = 8n\pi^2 m^2$.

Keywords: Anomaly, Quantum Field Theory

Introduction

Anomaly plays an important role in quantum field theory (QFT) [?]. An anomaly arises when a symmetry group G that leaves the classical action invariant is violated in the full quantum theory. The primary example is the anomaly for the decay rate $\pi^0 \rightarrow \gamma\gamma$. This decay rate confused people for many years until a gauge-invariant result was proposed by Steinberger [?] and Schwinger [?]. Steinberger considered a field theory in which pions are coupled to massive fermions (proton and neutron at the time, although they could just as well have been quarks). The connection to anomalous symmetries was finally understood in 1969 through the work of Adler [?], Bell and Jackiw [?]. The anomaly was calculated through Feynman diagrams. Fujikawa [?] later pointed out that anomalies arise when the measure in the path integral is not invariant under the group G that leaves the classical action invariant. Anomalies have many other applications, including a successful explanation for the $U(1)$ problem [?].

As is well known, the lightest three neutral pseudoscalar mesons are π^0 , η and η' (Table 1). Gell-Mann and Levy [?] constructed an $SU(2)$ chiral symmetry model with spontaneous symmetry breaking that yields a nucleon mass. In this model, the pion is interpreted as a Goldstone boson. This model can be extended to encompass an $SU(3) \times SU(3)$ chiral symmetry. However, we encounter the $U(1)$ problem: the η' is unexpectedly heavy. The Witten-Veneziano relation [?, ?] provides a way to solve the $U(1)$ problem, which cannot be resolved within a perturbative framework. A non-perturbative method is required.

In this note, we study anomalies in field theories. Our new result is that there are many single-valued branches of multi-valued functions that have not been considered previously. These single-valued branches have physical consequences, so we reconsider the well-known results of anomalies. Our approach also provides a new method to study the mass of the η' . In our previous work [?, ?], we connected the single-valued branches of multi-valued functions in correlation functions with bound states.

To study bound states, we define the integral of a complex function $f(z)$ along a smooth contour $C[a, b]$ in the complex plane. Suppose the function $f(z)$ has poles and branch cuts (Figure 1 [Figure 1: see original paper]), then the integral of $f(z)$ along the contour $C[a, b]$ can be expressed as

$$\int_{C[a,b]} f(z)dz = P \int_{C[a,b]} f(z)dz + \sum_i n_i \oint_{C_i} f(z)dz.$$

Where C_i is a closed curve circling the pole or branch cut (Figure 1). The P denotes the principal value which takes values in the main single-valued branch. The n_i are winding numbers of contours circling n_i times around the pole or branch cut.

The paper is organized as follows. In Section 2, we study the axial (ABJ) anomaly in (3+1) dimensions. We obtain a new anomaly-free condition and find the non-perturbative mass spectrum of neutral pseudoscalar mesons. In Section 3, we discuss the anomaly in 1+1 dimensional QFT. We conclude with a summary and discussion.

2 Axial (ABJ) Anomaly in (3+1) Dimensions

We first review the well-known axial (ABJ) anomaly [?, ?] in (3+1) dimensions, then present our new results. The 3+1 dimensional QED Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\cancel{\partial} - m)\psi.$$

The vector and axial vector currents are

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \quad j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi.$$

From the classical equations of motion, the divergences of these currents are

$$\partial_\mu j^\mu = 0, \quad \partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\psi.$$

Thus, classically the vector symmetry is exactly conserved, while the chiral symmetry is only conserved in the massless limit. We calculate the one-loop amplitude to describe how the classical equations (4) are modified in quantum theory. We consider the correlation function at 1-loop: $\langle j^{\rho 5} j^\mu j^\nu \rangle$. In momentum space,

$$iM^{\rho\mu\nu} = (-1)(-ie)^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[\frac{i}{\cancel{p} + \cancel{k}_1 - m} \gamma^\mu \frac{i}{\cancel{p} - \cancel{k}_2 - m} \gamma^\nu \gamma^5 \frac{i}{\cancel{p} - m} \right].$$

Contracting the axial current with its momentum q_ρ yields [?, ?]

$$q_\rho M^{\rho\mu\nu} = (-1)(-ie)^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[\frac{1}{\cancel{p} + \cancel{k}_1 - m} \gamma^\mu \frac{1}{\cancel{p} - \cancel{k}_2 - m} \gamma^\nu \frac{1}{\cancel{p} - m} \cancel{q} \gamma^5 \right].$$

The integrand is linearly divergent, which requires careful treatment. The result is

$$q_\rho M^{\rho\mu\nu} = \frac{e^2}{2\pi^2} \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}.$$

In the massless limit, we obtain

$$\lim_{m \rightarrow 0} q_\rho M^{\rho\mu\nu} = \frac{e^2}{2\pi^2} \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}.$$

Thus, the axial current is not conserved even in the massless limit. This is the ABJ anomaly.

Our new result is that there are many single-valued branches of multi-valued functions that have not been considered previously. These single-valued branches in correlation functions are connected with bound states. In physical processes, the external fields are on-shell, meaning $k_1^2 = k_2^2 = 0$ and $(k_1 + k_2)^2 = q^2$. Then expression (5) becomes

$$q_\rho M^{\rho\mu\nu} = \frac{e^2}{2\pi^2} \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} = \frac{e^2}{4\pi^2} \epsilon^{\mu\nu\alpha\beta} \frac{xyq^2}{k_1^2 k_2^2} k_{1\alpha} k_{2\beta}.$$

According to formula (1), the integral can be calculated as

$$\int \frac{xyq^2}{k_1^2 k_2^2} = \int \frac{xyq^2}{(k_1^2 - xq^2)(k_2^2 - yq^2)} = \frac{1}{q^2} \left[2\pi i k \ln \frac{k_1^2 k_2^2}{(k_1^2 - xq^2)(k_2^2 - yq^2)} + (2\pi i)^2 kl \right].$$

Then we obtain

$$q_\rho M^{\rho\mu\nu} = \frac{e^2}{4\pi^2} \epsilon^{\mu\nu\alpha\beta} \frac{k_{1\alpha} k_{2\beta}}{q^2} \left[2\pi i k \ln \frac{k_1^2 k_2^2}{(k_1^2 - xq^2)(k_2^2 - yq^2)} + (2\pi i)^2 n \right],$$

where we have defined $n = kl$ with $l, k \in \mathbb{Z}$.

Generally, anomalies must vanish for a local gauge symmetry. Traditionally, the anomalous contributions from various multiplets cancel completely, making the gauge current anomaly-free. That is, $\text{tr}[\{t^a, t^b\}t^c] = 0$ is a fundamental consistency condition for chiral gauge theories [?]. This is crucial for building models of particle interactions.

In our approach, we use the extra term $m^2/q^2 \cdot n$ in equation (6) to cancel the anomaly term $2\pi^2 \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta}$. This leads to the anomaly-free condition

$$\frac{m^2}{q^2} = 8n\pi^2.$$

We argue that this is the on-shell condition for an external neutral pseudoscalar meson P . That is, q^2 is related to the neutral pseudoscalar meson mass m_P for a given quantum number n by $q^2(n) = m_P^2(n)$. Their masses fall on a straight line, similar to a Regge trajectory [?]. The anomaly-free condition then

requires that all external fields be on-shell. The decay $P \rightarrow 2\gamma$ corresponds to the transition between bound states, where P denotes the n -th bound state of the neutral pseudoscalar meson. On the other hand, the anomaly-free condition gives the mass spectrum of neutral pseudoscalar mesons.

To illustrate this, we discuss the mass of the neutral pseudoscalar meson η' . The wave function of meson P can be expressed as $P = a_P^2(u\bar{u} + d\bar{d}) + b_P^2(2s\bar{s}) + c_P^2(u\bar{u} + d\bar{d} + s\bar{s})$ (Table 1). We denote the masses of u , d and s quarks as m_u , m_d and m_s in the pseudoscalar meson. To discuss neutral pseudoscalar mesons in the real world, we generalize the results (7) to three quarks. For a meson wave function $P = a_P^2(u\bar{u} + d\bar{d}) + b_P^2(2s\bar{s}) + c_P^2(u\bar{u} + d\bar{d} + s\bar{s})$, the mass of the ground state ($n = 1$) is

$$m_{\pi^0, \eta, \eta'} = 2\sqrt{2}\pi(a_{\pi^0, \eta, \eta'}^2 m_u + b_{\pi^0, \eta, \eta'}^2 m_d + c_{\pi^0, \eta, \eta'}^2 m_s).$$

We deduce the mass $m_{\eta'}$ from the masses of m_{π^0} and m_{η} using the equation

$$\begin{aligned} m_{\pi^0} &= 2\sqrt{2}\pi(a_{\pi^0}^2 m_u + b_{\pi^0}^2 m_d + c_{\pi^0}^2 m_s), \\ m_{\eta} &= 2\sqrt{2}\pi(a_{\eta}^2 m_u + b_{\eta}^2 m_d + c_{\eta}^2 m_s), \\ m_{\eta'} &= 2\sqrt{2}\pi(a_{\eta'}^2 m_u + b_{\eta'}^2 m_d + c_{\eta'}^2 m_s). \end{aligned}$$

We first compute the quark masses m_u , m_d and m_s from the experimental data for π^0 and η . Then we can obtain the mass of η' . From equation (8), we find that we need to exchange the wave function of η with the wave function of η' in Table 1. We show this new identification in Table 2.

From equation (8), we find

$$\begin{aligned} 2\sqrt{2}\pi(m_u + m_d) &= 135 \text{ MeV} \quad (\pi^0), \\ 3\sqrt{2}\pi(m_u + m_d + m_s) &= 548 \text{ MeV} \quad (\eta). \end{aligned}$$

This gives the quark mass combinations

$$m_u + m_d = 30.4 \text{ MeV}, \quad m_s = 154.6 \text{ MeV}.$$

Because quarks are confined in QCD, their masses cannot be directly measured. There are different mass definitions for quarks, with the popular scheme being the $\overline{\text{MS}}$ scheme where running masses $m_a(\mu)$ ($a = u, d, s$) decrease with increasing μ . These properties illustrate that the above result for quark masses is appropriate for neutral pseudoscalar mesons.

Applying equation (8), we obtain the theoretical value for the η' mass:

$$m_{\eta'} = 961 \text{ MeV}.$$

This is almost identical to the experimental value $m_{\eta'}^{\text{ex}} = 958 \text{ MeV}$ (Table 1).

We now discuss the massless limit. When $m^2 \rightarrow 0$, we require that m^2 and q^2 satisfy the anomaly-free condition (7), which gives

$$\lim_{(m^2, q^2) \rightarrow (0, 0)} \frac{m^2}{q^2} = 8n\pi^2.$$

Then the axial anomaly vanishes in the massless limit. We conclude that conservation of the axial current is restored if all external particles are on-shell when bound state contributions are included. In the next section, we consider the anomaly in 1+1 dimensional QFT.

3 Anomaly in 1+1 Dimensional QFT

The 1+1 dimensional QED Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\cancel{\partial} - m)\psi.$$

We can choose the two Dirac matrices as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The γ^5 matrix is the product of the Dirac matrices $\gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The vector j^μ and axial vector $j^{\mu 5}$ currents are defined as in four-dimensional QED (equations (3) and (4)).

We calculate the one-loop amplitude involving a photon and an axial vector current (Fig. 3 [Figure 3: see original paper]) to describe how the classical equations (4) are modified by quantum theory. The Fourier transform of the matrix element is

$$I^{5\mu\nu} = e^2 \int \frac{d^2p}{(2\pi)^2} \text{Tr} \left[\gamma^5 \gamma^\mu \frac{i}{\not{p} - m} \gamma^\nu \frac{i}{\not{p} + \not{q} - m} \right].$$

The two-dimensional Dirac matrices obey the identity $\gamma^5 \gamma^\mu = \epsilon^{\mu\nu} \gamma_\nu$. Then $I^{5\mu\nu}$ can be expressed as

$$I^{5\mu\nu} = e^2 \epsilon^{\mu\rho} \int \frac{d^2 p}{(2\pi)^2} \text{Tr} \left[\gamma_\rho \frac{i}{\not{p} - m} \gamma^\nu \frac{i}{\not{p} + \not{q} - m} \right] = \epsilon^{\mu\rho} i \Pi_\rho{}^\nu,$$

where $i\Pi^{\mu\nu}$ is the lowest-order vacuum polarization of QED [?], which is

$$i\Pi^{\mu\nu} = \frac{4ie^2}{2\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu) \int_0^1 dx \frac{x(1-x)}{m^2 - x(1-x)q^2}.$$

Finally, we obtain

$$q_\mu I^{5\mu\nu} = iq^2 \epsilon^{\mu\nu} \frac{e^2}{\pi} \int_0^1 dx \frac{x(1-x)}{m^2 - x(1-x)q^2}.$$

We now consider the massless limit, which is known as the Schwinger model [?]. Equation (18) in the massless limit gives the anomaly term

$$\lim_{m \rightarrow 0} q_\mu I^{5\mu\nu} = iq_\mu \epsilon^{\mu\nu} \frac{e^2}{\pi} \frac{1}{q^2}.$$

In physical processes, the external fields are on-shell ($q^2 = 0$). There is a subtle point here. Taking $q_\mu I^{5\mu\nu}$ as a function of q^2 and m^2 , we find

$$\lim_{(q^2, m^2) \rightarrow (0, 0)} q_\mu I^{5\mu\nu} = \lim_{q^2 \rightarrow 0} \lim_{m^2 \rightarrow 0} q_\mu I^{5\mu\nu}.$$

Then the function $q_\mu I^{5\mu\nu}(q^2, m^2)$ is actually undefined at the point $(q^2, m^2) = (0, 0)$. We now consider the pole contributions. According to formula (1), we get

$$q_\mu I^{5\mu\nu} = iq_\mu \epsilon^{\mu\nu} \frac{e^2}{\pi} \frac{1}{q^2} \arctan \left[\frac{q^2}{2m^2} \sqrt{1 + \frac{4m^2}{q^2}} \right] = iq_\mu \epsilon^{\mu\nu} \frac{e^2}{\pi} \frac{1}{q^2} \left[\pi n + \arctan \left(\frac{q^2}{2m^2} \right) \right].$$

By the same argument as for the axial anomaly in (3+1) dimensions, we find the anomaly-free condition

$$\frac{1}{q^2} \arctan \left(\frac{q^2}{2m^2} \right) = \pi n.$$

The solution of equation (22) is

$$1 + \frac{4m^2}{q^2} = 2(1 - 4n^2\pi^2)m^2, \quad n \in \mathbb{Z},$$

where q^2 takes complex values for m with $n > 0$. The m^2 and q^2 need to satisfy relation (23) in the limit $(q^2, m^2) \rightarrow (0, 0)$, which gives

$$\lim_{(m^2, q^2) \rightarrow (0, 0)} \frac{m^2}{q^2} = 0.$$

Then equation (21) becomes

$$\lim_{(q^2, m^2) \rightarrow (0, 0)} q_\mu T^{5\mu\nu} = 0.$$

We obtain that the axial anomaly vanishes if all external particles are on-shell by taking into account relation (23) in the massless limit.

4 Conclusions and Discussions

In this note, we have studied anomalies in field theories. The pole contributions in Feynman diagram calculations lead to multi-valued functions. Similar to our previous work [?, ?], these single-valued branches of multi-valued functions are related to the bound states of pseudoscalar mesons. Using this connection, we cancel the anomaly term with bound state contributions. We present a new anomaly-free condition requiring that all external particles be on-shell. We find that the non-perturbative mass spectrum of neutral pseudoscalar mesons follows $m_P^2(n) = 8n\pi^2 m^2$. This formula can be used to study the mass of the η' meson. We obtain $m_{\eta'} = 961$ MeV, which is almost identical to the experimental value of 958 MeV. We also discussed the anomaly in 1+1 dimensional QFT. In future work, we will study anomalies with bound state contributions using the Fujikawa method [?].

Acknowledgements

This work is supported by Chinese Universities Scientific Fund Grant No. 2452018158. We would like to thank Dr. Wei He, Youwei Li and Suzhi Wu for helpful discussions.

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