Approximately Sparse Low-Rank Tensor Completion

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Abstract

To address the phenomenon that most existing low-rank tensor completion (LRTC) models suffer from excessive sparsity, leading to the overlooking of subtle features in the data, this paper proposes an approximate sparsity-based low-rank tensor completion (AS-LRTC) model by leveraging framelet transform and low-rank matrix factorization, and further designs a block successive upper-bound minimization (BSUM) algorithm to solve the proposed model. The convergence of the algorithm can be proven under certain conditions, and extensive experimental results demonstrate that the proposed algorithm exhibits significant advantages over some existing classical algorithms.

Full Text

Preamble

This paper addresses the problem of low-rank tensor completion (LRTC). The fundamental LRTC problem can be formulated as minimizing the rank of a tensor Y subject to observed entries: rank(Y) subject to PΩ(Y) = F, where Y ^{d × ×d_N} is the target tensor, F represents the observed data, and Ω denotes the index set of observed entries, with P(·) being the projection operator.

Due to the NP-hard nature of rank minimization, practical methods often employ surrogate formulations. One common approach uses Tucker decomposition, representing the tensor through factor matrices and a core tensor. The optimization problem can be written as:

Y,A,X ‖Y − A X ‖^2 subject to PΩ(Y) = F, where Y A X , A = (A , , A_N), X = (X , , X_N). This formulation underlies methods such as TMac [21] and FaLRTC.

To incorporate spatial regularization, previous work has proposed adding total variation (TV) penalties. One such formulation introduces a TV term on the core tensor:

Y,X,A ‖Y − A X ‖²_F + TV(X) subject to PΩ(Y) = F, where > 0 is a regularization parameter, and TV denotes the total variation functional that promotes piecewise smoothness.

Building upon this, Jiang et al. [27] proposed a framelet-based regularization approach. Their method replaces the TV term with a sparse representation in the framelet domain:

Y,X,A ‖Y − A X ‖²_F + ‖WX ‖ , subject to PΩ(Y) = F, where > 0 is a regularization parameter, W represents the framelet transform matrix, and ‖ · ‖ , encourages sparsity. The constraint W W = I ensures the transform is unitary.

These formulations treat the problem as a joint optimization over the tensor Y, factor matrices A, and core tensors X. However, the norm may not provide sufficient sparsity for certain applications. To address this limitation, we propose an adaptive sparsity-enhanced LRTC (AS-LRTC) framework that employs a weighted penalty to better approximate the norm.

2.3 BSUM Framework

The Block Successive Upper-bound Minimization (BSUM) method provides a general optimization framework for non-convex problems. Consider a problem of the form:

min f(x) subject to x X, where X = X × X × × X with each X ^{m }, and x = (x , x , , x) with x X .

The BSUM algorithm constructs a surrogate function for each block:

u (x , x) = f(x , , x , , x) + (/2)‖x − x ‖² , where > 0 and i = 1, , n.

The algorithm proceeds iteratively by minimizing each surrogate function:

Algorithm 1 (BSUM):

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Step 1: x ¹ = argmin_{x X } u (x , x ¹ )
Step 2: x ¹ = argmin_{x X } u (x , x ² )
...
Step n: x ¹ = argmin_{x X } u (x , x  )
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where x = (x ¹ , , x ¹_{r−1}, x _{r}, , x).

Theorem 2.1 ([30]) establishes convergence guarantees: If the surrogate functions u (x , y) satisfy: 1. u (y , y) = f(y) for all y X and all i; 2. u (x , y) f(y , , y_{i−1}, x , y_{i+1}, , y) for all x X , y X, and i; 3. The directional derivatives match at the current iterate; 4. u (x , y) is continuous in both arguments;

then the BSUM algorithm converges to a stationary point of the original problem.

2.4 Framelet Transform

Framelets provide a multi-scale representation for signals. Let Ψ = { , , , } L²( ) be a set of framelet generators. The associated framelet system is:

X(Ψ) = {2^{k/2} (2 · − j) : Ψ; 1 l r; k, j }.

When X(Ψ) forms a tight frame for L²( ), the framelet transform W satisfies the unitary extension principle (UEP): W W = I. For discrete signals f ^{m×n}, the framelet coefficients are computed as Wf, where W ^{k×mn} represents the decomposition matrix. The UEP ensures perfect reconstruction: f = W (Wf).

Framelet regularization has proven effective for image restoration tasks [31-33], as it captures geometric structures through multi-scale, multi-directional analysis.

2.5-2.7 Proximal Operators and Subdifferentials

Definition 2.6 (Proximity Operator). For a proper, closed, convex function f: ^m → (−∞, +∞], the proximity operator is defined as:

prox_f(x) = argmin_{u ^n} {f(u) + ‖u − x‖² }.

For a positive definite matrix H S , the weighted proximity operator is:

prox_{f,H}(x) = argmin_{u ^n} {f(u) + ‖u − x‖²_H}.

Definition 2.7 (Subdifferential). The subdifferential of a convex function f at x ^n is:

f(x) = {y ^n : f(z) f(x) + y, z − x , z ^n}.

These tools are fundamental for analyzing optimization algorithms involving non-smooth regularizers.

3.1 Problem Formulation

We propose the following AS-LRTC model that incorporates adaptive sparsity in the framelet domain:

Y,A,X ‖Y − A X ‖²_F + ‖S_ (WX)‖ subject to PΩ(Y) = F, where ‖ · ‖ denotes the pseudo-norm counting non-zero entries, and S_ is the soft-thresholding operator: (S_ (u)) = sign(u)max{|u | − , 0}.

Since minimization is NP-hard, we approximate it using a weighted norm. Define a continuous approximation F_ (x) = 1/|x| for |x| > and 0 otherwise, which approaches the indicator function as → 0 . This yields the weighted formulation:

Y,A,X ‖Y − A X ‖²_F + ‖ΞS_ (WX)‖ subject to PΩ(Y) = F, where Ξ is a diagonal weight matrix with entries = F_ ((S_ (WX))).

The complete optimization problem becomes:

min_{Y,A,X} f(Y, A, X) = _{n=1}³ ‖Y − A X ‖²_F + ‖ΞS_ (WX)‖ + (Y),

where (Y) is the indicator function enforcing the observation constraint: (Y) = 0 if PΩ(Y) = F, and +∞ otherwise.

This formulation adaptively reweights the sparsity penalty based on the current estimate, better approximating the norm than standard regularization.

Experimental Results

We evaluate AS-LRTC against MF-TV and MF-Framelet baselines on three types of data: video sequences, MRI volumes, and color images. Performance is measured using PSNR and SSIM metrics.

Video Completion: Table 1 compares results on standard video sequences with 10% sampling rate. AS-LRTC consistently achieves the highest PSNR and SSIM values across all test videos (salesman, coastguard, suzie, carphone, foreman), demonstrating superior reconstruction quality.

MRI Reconstruction: For a 350×350×188 MRI volume with 10% sampling, AS-LRTC produces significantly better structural preservation compared to MF-TV and MF-Framelet, as shown in Figure 5. The method better captures anatomical details in the reconstructed slices.

Color Image Inpainting: On 256×256×3 color images with 40% missing data, AS-LRTC again outperforms the baselines (Table 2). The adaptive sparsity mechanism proves particularly effective for natural image textures and edges.

Figure 7 visualizes qualitative improvements on MRI data, where AS-LRTC recovers finer details with fewer artifacts. Figure 9 demonstrates the effectiveness on color images, showing better color consistency and edge preservation.

Conclusion

We have presented an Adaptive Sparsity-enhanced Low-Rank Tensor Completion (AS-LRTC) framework that integrates framelet regularization with a weighted approximation of the norm. The BSUM algorithm provides an efficient optimization scheme with convergence guarantees. Experimental results on video, MRI, and color image data demonstrate consistent improvements over existing MF-TV and MF-Framelet methods in terms of PSNR and SSIM metrics. Future work will explore extensions to higher-order tensors and applications to other imaging modalities.

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