Copositivity for 3rd order symmetric tensors and applications

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Date: 2019-11-23T00:00:00+00:00

Abstract

The strict copositivity of 4th order symmetric tensors may be applied to detect the vacuum stability of general scalar potentials. To find analytical expressions for the (strict) copositivity of 4th order symmetric tensors, we may reduce their order to 3rd order to better deal with it. Accordingly, several analytical sufficient conditions for the copositivity of 3rd order 2-dimensional (3-dimensional) symmetric tensors are derived. Subsequently, applying these conclusions to 4th order tensors, the analytical sufficient conditions for copositivity are proved for 4th order 2-dimensional and 3-dimensional symmetric tensors. Finally, we apply these results to present analytical vacuum stability conditions for $\mathbb{Z}_3$ scalar dark matter.

Full Text

Preamble

Copositivity for 3rd Order Symmetric Tensors and Applications

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November 23, 2019

Abstract. The strict copositivity of 4th order symmetric tensors may be applied to detect vacuum stability of general scalar potentials. To find analytical expressions for the (strict) copositivity of 4th order symmetric tensors, we may reduce their order to 3rd order to better deal with them. We provide several analytically sufficient conditions for the copositivity of 3rd order 2-dimensional and 3-dimensional symmetric tensors. Subsequently, applying these conclusions to 4th order tensors, we prove analytical sufficient conditions of copositivity for 4th order 2-dimensional and 3-dimensional symmetric tensors. Finally, we apply these results to present analytical vacuum stability conditions for Z3 scalar dark matter.

Key Words and Phrases: Copositive Tensors, Symmetric Tensor, Homogeneous Polynomial, Analytical Expression.

2010 AMS Subject Classification: 15A18, 15A69, 90C20, 90C30

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Introduction

Recently, Kannike [12-14] studied the vacuum stability of general scalar potentials of a few fields. The most general scalar quartic potential of the SM Higgs H1, an inert doublet H2 and a complex singlet S is given by

$$V(h_1,h_2,s)=\Gamma\phi^4=\sum_{i,j,k,l=1}^{3}\gamma_{ijkl}z_iz_jz_kz_l, \tag{1.1}$$

where $z=(z_1,z_2,z_3)^\top=(h_1,h_2,s)^\top$, $h_1=|H_1|$, $h_2=|H_2|$, $H_2^\dagger H_1=h_1h_2\rho e^{i\phi}$, $S=se^{i\phi_S}$, $\Gamma=(\gamma_{ijkl})$ is a coupling tensor, a 4th order 3-dimensional real symmetric tensor. Clearly, $h_1\geq 0$, $h_2\geq 0$, $s\geq 0$. Therefore, the vacuum stability of Z3 scalar dark matter $V(h_1,h_2,s)$ is equivalent to the strict copositivity of the coupling tensor $\Gamma=(\gamma_{ijkl})$.

The concept of (strict) copositivity of symmetric tensors was introduced by Qi [20] in 2013. A symmetric tensor Γ with order m and dimension n is called (i) copositive if $\Gamma x^m=\sum_{i_1,i_2,\cdots,i_m=1}^{n}\gamma_{i_1i_2\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m}\geq 0$ for all nonnegative vectors $x=(x_1,\cdots,x_n)^\top$; (ii) strictly copositive if $\Gamma x^m>0$ for all nonnegative and nonzero vectors $x=(x_1,\cdots,x_n)^\top$.

Since practical matters such as the vacuum stability of general scalar potentials of a few fields require precise expressions, it is necessary to find analytical expressions for the strict copositivity of a symmetric tensor. Qi [20] showed that a symmetric tensor is strictly copositive if for all $i=1,2,\cdots,n$,

$$\gamma_{ii\cdots i}+\sum_{\{\gamma_{i_1i_2\cdots i_m}<0,(i_1,i_2,\cdots,i_m)\neq(i,i,\cdots,i)\}}>0.$$

Song and Qi [23] gave a necessary and sufficient condition for strictly copositive tensors using H-eigenvalues and Z-eigenvalues. Song and Qi [28] studied the (strict) copositivity of symmetric tensors through constrained minimization problems on the unit sphere. Song and Qi [26] showed that the (strict) copositivity of a symmetric tensor is equivalent to its (strict) semi-positiveness. A tensor Γ is said to be (strictly) semipositive (Song and Qi [24]) if for each nonzero and nonnegative vector $x = (x_1, x_2, \cdots, x_n)^\top$, there exists $k \in \{1, 2, \cdots, n\}$ such that $x_k > 0$ and

$$(\Gamma x^{m-1})_k = \sum_{i_2,\cdots,i_m=1}^{n} \gamma_{ki_2\cdots i_m} x_{i_2} \cdots x_{i_m} \geq 0 (> 0).$$

This class of tensors has close relationships with tensor complementarity problems (TCP) [29-31]. For more details about TCP, see also [1-4, 8-11, 16, 33-35] and references cited therein.

Recently, based on the main results of Song and Qi [26], an analytical expression for detecting the strict copositivity of a symmetric tensor was obtained by summarizing conclusions from Qi and Song [18], Song and Qi [25], Song and Mei [31], and Yuan and You [36]. Specifically, a symmetric tensor Γ is strictly copositive if it satisfies

$$\sum_{i_2,\cdots,i_m=1}^{n} \gamma_{ii_2i_3\cdots i_m} > 0 \text{ for all } i \in \{1, 2, \cdots, n\},$$

and

$$n^{m-1} \left(\sum_{i_2,\cdots,i_m=1}^{n} \gamma_{ii_2i_3\cdots i_m} \right) > \gamma_{ij_2j_3\cdots j_m} \text{ for all } (j_2, j_3, \cdots, j_m) \neq (i, i, \cdots, i).$$

Numerical algorithms for copositivity of tensors were also presented by Chen, Huang, and Qi [5, 6], Nie, Yang, and Zhang [17], and Li, Zhang, Huang, and Qi [15]; see also [7, 21, 22]. Very recently, Song and Qi [27] provided several analytically sufficient conditions for checking (strict) copositivity of 4th order 3-dimensional symmetric tensors by reducing the order or dimension of the tensor.

In this paper, we study analytical expressions for certifying copositivity of a 3rd order 3-dimensional (or 2-dimensional) symmetric tensor. We then apply these results to derive sufficient conditions for copositivity of 4th order 3-dimensional tensors, which differ from those in Song and Qi [27]. Furthermore, we use these conclusions to check vacuum stability for Z3 scalar dark matter.

2 Preliminaries and Basic Facts

Let $\|\cdot\|$ denote any norm on $\mathbb{R}^n$. The equivalent definition of (strict) copositivity and semipositive (positive) definiteness of a symmetric tensor was presented in the sense of any norm on $\mathbb{R}^2$ [19-23].

Lemma 2.1. [23] Let Γ be a symmetric tensor of order m and dimension n. Then (i) Γ is copositive if and only if $\Gamma x^m \geq 0$ for all nonnegative vectors

$x \in \mathbb{R}^n$ with $\|x\| = 1$; (ii) Γ is strictly copositive if and only if $\Gamma x^m > 0$ for all nonnegative vectors $x \in \mathbb{R}^n$ with $\|x\| = 1$.

For a cubic univariate polynomial $P(t)$ with real coefficients,

$$P(t) = at^3 + bt^2 + ct + d, \tag{2.1}$$

Schmidt and Heß [32] proved its nonnegative conditions.

Lemma 2.2. [32, Proposition 2] Let $P(t)$ be a cubic univariate polynomial given by (2.1). Then (i) $P(t) \geq 0$ for all $t \geq 0$ if and only if either inequality system (1) or (2) holds: (1) $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$; (2) $\max\{a, d\} > 0$, $a \geq 0$, $d \geq 0$, $4ac^3 + 4b^3d + 27a^2d^2 - 18abcd - b^2c^2 \geq 0$. (ii) $P(t) \geq 0$ for all $t \geq 0$ if the inequalities $ad \geq 0$, $c \geq 0$, and $b + 2\sqrt{ac} \geq 0$ hold simultaneously.

Lemma 2.3. [32, Proposition 3] Let a quadratic univariate polynomial $p(t)$ be given by $p(t) = \alpha t^2 + \beta t + \gamma$. Then $p(t) \geq 0$ for all $t \geq 0$ if and only if the inequalities

$$\alpha \geq 0, \quad \gamma \geq 0, \quad \beta + 2\sqrt{\alpha\gamma} \geq 0 \tag{2.2}$$

hold simultaneously.

3 Copositivity of 3rd Order Tensors

Let $\Gamma = (\gamma_{ijk})$ be a 3rd order 2-dimensional symmetric tensor. Then for a vector $x = (x_1, x_2)^\top$,

$$\Gamma x^3 = \sum_{i,j,k=1}^{2} \gamma_{ijk} x_i x_j x_k = \gamma_{111} x_1^3 + 3\gamma_{112} x_1^2 x_2 + 3\gamma_{122} x_1 x_2^2 + \gamma_{222} x_2^3. \tag{3.1}$$

Theorem 3.1. A 3rd order 2-dimensional symmetric tensor is copositive if and only if either inequality system (1) or (2) holds: (1) $\gamma_{111} \geq 0$, $\gamma_{222} \geq 0$, $\gamma_{112} \geq 0$, $\gamma_{122} \geq 0$; (2) $\max\{\gamma_{111}, \gamma_{222}\} > 0$, $\gamma_{111} \geq 0$, $\gamma_{222} \geq 0$, $4\gamma_{111}\gamma_{122}^3 + 4\gamma_{112}^3\gamma_{222} + \gamma_{111}^2\gamma_{222}^2 - 6\gamma_{111}\gamma_{112}\gamma_{122}\gamma_{222} - 3\gamma_{112}^2\gamma_{122}^2 \geq 0$.

Proof. It follows from Lemma 2.1 that we can restrict $x = (x_1, x_2)^\top$ to $\|x\| = |x_1| + |x_2| = 1$ for all x with $x_1 \geq 0$, $x_2 \geq 0$. Clearly, $\Gamma x^3 = \gamma_{111} x_1^3 \geq 0$ for $x = (x_1, 0)$ and $\Gamma x^3 = \gamma_{222} x_2^3 \geq 0$ for $x = (0, x_2)$. Now assume both x_1 and x_2 are nonzero. Then the homogeneous polynomial Γx^3 can be divided by x_2^3 to yield

$$\frac{\Gamma x^3}{x_2^3} = \gamma_{111}\left(\frac{x_1}{x_2}\right)^3 + 3\gamma_{112}\left(\frac{x_1}{x_2}\right)^2 + 3\gamma_{122}\left(\frac{x_1}{x_2}\right) + \gamma_{222}.$$

Let $t = \frac{x_1}{x_2} \geq 0$ and define $P(t) = \Gamma x^3 / x_2^3$, i.e.,

$$P(t) = \gamma_{111} t^3 + 3\gamma_{112} t^2 + 3\gamma_{122} t + \gamma_{222}. \tag{3.2}$$

Clearly, $P(t) \geq 0$ if and only if $\Gamma x^3 \geq 0$. Applying Lemma 2.2(i) to the polynomial $P(t)$ with $(a, b, c, d)^\top = (\gamma_{111}, 3\gamma_{112}, 3\gamma_{122}, \gamma_{222})^\top$, we obtain that $P(t) \geq 0$ for all $t \geq 0$ if and only if either inequality system holds: $a = \gamma_{111} \geq 0$, $d = \gamma_{222} \geq 0$, $b = 3\gamma_{112} \geq 0$, $c = 3\gamma_{122} \geq 0$; or $\max\{\gamma_{111}, \gamma_{222}\} > 0$, $\gamma_{111} \geq 0$, $\gamma_{222} \geq 0$, $4\gamma_{111}(3\gamma_{122})^3 + 4(3\gamma_{112})^3\gamma_{222} + 27\gamma_{111}^2\gamma_{222}^2 - 18\gamma_{111}(3\gamma_{112})(3\gamma_{122})\gamma_{222} - (3\gamma_{112})^2(3\gamma_{122})^2 \geq 0$. This simplifies to the two systems stated in the theorem, completing the proof.

Remark 3.1. From the proof of Theorem 3.1, we see that Γx^3 may alternatively be divided by x_1^3 (when $x_1 \neq 0$), yielding $P(t) = \gamma_{111} + 3\gamma_{112} t + 3\gamma_{122} t^2 + \gamma_{222} t^3$ where $t = \frac{x_2}{x_1}$. The conclusions remain the same.

Theorem 3.2. Let Γ be a 3rd order 2-dimensional symmetric tensor. Assume that $\gamma_{111} \geq 0$, $\gamma_{222} \geq 0$, $\gamma_{112} \geq \frac{1}{3}\sqrt{\gamma_{111}\gamma_{222}}$, and $\gamma_{122} \geq \frac{1}{3}\sqrt{\gamma_{111}\gamma_{222}}$. Then Γ is copositive.

Proof. Using the proof technique of Theorem 3.1, we only need to show the nonnegativity of the polynomial $P(t)$ given by (3.2) for all $t \geq 0$, where $P(t) = at^3 + bt^2 + ct + d$ with $a = \gamma_{111}$, $b = 3\gamma_{112}$, $c = 3\gamma_{122}$, and $d = \gamma_{222}$. The assumptions mean that $a = \gamma_{111} \geq 0$, $d = \gamma_{222} \geq 0$, $b = 3\gamma_{112} \geq 2\sqrt{\gamma_{111}\gamma_{222}}$, and $c = 3\gamma_{122} \geq 2\sqrt{\gamma_{111}\gamma_{222}}$. From Lemma 2.2(ii), it follows that $P(t) \geq 0$ for all $t \geq 0$, and thus the tensor Γ is copositive, as required.

Theorem 3.3. Let Γ be a 3rd order 2-dimensional symmetric tensor. Assume that one of the following inequality systems holds: (1) $\gamma_{111} \geq 0$, $\gamma_{222} \geq 0$, $\gamma_{122} \geq 0$, $\gamma_{112} \geq -\frac{2}{3}\sqrt{3\gamma_{122}\gamma_{111}}$; or (2) $\gamma_{111} \geq 0$, $\gamma_{222} \geq 0$, $\gamma_{112} \geq 0$, $\gamma_{122} \geq -\frac{2}{3}\sqrt{3\gamma_{112}\gamma_{222}}$. Then Γ is copositive.

Proof. Using the proof technique of Theorem 3.1, the copositivity of Γ is equivalent to the nonnegativity of the polynomial $P(t)$ for all $t \geq 0$, where $P(t) = \gamma_{111} t^3 + 3\gamma_{112} t^2 + 3\gamma_{122} t + \gamma_{222}$. We show the nonnegativity of $P(t)$ by rewriting it as $P(t) = \gamma_{111} t^3 + (3\gamma_{112} t^2 + 3\gamma_{122} t + \gamma_{222}) = t(\gamma_{111} t^2 + 3\gamma_{112} t + 3\gamma_{122}) + \gamma_{222}$. Let $f(t) = 3\gamma_{112} t^2 + 3\gamma_{122} t + \gamma_{222}$ and $g(t) = \gamma_{111} t^2 + 3\gamma_{112} t + 3\gamma_{122}$. From Lemma 2.3, the nonnegativity of $f(t)$ is equivalent to the inequality system $\gamma_{222} \geq 0$, $\gamma_{112} \geq 0$, and $3\gamma_{122} + 2\sqrt{3\gamma_{112}\gamma_{222}} \geq 0$. The inequality $\gamma_{111} \geq 0$ along with this system implies $P(t) = \gamma_{111} t^3 + f(t) \geq 0$ for all $t \geq 0$, meaning Γ is copositive. For assumption (1), the same technique applied to $g(t)$ yields the desired conclusion.

For a 3rd order 3-dimensional symmetric tensor Γ and a vector $x = (x_1, x_2, x_3)^\top$,

$$\Gamma x^3 = \sum_{i,j,k=1}^{3} \gamma_{ijk} x_i x_j x_k = \gamma_{111}x_1^3+\gamma_{222}x_2^3+\gamma_{333}x_3^3+3\gamma_{112}x_1^2x_2+3\gamma_{122}x_1x_2^2+3\gamma_{113}x_1^2x_3+3\gamma_{133}x_1x_3^2+3\gamma_{223}x_2^2x_3+3 \tag{3.4}$$

Theorem 3.4. A 3rd order 3-dimensional symmetric tensor Γ is copositive if the following inequality system holds: $\gamma_{111} \geq 0$, $\gamma_{222} \geq 0$, $\gamma_{333} \geq 0$, $\gamma_{123} \geq 0$, $32\gamma_{111}\gamma_{122}^3 + 32\gamma_{112}^3\gamma_{222} + \gamma_{111}^2\gamma_{222}^2 - 24\gamma_{111}\gamma_{112}\gamma_{122}\gamma_{222} - 48\gamma_{112}^2\gamma_{122}^2 \geq 0$, $32\gamma_{111}\gamma_{133}^3 + 32\gamma_{113}^3\gamma_{333} + \gamma_{111}^2\gamma_{333}^2 - 24\gamma_{111}\gamma_{113}\gamma_{133}\gamma_{333} - 48\gamma_{113}^2\gamma_{133}^2 \geq 0$, and $32\gamma_{222}\gamma_{233}^3 + 32\gamma_{223}^3\gamma_{333} + \gamma_{222}^2\gamma_{333}^2 - 24\gamma_{222}\gamma_{223}\gamma_{233}\gamma_{333} - 48\gamma_{223}^2\gamma_{233}^2 \geq 0$.

Proof. Rewrite Γx^3 as follows:

$$\Gamma x^3 = (\gamma_{111}x_1^3 + 3\gamma_{112}x_1^2x_2 + 3\gamma_{122}x_1x_2^2 + \gamma_{222}x_2^3)+(\gamma_{111}x_1^3 + 3\gamma_{113}x_1^2x_3 + 3\gamma_{133}x_1x_3^2 + \gamma_{333}x_3^3)+(\gamma_{222}x_2^3 + 3\gamma_{22}$$

where $y = (x_1, x_2)^\top$, $z = (x_1, x_3)^\top$, $w = (x_2, x_3)^\top$, and $A = (a_{ijk})$, $B = (b_{ijk})$, $C = (c_{ijk})$ are three 3rd order 2-dimensional symmetric tensors with entries $a_{111} = b_{111} = c_{111} = \gamma_{111}$, $a_{112} = \gamma_{112}$, $a_{122} = \gamma_{122}$, $a_{222} = \gamma_{222}$; $b_{112} = \gamma_{113}$, $b_{122} = \gamma_{133}$, $b_{222} = \gamma_{333}$; and $c_{112} = \gamma_{223}$, $c_{122} = \gamma_{233}$, $c_{222} = \gamma_{333}$. For the polynomial $Ay^3 = \gamma_{111}x_1^3 + 3\gamma_{112}x_1^2x_2 + 3\gamma_{122}x_1x_2^2 + \gamma_{222}x_2^3$, the assumptions imply that $a_{111} = \gamma_{111} \geq 0$, $a_{222} = \gamma_{222} \geq 0$, and $4a_{111}a_{122}^3 + 4a_{112}^3a_{222} + a_{111}^2a_{222}^2 - 6a_{111}a_{112}a_{122}a_{222} - 3a_{112}^2a_{122}^2 = (32\gamma_{111}\gamma_{122}^3 + 32\gamma_{112}^3\gamma_{222} + \gamma_{111}^2\gamma_{222}^2 - 24\gamma_{111}\gamma_{112}\gamma_{122}\gamma_{222} - 48\gamma_{112}^2\gamma_{122}^2) \geq 0$. By Theorem 3.1, tensor A is copositive, i.e., $Ay^3 \geq 0$ for all $y \geq 0$. Similarly, $Bz^3 \geq 0$ for all $z \geq 0$ and $Cw^3 \geq 0$ for all $w \geq 0$. Since $\gamma_{123} \geq 0$, we have $\Gamma x^3 = Ay^3 + Bz^3 + Cw^3 + 6\gamma_{123}x_1x_2x_3 \geq 0$ for all $x \geq 0$, proving the desired conclusion.

Using a similar proof technique as in Theorem 3.4, we may apply Theorem 3.2 to the three tensors $A = (a_{ijk})$, $B = (b_{ijk})$, and $C = (c_{ijk})$ to obtain the following result easily.

Theorem 3.5. Let Γ be a 3rd order 3-dimensional symmetric tensor. Assume that $\gamma_{111} \geq 0$, $\gamma_{222} \geq 0$, $\gamma_{333} \geq 0$, $\gamma_{123} \geq 0$, $\gamma_{112} \geq \frac{1}{3}\sqrt{\gamma_{111}\gamma_{222}}$, $\gamma_{122} \geq \frac{1}{3}\sqrt{\gamma_{111}\gamma_{222}}$, $\gamma_{113} \geq \frac{1}{3}\sqrt{\gamma_{111}\gamma_{333}}$, $\gamma_{133} \geq \frac{1}{3}\sqrt{\gamma_{111}\gamma_{333}}$, $\gamma_{223} \geq \frac{1}{3}\sqrt{\gamma_{222}\gamma_{333}}$, and $\gamma_{233} \geq \frac{1}{3}\sqrt{\gamma_{222}\gamma_{333}}$. Then Γ is copositive.

4 Copositivity of 4th Order Tensors

Let A be a 4th order 2-dimensional symmetric tensor. Then for a vector $x = (x_1, x_2)^\top$,

$$Ax^4 = \sum_{i,j,k,l=1}^{2} a_{ijkl}x_ix_jx_kx_l = a_{1111}x_1^4+4a_{1211}x_1^3x_2+6a_{1221}x_1^2x_2^2+4a_{1222}x_1x_2^3+a_{2222}x_2^4. \tag{4.1}$$

Theorem 4.1. Let A be a 4th order 2-dimensional symmetric tensor with $a_{1111} > 0$ and $a_{2222} > 0$. Assume that either condition (1) or (2) holds: (1) $a_{1222} \geq 0$, $54a_{1111}a_{1121}^3 + 64a_{1121}^3a_{1222} + 27a_{1111}^2a_{1222}^2 - 36a_{1221}^2 - 108a_{1111}a_{1121}a_{1221}a_{1222} \geq 0$; (2) $a_{1112} \geq 0$, $64a_{1211}a_{1221}^3 + 54a_{1221}^3a_{2222} + 27a_{1211}^2a_{2222}^2 - 36a_{1222}^2 - 108a_{1211}a_{1221}a_{1222}a_{2222} \geq 0$. Then A is copositive.

Proof. Rewrite Ax^4 as follows:

$$Ax^4 = x_1(a_{1111}x_1^3+4a_{1211}x_1^2x_2+6a_{1221}x_1x_2^2+4a_{1222}x_2^3)+a_{2222}x_2^4 = a_{1111}x_1^4+x_2(4a_{1211}x_1^3+6a_{1221}x_1^2x_2+4a_{1222}x_1$$

Define $f(x_1, x_2) = a_{1111}x_1^3 + 4a_{1211}x_1^2x_2 + 6a_{1221}x_1x_2^2 + 4a_{1222}x_2^3$. Then $f(x_1, x_2)$ can be expressed as a homogeneous polynomial defined by a 3rd order 2-dimensional symmetric tensor $\Gamma = (\gamma_{ijk})$ with entries $\gamma_{111} = a_{1111}$, $\gamma_{112} = a_{1211}$, $\gamma_{122} = 2a_{1221}$, $\gamma_{222} = 4a_{1222}$. From assumption (1), we have

$$4\gamma_{111}\gamma_{122}^3+4\gamma_{112}^3\gamma_{222}+\gamma_{111}^2\gamma_{222}^2-6\gamma_{111}\gamma_{112}\gamma_{122}\gamma_{222}-3\gamma_{112}^2\gamma_{122}^2 = 4a_{1111}(2a_{1221})^3+4(a_{1211})^3(4a_{1222})+a_{1111}^2(4a_{122}$$

By Theorem 3.1, we have $f(x_1, x_2) \geq 0$ for all $x \geq 0$, and hence $Ax^4 = x_1f(x_1, x_2) + a_{2222}x_2^4 \geq 0$ for all $x = (x_1, x_2)^\top \geq 0$. That is, A is copositive. Similarly, defining $g(x_1, x_2) = 4a_{1211}x_1^3 + 6a_{1221}x_1^2x_2 + 4a_{1222}x_1x_2^2 + a_{2222}x_2^3$, we also have $g(x_1, x_2) \geq 0$ for all $x \geq 0$ by assumption (2). Therefore, $Ax^4 = a_{1111}x_1^4 + x_2g(x_1, x_2) \geq 0$ for all $x = (x_1, x_2)^\top \geq 0$, proving the copositivity of A.

Similarly, using Theorem 3.2 (for the tensor $\Gamma = (\gamma_{ijk})$ in the above proof), we obtain the following theorem easily.

Theorem 4.2. Let A be a 4th order 2-dimensional symmetric tensor with $a_{1111} \geq 0$ and $a_{2222} \geq 0$. Assume that either condition (1) or (2) holds: (1) $a_{1222} \geq 0$, $a_{1112} \geq \sqrt{a_{1111}a_{1222}}$, $a_{1221} \geq \sqrt{a_{1112}a_{2222}}$, $a_{1222} \geq \sqrt{a_{2222}}(a_{1222} - 2\sqrt{a_{1111}a_{1222}})$; (2) $a_{1112} \geq 0$, $a_{1222} \geq \sqrt{a_{1112}a_{2222}}$, $a_{1221} \geq \sqrt{a_{1111}}(a_{1112} - 2\sqrt{a_{1112}a_{2222}})$. Then A is copositive.

Let A be a 4th order 3-dimensional symmetric tensor. Then for a vector $x = (x_1, x_2, x_3)^\top$,

$$Ax^4 = \sum_{i,j,k,l=1}^{3} a_{ijkl}x_ix_jx_kx_l = a_{1111}x_1^4+a_{2222}x_2^4+a_{3333}x_3^4+4a_{2111}x_1^3x_2+4a_{1222}x_1x_2^3+6a_{1122}x_1^2x_2^2+4a_{3111}x_1^3x_3+4a \tag{4.2}$$

Now we present several sufficient conditions for 4th order copositive tensors using the results for 3rd order copositive tensors from Section 3.

Theorem 4.3. Let A be a 4th order 3-dimensional symmetric tensor. Assume that $a_{1111} \geq 0$, $a_{2222} \geq 0$, $a_{3333} \geq 0$, $a_{1112} \geq 0$, $a_{1113} \geq 0$,

$a_{1222} \geq 0$, $a_{2223} \geq 0$, $a_{1333} \geq 0$, $a_{2333} \geq 0$, $\max\{a_{1222}, a_{1333}\} > 0$, $\max\{a_{1112}, a_{2333}\} > 0$, $\max\{a_{1113}, a_{2223}\} > 0$, $6a_{1122} + 8a_{1222}a_{1121}^3 + 16a_{2111}^2 a_{1222}^2 - 24a_{1222}a_{1223}a_{1233}a_{1333} - 3a_{1123}^2 a_{2333}^2 \geq 0$, $6a_{1133} + 8a_{3111}a_{1133}^3 + 16a_{3111}^2 a_{1333}^2 - 24a_{3111}a_{1123}a_{1233}a_{2333} - 3a_{1123}^2 a_{2333}^2 \geq 0$, and $6a_{2233} + 8a_{3222}a_{2233}^3 + 16a_{3222}^2 a_{2333}^2 - 24a_{3222}a_{1123}a_{1223}a_{2333} - 3a_{1123}^2 a_{2333}^2 \geq 0$. Then A is copositive.

Proof. Rewrite the homogeneous polynomial Ax^4 from (4.2) as follows:

$$Ax^4 = (a_{1111}x_1^4 + 6a_{1122}x_1^2x_2^2 + a_{2222}x_2^4) + (a_{2222}x_2^4 + 6a_{2233}x_2^2x_3^2 + a_{3333}x_3^4) + (a_{1111}x_1^4 + 6a_{1133}x_1^2x_3^2 + a_{3333}x_3^4) +$$

Define $p_1(x_1, x_2) = a_{1111}x_1^4 + 6a_{1122}x_1^2x_2^2 + a_{2222}x_2^4$. By Lemma 2.1, we can restrict x to $\|x\| = x_1 + x_2 + x_3 = 1$ with $x_i \geq 0$ for $i = 1, 2, 3$. Clearly, $p_1(0, 0) = 0$, $p_1(x_1, 0) = \frac{1}{2}a_{1111}x_1^4 \geq 0$, and $p_1(0, x_2) = \frac{1}{2}a_{2222}x_2^4 \geq 0$. Assume both x_1 and x_2 are nonzero. Without loss of generality, let $x_2 > 0$ and set $t = \frac{x_1}{x_2} \geq 0$. Then $p_1(x_1, x_2) = \alpha t^2 + \beta t + \gamma$ with $\alpha = a_{1111}$, $\beta = 6a_{1122}$, and $\gamma = a_{2222}$. By Lemma 2.3, the assumptions $a_{1111} \geq 0$, $a_{2222} \geq 0$, and $6a_{1122} + 2\sqrt{a_{1111}a_{2222}} \geq 0$ imply $p(t) \geq 0$ for all $t \geq 0$, and hence $p_1(x_1, x_2) \geq 0$ for all $x = (x_1, x_2)^\top \geq 0$. Similarly, $p_2(x_2, x_3) = a_{2222}x_2^4 + 6a_{2233}x_2^2x_3^2 + a_{3333}x_3^4 \geq 0$ and $p_3(x_1, x_3) = a_{1111}x_1^4 + 6a_{1133}x_1^2x_3^2 + a_{3333}x_3^4 \geq 0$.

Define $F_1(x_2, x_3) = 4a_{1222}x_2^3 + 6a_{1223}x_2^2x_3 + 6a_{1233}x_2x_3^2 + 4a_{1333}x_3^3$. This homogeneous polynomial can be written as $F_1(x_2, x_3) = \Gamma y^3 = \sum_{i,j,k=1}^2 \gamma_{ijk}y_iy_jy_k$ for $y = (x_2, x_3)^\top$, where $\Gamma = (\gamma_{ijk})$ is a 3rd order 2-dimensional symmetric tensor with entries $\gamma_{111} = 4a_{1222}$, $\gamma_{112} = 2a_{1223}$, $\gamma_{122} = 2a_{1233}$, $\gamma_{222} = 4a_{1333}$. By the assumptions, we have

$$4\gamma_{111}\gamma_{122}^3 + 4\gamma_{112}^3\gamma_{222} + \gamma_{111}^2\gamma_{222}^2 - 6\gamma_{111}\gamma_{112}\gamma_{122}\gamma_{222} - 3\gamma_{112}^2\gamma_{122}^2 = 16(8a_{1222}a_{1223}^3 + 8a_{1223}^3a_{1333} + 16a_{1222}^2a_{1333}^2 - 24a$$

By Theorem 3.1, Γ is copositive, i.e., $F_1(x_2, x_3) = \Gamma y^3 \geq 0$ for all $y = (x_2, x_3)^\top \geq 0$. Similarly, $F_2(x_1, x_3) = 4a_{2111}x_1^3 + 6a_{1123}x_1^2x_3 + 6a_{1233}x_1x_3^2 + 4a_{2333}x_3^3 \geq 0$ and $F_3(x_1, x_2) = 4a_{3111}x_1^3 + 6a_{1123}x_1^2x_2 + 6a_{1223}x_1x_2^2 + 4a_{3222}x_2^3 \geq 0$. Thus, for all $x = (x_1, x_2, x_3)^\top \geq 0$, we have

$$Ax^4 = p_1(x_1, x_2) + p_2(x_2, x_3) + p_3(x_1, x_3) + x_1F_1(x_2, x_3) + x_2F_2(x_1, x_3) + x_3F_3(x_1, x_2) \geq 0,$$

meaning A is copositive, as required.

Now we give a simpler sufficient condition for copositive tensors using Theorem 3.2 (for the tensor $\Gamma = (\gamma_{ijk})$ in the above proof).

Theorem 4.4. Let A be a 4th order 3-dimensional symmetric tensor. Assume that $a_{1111} \geq 0$, $a_{2222} \geq 0$, $a_{3333} \geq 0$, $a_{1112} \geq 0$, $a_{1113} \geq 0$, $a_{1222} \geq 0$, $a_{2223} \geq 0$,

$a_{1333} \geq 0$, $a_{2333} \geq 0$, $a_{1122} \geq -\sqrt{a_{1111}a_{2222}}$, $a_{1133} \geq -\sqrt{a_{1111}a_{3333}}$, $a_{2233} \geq -\sqrt{a_{3333}a_{2222}}$, $a_{1223} \geq \max\{a_{1222} - 2\sqrt{a_{1222}a_{1333}}, a_{2223} - 2\sqrt{a_{2333}a_{2223}}\}$, $a_{1233} \geq \max\{a_{1333} - 2\sqrt{a_{1113}a_{2223}}, a_{1112} - 2\sqrt{a_{1112}a_{2333}}\}$, and $a_{1123} \geq \max\{a_{1113} - 2\sqrt{a_{1112}a_{2333}}, a_{1112} - 2\sqrt{a_{1112}a_{2223}}\}$. Then A is copositive.

Theorem 4.5. Let A be a 4th order 3-dimensional symmetric tensor. Assume that $a_{1111} > 0$, $a_{2222} > 0$, $a_{3333} > 0$, $a_{1113} \geq 0$, $a_{1222} \geq 0$, $a_{2333} \geq 0$, $9a_{1122} + \sqrt{a_{1111}a_{2222}} \geq 0$, $9a_{2233} + \sqrt{a_{3333}a_{2222}} \geq 0$, $9a_{1133} + \sqrt{a_{1111}a_{3333}} \geq 0$, $27a_{1123} + \sqrt{(9a_{1122} + a_{1111}a_{2222})(9a_{1133} + a_{1111}a_{3333})} \geq 0$, $27a_{1223} + \sqrt{(9a_{1122} + a_{1111}a_{2222})(9a_{2233} + a_{3333}a_{2222})} \geq 0$, $27a_{1233} + \sqrt{(9a_{1133} + a_{1111}a_{3333})(9a_{2233} + a_{3333}a_{2222})} \geq 0$, and

$$2a_{1111}(9a_{1122}+a_{1111}a_{2222})^3+37\cdot 4^3a_{1222}a_{1112}^3+38a_{1111}^2a_{1112}^2(9a_{1122}+a_{1111}a_{2222})^2-36\cdot 4a_{1111}a_{1112}a_{1222}(9a_{1122}+a_{1}$$

$$2a_{2222}(9a_{2233}+a_{3333}a_{2222})^3+37\cdot 4^3a_{2223}a_{2333}^3+38a_{2222}^2a_{2333}^2(9a_{2233}+a_{3333}a_{2222})^2-36\cdot 4a_{2222}a_{2223}a_{2333}(9a_{2233}+a_{3}$$

$$2a_{3333}(9a_{1133}+a_{1111}a_{3333})^3+37\cdot 4^3a_{1333}a_{1113}^3+38a_{3333}^2a_{1333}^2(9a_{1133}+a_{1111}a_{3333})^2-36\cdot 4a_{3333}a_{1113}a_{1333}(9a_{1133}+a_{1}$$

Then A is copositive.

Proof. Rewrite the homogeneous polynomial Ax^4 from (4.2) as follows:

$$Ax^4 = (\sqrt{a_{1111}}x_1^2)^2+(\sqrt{a_{2222}}x_2^2)^2+(\sqrt{a_{3333}}x_3^2)^2+(3a_{1111}x_1^3 + 36a_{1112}x_1^2x_2 + 2(9a_{1122} + a_{1111}a_{2222})x_1x_2^2 + 36a_{1}$$

Define $F_1(x_1, x_2) = 3a_{1111}x_1^3 + 36a_{1112}x_1^2x_2 + 2(9a_{1122} + a_{1111}a_{2222})x_1x_2^2 + 36a_{1222}x_2^3$. This homogeneous polynomial can be written as $F_1(x_1, x_2) = \Gamma y^3 = \sum_{i,j,k=1}^2 \gamma_{ijk}y_iy_jy_k$ for $y = (x_1, x_2)^\top$, where $\Gamma = (\gamma_{ijk})$ is a 3rd order 2-dimensional symmetric tensor with entries $\gamma_{111} = 3a_{1111}$, $\gamma_{112} = 12a_{1112}$, $\gamma_{122} = (9a_{1122} + a_{1111}a_{2222})$, $\gamma_{222} = 36a_{1222}$. By the assumptions, we have

$$4\gamma_{111}\gamma_{122}^3+4\gamma_{112}^3\gamma_{222}+\gamma_{111}^2\gamma_{222}^2-6\gamma_{111}\gamma_{112}\gamma_{122}\gamma_{222}-3\gamma_{112}^2\gamma_{122}^2 = 4(3a_{1111})\left(\frac{9a_{1122} + a_{1111}a_{2222}}{2}\right)^3+4(12a_{1112})^3($$

By Theorem 3.1, Γ is copositive, i.e., $F_1(x_1, x_2) = \Gamma y^3 \geq 0$ for all $y = (x_1, x_2)^\top \geq 0$. Similarly, we must have $F_2(x_2, x_3) = 3a_{2222}x_2^3 + 36a_{2223}x_2^2x_3 + 2(9a_{2233} + a_{3333}a_{2222})x_2x_3^2 + 36a_{2333}x_3^3 \geq 0$ and $F_3(x_1, x_3) = 3a_{3333}x_3^3 + 36a_{1333}x_1x_3^2 + 2(9a_{1133} + a_{1111}a_{3333})x_1^2x_3 + 36a_{1113}x_1^3 \geq 0$.

Define $p_1(x_2, x_3) = (9a_{1122} + a_{1111}a_{2222})x_2^2 + 54a_{1123}x_2x_3 + (9a_{1133} + a_{1111}a_{3333})x_3^2$. By the (strict) copositivity of a 2×2 matrix (or Lemma 2.3), $p_1(x_2, x_3) \geq 0$ for all $(x_2, x_3)^\top \geq 0$ if and only if $9a_{1122} + a_{1111}a_{2222} \geq 0$, $9a_{1133} + a_{1111}a_{3333} \geq 0$, and $27a_{1123} + \sqrt{(9a_{1122} + a_{1111}a_{2222})(9a_{1133} + a_{1111}a_{3333})} \geq 0$. Similarly, we have $p_2(x_1, x_3) = (9a_{1122} + a_{1111}a_{2222})x_1^2 + 54a_{1223}x_1x_3 + (9a_{2233} + a_{3333}a_{2222})x_3^2 \geq 0$ and $p_3(x_1, x_2) = (9a_{1133} + a_{1111}a_{3333})x_1^2 + 54a_{1233}x_1x_2 + (9a_{2233} + a_{3333}a_{2222})x_2^2 \geq 0$.

Thus, for all $x = (x_1, x_2, x_3)^\top \geq 0$, we have

$$Ax^4 = (\sqrt{a_{1111}}x_1^2)^2 + (\sqrt{a_{2222}}x_2^2)^2 + (\sqrt{a_{3333}}x_3^2)^2 + \frac{1}{2}x_1p_1(x_2, x_3) + \frac{1}{2}x_2p_2(x_1, x_3) + \frac{1}{2}x_3p_3(x_1, x_2) + \frac{1}{2}x_1F_1(x_1, x_2)$$

meaning A is copositive, as required.

Remark 4.1. In this section, the conclusions are proved by reducing dimensions or orders of tensors. For example, in Theorems 4.5 and 4.3, a 4th order 3-dimensional tensor is decomposed into three 3rd order 2-dimensional tensors and three 2nd order 2-dimensional tensors, and then by studying the copositivity of these lower-dimensional and lower-order tensors, the desired conclusions are proved. Similarly, we may decompose a 4th order 3-dimensional tensor into three 3rd order 3-dimensional tensors $\Gamma_1, \Gamma_2, \Gamma_3$ such that

$$Ax^4 = x_1\Gamma_1x^3 + x_2\Gamma_2x^3 + x_3\Gamma_3x^3,$$

where

$$\Gamma_1x^3 = a_{1111}x_1^3 + 3a_{1122}x_1x_2^2 + 2a_{1333}x_2^3 + 2a_{1222}x_3^3 + 3a_{1133}x_1x_3^2 + 2a_{1113}x_1^2x_3 + 4a_{1123}x_1x_2x_3 + 4a_{1223}x_2^2x_3 + 4a_{1233}x$$

$$\Gamma_2x^3 = 2a_{1112}x_1^3 + 3a_{1122}x_1^2x_2 + 2a_{1222}x_1x_2^2 + a_{2222}x_2^3 + 2a_{2333}x_3^3 + 4a_{1233}x_1x_2x_3 + 4a_{1123}x_1^2x_3 + 4a_{1223}x_1x_2x_3 + 2a_{222}$$

$$\Gamma_3x^3 = 2a_{1113}x_1^3 + 4a_{1123}x_1^2x_2 + 4a_{1223}x_1x_2^2 + 2a_{2223}x_2^3 + a_{3333}x_3^3 + 2a_{1333}x_1x_2x_3 + 3a_{1133}x_1^2x_3 + 4a_{1233}x_1x_2x_3 + 3a_{223}$$

Then we can obtain other sufficient conditions for copositivity of a 4th order 3-dimensional tensor A by applying Theorems 3.4 and 3.5. We omit the details here. This approach of reducing dimensions or orders of tensors may be a very important method for analyzing higher-order tensors in the future.

5 Checking Vacuum Stability of Z3 Scalar Dark Matter

Kannike [12, 13] presented a physical example defined by scalar dark matter stable under a Z3 discrete group. The most general scalar quartic potential of the SM Higgs H_1, an inert doublet H_2, and a complex singlet S which is symmetric under a Z3 group is

$$V(h_1, h_2, s) = \lambda_1|H_1|^4+\lambda_2|H_2|^4+\lambda_3|H_1|^2|H_2|^2+\lambda_4(H_1^\dagger H_2)(H_2^\dagger H_1)+\lambda_S|S|^4+\lambda_{S1}|S|^2|H_1|^2+\lambda_{S2}|S|^2|H_2|^2+(\lambda_{S12}$$

$$= \lambda_1 h_1^4 + \lambda_2 h_2^4 + \lambda_3 h_1^2 h_2^2 + \lambda_4 \rho^2 h_1^2 h_2^2 + \lambda_S s^4 + \lambda_{S1} s^2 h_1^2 + \lambda_{S2} s^2 h_2^2 - |\lambda_{S12}|\rho s^2 h_1 h_2$$

$$= \Gamma z^4 = \sum_{i,j,k,l=1}^{3} \gamma_{ijkl} z_i z_j z_k z_l,$$

where $z = (z_1, z_2, z_3)^\top = (h_1, h_2, s)^\top$, the orbit space parameter $\rho \in [0, 1]$, $h_1 = |H_1|$, $h_2 = |H_2|$, $H_2^\dagger H_1 = h_1 h_2 \rho e^{i\phi}$, $S = se^{i\phi_S}$, $\lambda_{S12} = -|\lambda_{S12}|$, and $\Gamma = (\gamma_{ijkl})$ is a 4th order 3-dimensional real symmetric tensor with $\gamma_{1111} = \lambda_1$, $\gamma_{2222} = \lambda_2$, $\gamma_{3333} = \lambda_S$, $\gamma_{1122} = \frac{1}{6}(\lambda_3 + \lambda_4\rho^2)$, $\gamma_{1133} = \frac{1}{6}\lambda_{S1}$, $\gamma_{2233} = \frac{1}{6}\lambda_{S2}$, $\gamma_{1233} = -\frac{1}{12}|\lambda_{S12}|\rho$, and $\gamma_{ijkl} = 0$ for all other entries. Clearly, $z \geq 0$. It follows from Theorem 4.5 that the conditions for (strict) copositivity of the tensor Γ (that is, $V(h_1, h_2, s) = \Gamma z^4 \geq 0(> 0)$) are

$$\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_S > 0,$$

$$3\lambda_3 + 3\lambda_4\rho^2 + 2\sqrt{\lambda_1\lambda_2} \geq 0(> 0),$$

$$3\lambda_{S1} + 2\sqrt{\lambda_1\lambda_S} \geq 0(> 0), \quad 3\lambda_{S2} + 2\sqrt{\lambda_S\lambda_2} \geq 0(> 0),$$

$$|\lambda_{S12}|\rho + \sqrt{\left(3\lambda_{S1} + 2\sqrt{\lambda_1\lambda_S}\right)\left(3\lambda_{S2} + 2\sqrt{\lambda_S\lambda_2}\right)} \geq 0(> 0).$$

These conditions ensure that the potential $V(h_1, h_2, s)$ under a Z3 group is bounded from below, thereby guaranteeing vacuum stability for Z3 scalar dark matter.

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Note: Figure translations are in progress. See original paper for figures.