Analytical Expressions of Copositivity for 4th Order Symmetric Tensors and Applications

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Abstract

In particle physics, scalar potentials have to be bounded from below in order for the physics to make sense. The precise expressions of checking lower bound of scalar potentials are essential, which is an analytical expression of checking copositivity and positive definiteness of tensors given by such scalar potentials. Because the tensors given by general scalar potential are 4th order and symmetric, our work mainly focuses on finding precise expressions to test copositivity and positive definiteness of 4th order tensors in this paper. First of all, an analytically sufficient and necessary condition of positive definiteness is provided for 4th order 2 dimensional symmetric tensors. For 4th order 3 dimensional symmetric tensors, we give two analytically sufficient conditions of (strictly) cpositivity by using proof technique of reducing orders or dimensions of such a tensor. Furthermore, an analytically sufficient and necessary condition of copositivity is showed for 4th order 2 dimensional symmetric tensors. We also give several distinctly analytically sufficient conditions of (strict) copositivity for 4th order 2 dimensional symmetric tensors. Finally, we apply these results to check lower bound of scalar potentials, and to present analytical vacuum stability conditions for potentials of two real scalar fields and the Higgs boson.

Full Text

Preamble

Analytical Expressions of Copositivity for 4th Order Symmetric Tensors and Applications

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Abstract. In particle physics, scalar potentials must be bounded from below for the physical theory to be sensible. Precise expressions for checking the lower bound of scalar potentials are essential, which correspond to analytical expressions for checking copositivity and positive definiteness of tensors arising from such scalar potentials. Since tensors from general scalar potentials are 4th order and symmetric, this work focuses on finding precise expressions to test copositivity and positive definiteness of 4th order tensors. First, we provide an analytically sufficient and necessary condition for positive definiteness of 4th order 2-dimensional symmetric tensors. For 4th order 3-dimensional symmetric tensors, we give two analytically sufficient conditions for (strict) copositivity using proof techniques that reduce the order or dimensions of such tensors. Furthermore, an analytically sufficient and necessary condition for copositivity is established for 4th order 2-dimensional symmetric tensors. We also give several distinct analytically sufficient conditions for (strict) copositivity of 4th order 2-dimensional symmetric tensors. Finally, we apply these results to check the lower bound of scalar potentials and present analytical vacuum stability conditions for potentials of two real scalar fields and the Higgs boson.

Key Words and Phrases: Copositive Tensors, Positive Definiteness, Homogeneous Polynomial, Analytical Expression.

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Introduction

Recently, Kannike [25, 26] studied the vacuum stability of general scalar potentials of a few fields. The most general scalar potential of n real singlet scalar fields ϕ_i $(i = 1, 2, \cdots, n)$ can be expressed as

$$V(\phi) = \sum_{i,j,k,l=1}^{n} \lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l = \Lambda\phi^4, \tag{1.1}$$

where $\Lambda = (\lambda_{ijkl})$ is the symmetric tensor of scalar couplings and $\phi = (\phi_1, \phi_2, \cdots, \phi_n)^\top$ is the vector of fields. Thus, the vacuum stability of such a system is equivalent to the positivity of the polynomial (1.1) [25], i.e., the positive definiteness of the tensor $\Lambda = (\lambda_{ijkl})$. However, it is NP-hard to determine the non-negativity of a given polynomial if its degree is larger than or equal to 4 [18,31]. A significant special case [27] is the quartic potentials of quadratic scalar fields ϕ_i^2 $(i = 1, 2, \cdots, n)$, presented by

$$V(\phi) = \sum_{i,j=1}^{n} \lambda_{ij}\phi_i^2\phi_j^2 = (\phi_1^2, \phi_2^2, \cdots, \phi_n^2)^\top A(\phi_1^2, \phi_2^2, \cdots, \phi_n^2), \tag{1.2}$$

where $A = (\lambda_{ij})$ is a symmetric matrix. Then the positivity of the polynomial (1.2) becomes the strict copositivity of matrix A. In 2012, Kannike [27] first obtained the vacuum stability conditions of such a special case by testing copositivity of matrices. The vacuum stability conditions of the general potential of two real scalars (without or with the Higgs boson included in the potential) were obtained in [25, 26] with the help of matrix copositivity and polynomial positivity.

The concept of copositive matrices was introduced by Motzkin [33] in 1952. A real symmetric matrix A is said to be (i) copositive if $x^T Ax \geq 0$ for all vectors $x \geq 0$ in the non-negative orthant $\mathbb{R}^n_+$ ($x \geq 0$ implies that $x_i \geq 0$ for each $i = 1, 2, \cdots, n$); (ii) strictly copositive if $x^T Ax > 0$ for all nonzero vectors $x \geq 0$. Hadeler [20] and Nadler [34] established the copositive conditions for a 2×2 matrix A (also see Andersson-Chang-Elfving [1]). A real symmetric 2×2 matrix $A = (a_{ij})$ is (strictly) copositive if and only if $a_{11} \geq 0 (> 0)$, $a_{22} \geq 0 (> 0)$, and $a_{12} + \sqrt{a_{11}a_{22}} \geq 0 (> 0)$.

The copositive conditions for a 3×3 matrix A were obtained by Hadeler [20] and Chang-Sederberg [8]. A real symmetric 3×3 matrix $A = (a_{ij})$ is copositive if and only if

$$\alpha = a_{12} + \sqrt{a_{11}a_{22}} \geq 0, \quad \beta = a_{13} + \sqrt{a_{33}} + \frac{a_{13}a_{22} + a_{23}\sqrt{a_{11}a_{33}}}{\sqrt{a_{22}}} \geq 0,$$

$$\gamma = a_{23} + \sqrt{a_{11}} \geq 0, \quad a_{22} \geq 0, \quad a_{33} \geq 0, \quad a_{33}a_{22} \geq 0, \quad a_{11}a_{22}a_{33} + \sqrt{2\alpha\beta\gamma} \geq 0.$$

Ping-Yu [36] gave criteria for 4×4 copositive matrices, though the expression is not simpler than the above. They also proved an equivalent condition for $n \times n$ copositive matrices. Cottle-Habetler-Lemke [7] presented analytical conditions for copositive matrices using determinants and adjugates. Väliaho [50] discussed criteria for (strictly) copositive matrices using properties of principal submatrices. Kaplan [24] proved a method to test matrix copositivity using eigenvalues and eigenvectors of principal submatrices. Haynsworth-Hoffman [21] showed Perron properties of a class of copositive matrices. Johnson-Reams [22] discussed spectral theory of copositive matrices. For more properties and applications such as copositive programs, see [2–4, 32] and relevant literature.

Recently, Kannike [25,26] gave another physical example defined by scalar dark matter stable under a Z_3 discrete group. The most general scalar quartic potential of the SM Higgs H_1, an inert doublet H_2, and a complex singlet S is

$$V(h_1, h_2, s) = V(\phi) = V\phi^4 = \sum_{i,j,k,l=1}^{3} v_{ijkl}\phi_i\phi_j\phi_k\phi_l, \tag{1.3}$$

where $\phi = (\phi_1, \phi_2, \phi_3)^\top = (h_1, h_2, s)^\top$, $h_1 = |H_1|$, $h_2 = |H_2|$, $H_2^\dagger H_1 = h_1h_2\rho e^{i\phi}$, $S = se^{i\phi_S}$, $V = (v_{ijkl})$ is a 4th order 3-dimensional real symmetric tensor with entries:

$$v_{1111} = \lambda_1, \quad v_{2222} = \lambda_2, \quad v_{3333} = \lambda_S, \quad v_{1122} = \frac{1}{6}(\lambda_3 + \lambda_4 \rho^2),$$

$$v_{1133} = \frac{1}{6}\lambda_{S2}, \quad v_{1233} = -\frac{1}{12}|\lambda_{S12}|\rho, \quad v_{ijkl} = 0 \text{ for others.}$$

Clearly, $h_1 \geq 0$, $h_2 \geq 0$, $s \geq 0$. Thus, the vacuum stability for Z_3 scalar dark matter $V(h_1, h_2, s)$ is equivalent to the (strict) copositivity of the tensor $V = (v_{ijkl})$ [25, 26].

An mth order n-dimensional real symmetric tensor $\mathcal{A}$ is said to be (i) copositive if $\mathcal{A}x^m = x^T(\mathcal{A}x^{m-1}) = \sum_{i_1,i_2,\cdots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \geq 0$ for all $x \in \mathbb{R}^n_+$; (ii) strictly copositive if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$; (iii) semipositive definite if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n$ and even m; (iv) positive definite if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and even m.

These concepts were first introduced by Qi [37,38] for higher order symmetric tensors. Qi [37] showed that an even order symmetric tensor is positive definite if and only if all its H-(Z-)eigenvalues are positive. Qi [38] proved that a symmetric tensor is strictly copositive if the sum of the main diagonal element and negative elements in each row is positive. Song-Qi [41] extended Kaplan' s test for copositive matrices [24] to copositive tensors and presented structural properties of this class. Recently, checking copositivity of tensors has attracted mathematical attention. For example, Chen-Huang-Qi [11] studied basic theory of copositivity detection for symmetric tensors and gave numerical algorithms based on standard simplex and simplicial partitions; Chen-Huang-Qi [12] revised the algorithm with a proper convex subcone of the copositive tensor cone; Nie-Yang-Zhang [35] proposed a complete semidefinite relaxation algorithm for detecting copositivity and showed this can be done by solving a finite number of semidefinite relaxations; Li-Zhang-Huang-Qi [28] presented an SDP relaxation algorithm to test copositivity of higher order tensors. For more properties and algorithms, see [13, 39, 40].

On the other hand, some structured tensors are closely related to strictly copositive tensors. Song-Qi [45] analyzed the relationship between constrained minimization problems on the unit sphere and (strict) copositivity of corresponding tensors. Song-Qi [44] proved that a symmetric tensor is (strictly) copositive if and only if it is (strictly) semipositive. A tensor is called (strictly) semipositive if for each nonzero vector $x = (x_1, x_2, \cdots, x_n)^\top \geq 0$, there exists an index $k \in \{1, 2, \cdots, n\}$ such that $x_k > 0$ and $(\mathcal{A}x^{m-1})_k = \sum_{i_2,\cdots,i_m=1}^{n} a_{k i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \geq 0 (> 0)$. This notion was first used by Song-Qi [42]. This class of tensors ensures solvability of corresponding tensor complementarity problems (TCP). Thus, we may investigate checking copositivity and its applications by studying semipositive tensors. For more properties and TCP applications, see [5, 6, 9, 10, 14, 15, 17, 19, 30, 46-48, 51-53].

Until now, there has been no analytical expression for checking copositivity and positive definiteness of tensors like those for 2×2 and 3×3 matrices. However, practical problems such as vacuum stability of general scalar potentials require precise expressions.

Motivated by these works, we study analytical expressions for certifying symmetric tensors to be copositive and positive definite. We confine our work to 4th order tensors since those from general scalar potentials are 4th order. We provide analytical expressions for testing copositivity and positive definiteness of 4th order 3- (or 2-) dimensional symmetric tensors. We employ argumentation techniques reducing orders or dimensions, which may be important for analyzing higher order tensors. These results can be applied to check vacuum stability of general scalar potentials of two real singlet scalar fields and Z_3 scalar dark matter.

2 Preliminaries and Basic Facts

A 4th order 3-dimensional real tensor $\mathcal{A}$ consists of 81 entries in $\mathbb{R}$, i.e., $\mathcal{A} = (a_{ijkl})$, $a_{ijkl} \in \mathbb{R}$, $i, j, k, l = 1, 2, 3$. Let $x^\top$ denote the transpose of vector x. For $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, $\mathcal{A}x^3$ is a vector in $\mathbb{R}^3$:

$$\mathcal{A}x^3 = \left(\sum_{j,k,l=1}^{3} a_{1jkl}x_j x_k x_l, \sum_{j,k,l=1}^{3} a_{2jkl}x_j x_k x_l, \sum_{j,k,l=1}^{3} a_{3jkl}x_j x_k x_l \right)^\top. \quad (2.1)$$

Then $x^\top(\mathcal{A}x^3)$ is a homogeneous polynomial, denoted $\mathcal{A}x^4$:

$$\mathcal{A}x^4 = x^\top(\mathcal{A}x^3) = \sum_{i,j,k,l=1}^{3} a_{ijkl}x_i x_j x_k x_l. \quad (2.2)$$

Similarly, a 4th order 2-dimensional real tensor $\mathcal{A}$ consists of 16 entries in $\mathbb{R}$, and for $x = (x_1, x_2)^\top \in \mathbb{R}^2$,

$$\mathcal{A}x^4 = x^\top(\mathcal{A}x^3) = \sum_{i,j,k,l=1}^{2} a_{ijkl}x_i x_j x_k x_l. \quad (2.3)$$

A tensor $\mathcal{A}$ is symmetric if its entries a_{ijkl} are invariant under any permutation of indices. Each 4th order 2-dimensional symmetric tensor $\mathcal{A}$ determines a homogeneous polynomial $\mathcal{A}x^4$ of degree 4 with 2 variables and vice versa.

Let $\|\cdot\|$ denote any norm on $\mathbb{R}^n$. The equivalent definitions of (strict) copositivity and semipositive (positive) definiteness of a symmetric tensor in the sense of any norm are given in [37, 39–41].

Lemma 2.1. ([37, 41]) Let $\mathcal{A}$ be a symmetric tensor of order 4. Then: (i) $\mathcal{A}$ is copositive iff $\mathcal{A}x^4 \geq 0$ for all $x \in \mathbb{R}^n_+$ with $\|x\| = 1$; (ii) $\mathcal{A}$ is strictly copositive iff $\mathcal{A}x^4 > 0$ for all $x \in \mathbb{R}^n_+$ with $\|x\| = 1$; (iii) $\mathcal{A}$ is semipositive definite iff $\mathcal{A}x^4 \geq 0$ for all $x \in \mathbb{R}^n$ with $\|x\| = 1$; (iv) $\mathcal{A}$ is positive definite iff $\mathcal{A}x^4 > 0$ for all $x \in \mathbb{R}^n$ with $\|x\| = 1$.

A quadratic Bernstein-Bézier polynomial $p(t)$ on $[0, 1]$ is

$$p(t) = at^2 + 2b(1-t)t + c(1-t)^2, \quad t \in [0, 1]. \tag{2.4}$$

Nadler [34] and Andersson-Chang-Elfving [1] independently showed:

Lemma 2.2. ([34, Lemma 1], [1, Lemma 2.1]) Let $p(t)$ be defined by (2.4). Then $p(t) \geq 0$ (> 0) for all $t \in [0, 1]$ iff

$$a \geq 0 (> 0), \quad c \geq 0 (> 0), \quad b + \sqrt{ac} \geq 0 (> 0) \tag{2.5}$$

hold simultaneously.

For a quartic univariate polynomial $f(t)$ with real coefficients,

$$f(t) = a_0 t^4 + 4a_1 t^3 + 6a_2 t^2 + 4a_3 t + a_4, \tag{2.6}$$

Gadenz-Li [16], Ku [23], and Jury-Mansor [29] independently obtained positivity conditions.

Lemma 2.3. ([16, 23, 29]) Let $f(t)$ be defined by (2.6) with $a_0 > 0$ and $a_4 > 0$. Define

$$F = 9a_2^2 + 12a_4 a_1^2 - 3a_0 a_4 + 4a_0 a_1 a_3,$$

$$G = a_2^2 - 24a_0 a_2 a_3 - 3a_0 a_1 a_2 + 2a_3^2,$$

$$H = a_0 a_2 - a_1^2,$$

$$I = a_0 a_4 - 4a_1 a_3 + 3a_2^2,$$

$$J = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_3^2 - a_0 a_3^2 - a_1^2 a_4,$$

$$\Delta = I^3 - 27J^2.$$

Then $f(t) > 0$ for all $0 < |t| < \infty$ iff: (1) $\Delta > 0$, $H \geq 0$; (2) $\Delta > 0$, $H < 0$, $F < 0$; (3) $\Delta = 0$, $H > 0$, $F = 0$, $G = 0$.

For a quartic univariate polynomial $g(t)$ with real coefficients,

$$g(t) = at^4 + bt^3 + ct^2 + dt + e, \tag{2.7}$$

Ulrich-Watson [49] proved nonnegativity conditions for $t > 0$.

Lemma 2.4. ([49, Theorem 2]) Let $g(t)$ be defined by (2.7) with $a > 0$ and $e > 0$. Define

$$\alpha = ba^{-3/4}e^{-1/4}, \quad \beta = ca^{-1/2}e^{-1/2}, \quad \gamma = da^{-1/4}e^{-3/4},$$

$$\Delta = 4(\beta^2 - 3\alpha\gamma + 12)^3 - (72\beta + 9\alpha\beta\gamma - 2\beta^3 - 27\alpha^2 - 27\gamma^2)^2,$$

$$\mu = (\alpha - \gamma)^2 - 16(\alpha + \beta + \gamma + 2),$$

$$\eta = (\alpha - \gamma)^2 - 4(\beta + 2)\sqrt{\beta - 2}(\alpha + \gamma + 4\sqrt{\beta - 2}).$$

Then: (i) $g(t) \geq 0$ for all $t > 0$ iff $\beta < -2$ and $\Delta \leq 0$ and $\alpha + \gamma > 0$; or $-2 \leq \beta \leq 6$ and $\Delta \leq 0$; or $\beta > 6$ and $\Delta \leq 0$ and $\alpha > 0$; or $\Delta \geq 0$ and $\alpha + \gamma > 0$ and $\gamma > 0$ and $\eta \leq 0$.

(ii) $g(t) \geq 0$ for all $t > 0$ if $\alpha > -\frac{\beta+2}{2}$ and $\gamma > -\frac{\beta+2}{2}$ for $\beta \leq 6$; or $\alpha > -2\sqrt{\beta - 2}$ and $\gamma > -2\sqrt{\beta - 2}$ for $\beta > 6$.

A quadratic multivariate polynomial $F(1-t, tv, tw)$ is defined by

$$F(1-t, tv, tw) = A(1-t)^2 + 2(bw+cv)t(1-t) + (Bv^2 + 2awv + Cw^2)t^2, \quad t, v \in [0, 1], \tag{2.8}$$

where $w = 1 - v$. Chang-Sederberg [8] provided the following result (see also Nadler [34]).

Lemma 2.5. ([8, Theorem 1]) Let $F(1-t, tv, tw)$ be defined by (2.8). Then $F(1-t, tv, tw) \geq 0$ (> 0) for all $t \in [0, 1]$, $v \in [0, 1]$, and $w = 1 - v$ iff

$$A \geq 0 (> 0), \quad B \geq 0 (> 0), \quad C \geq 0 (> 0),$$

$$b + \sqrt{AC} \geq 0 (> 0), \quad c + \sqrt{AB} \geq 0 (> 0),$$

$$\sqrt{AB} + \sqrt{AC} + \sqrt{BC} + a \geq 0 (> 0),$$

$$\sqrt{AB} + \sqrt{AC} + \sqrt{BC} - a \geq 0 (> 0),$$

$$2(a + \sqrt{BC})(b + \sqrt{AC})(c + \sqrt{AB}) \geq 0 (> 0). \tag{2.9}$$

3 Copositivity of 4th Order Symmetric Tensors

Let $\mathcal{A}$ be a 4th order 2-dimensional symmetric tensor. For vector $x = (x_1, x_2)^\top$,

$$\mathcal{A}x^4 = \sum_{i,j,k,l=1}^{2} a_{ijkl} x_i x_j x_k x_l = a_{1111} x_1^4 + 4a_{1211} x_1^3 x_2 + 6a_{1221} x_1^2 x_2^2 + 4a_{1222} x_1 x_2^3 + a_{2222} x_2^4. \tag{3.1}$$

Take $y = (1,0)^\top$ and $z = (0,1)^\top$. Then $\mathcal{A}y^4 = a_{1111}$ and $\mathcal{A}z^4 = a_{2222}$. Thus, $a_{1111} > 0$ and $a_{2222} > 0$ are necessary for positive definiteness of $\mathcal{A}$.

Theorem 3.1. Let $\mathcal{A}$ be a symmetric tensor of order 4 and dimension 2 with $a_{1111} > 0$ and $a_{2222} > 0$. Then $\mathcal{A}$ is positive definite iff:

(1) $a_{1111}a_{1221} \geq a_{1211}^2$, and $(a_{1111}a_{2222} - 4a_{1211}a_{1222} + 3a_{1221}^2 - a_{1111}a_{1222}^2 - a_{1211}^2 a_{2222})^2$;

(2) $a_{1111}a_{1221} < a_{1211}^2$, and $(a_{1111}a_{2222} - 4a_{1211}a_{1222} + 3a_{1221}^2 - a_{1111}a_{1222}^2 - a_{1211}^2 a_{2222})^2$, with $a_{1111}a_{1211}a_{1222} < a_{1111}^3 a_{2222} + 24a_{1111}^2 a_{1211}a_{1221}$;

(3) $a_{1111}a_{1221} > a_{1211}^2$, and $(a_{1111}a_{2222} - 4a_{1211}a_{1222} + 3a_{1221}^2 - a_{1111}a_{1222}^2 - a_{1211}^2 a_{2222})^2$, with $a_{1111}a_{1211}a_{1222} = a_{1111}^3 a_{2222} + 24a_{1111}^2 a_{1211}a_{1221}$.

Proof. By Lemma 2.1, we can restrict x to $\|x\| = |x_1| + |x_2| = 1$. Consider $\mathcal{A}x^4$ with $a_{1111} > 0$ and $a_{2222} > 0$ in three cases.

Case 1: $x_1 = 0$ and $x_2 \neq 0$. Then $|x_2| = 1$, so $\mathcal{A}x^4 = a_{2222} > 0$.

Case 2: $x_1 \neq 0$ and $x_2 = 0$. Then $|x_1| = 1$, so $\mathcal{A}x^4 = a_{1111} > 0$.

Case 3: $x_1 \neq 0$ and $x_2 \neq 0$. Divide $\mathcal{A}x^4$ by x_2^4 to obtain

$$\frac{\mathcal{A}x^4}{x_2^4} = a_{1111}\left(\frac{x_1}{x_2}\right)^4 + 4a_{1211}\left(\frac{x_1}{x_2}\right)^3 + 6a_{1221}\left(\frac{x_1}{x_2}\right)^2 + 4a_{1222}\left(\frac{x_1}{x_2}\right) + a_{2222}.$$

Let $t = \frac{x_1}{x_2}$ and $f(t) = \frac{\mathcal{A}x^4}{x_2^4}$, i.e.,

$$f(t) = a_{1111}t^4 + 4a_{1211}t^3 + 6a_{1221}t^2 + 4a_{1222}t + a_{2222}. \tag{3.2}$$

Clearly, $f(t) > 0$ iff $\mathcal{A}x^4 > 0$. Define

$$F = 9a_{1221}^2 + 12a_{2222}a_{1211}^2 - 3a_{1111}a_{2222} + 4a_{1111}a_{1211}a_{1222},$$

$$G = a_{1221}^2 - 24a_{1111}a_{1221}a_{1222} - 3a_{1111}a_{1211}a_{1221} + 2a_{1222}^2,$$

$$H = a_{1111}a_{1221} - a_{1211}^2,$$

$$I = a_{1111}a_{2222} - 4a_{1211}a_{1222} + 3a_{1221}^2,$$

$$J = a_{1111}a_{1221}a_{2222} + 2a_{1211}a_{1221}a_{1222} - a_{1222}^3 - a_{1111}a_{1222}^2 - a_{1211}^2a_{2222},$$

$$\Delta = I^3 - 27J^2.$$

The conclusions follow directly from Lemma 2.3 with $a_0 = a_{1111}$, $a_1 = a_{1211}$, $a_2 = a_{1221}$, $a_3 = a_{1222}$, and $a_4 = a_{2222}$.

Remark 3.1. From the proof of Theorem 3.1, $\mathcal{A}x^4$ can be divided by x_1^4, giving $f(t) = a_{1111} + 4a_{1211}t + 6a_{1221}t^2 + 4a_{1222}t^3 + a_{2222}t^4$, where $t = \frac{x_2}{x_1}$. Thus the conclusions still hold if a_{2222} and a_{1111} are interchanged.

Theorem 3.2. Let $\mathcal{A}$ be a symmetric tensor of order 4 and dimension 3. Assume:

$$a_{1111} \geq 0, \quad a_{2222} \geq 0, \quad a_{3333} \geq 0, \quad a_{1122} \geq 0, \quad a_{1133} \geq 0, \quad a_{2233} \geq 0,$$

$$\alpha_1 = 6a_{1231} + 3\sqrt{a_{1122}a_{1133}} \geq 0, \quad \beta_1 = 2a_{1113} + 3\sqrt{a_{1111}a_{1122}} \geq 0, \quad \gamma_1 = 2a_{2111} + \sqrt{a_{1122}a_{2233}} \geq 0,$$

$$\alpha_2 = 2a_{3222} + 3\sqrt{a_{2222}a_{2233}} \geq 0, \quad \beta_2 = 6a_{1223} + 3\sqrt{a_{1122}a_{2222}} \geq 0, \quad \gamma_2 = 2a_{1222} + \sqrt{a_{1122}a_{2233}} \geq 0,$$

$$\alpha_3 = 2a_{2333} + 3\sqrt{a_{3333}a_{2233}} \geq 0, \quad \beta_3 = 2a_{1333} + 3\sqrt{a_{1111}a_{1133}} \geq 0, \quad \gamma_3 = 6a_{1233} + 3\sqrt{a_{1133}a_{2233}} \geq 0,$$

$$\tau_1 = 3\sqrt{a_{1111}a_{1122}a_{1133}} + 6a_{1231} + \sqrt{2\alpha_1\beta_1\gamma_1} \geq 0,$$

$$\tau_2 = 3\sqrt{a_{1122}a_{2222}a_{2233}} + 2a_{3222} + \sqrt{2\alpha_2\beta_2\gamma_2} \geq 0,$$

$$\tau_3 = 3\sqrt{a_{1133}a_{3333}a_{2233}} + 2a_{2333} + \sqrt{2\alpha_3\beta_3\gamma_3} \geq 0.$$

Then $\mathcal{A}$ is copositive.

Proof. By Lemma 2.1, we can restrict x to $\|x\| = x_1 + x_2 + x_3 = 1$ with $x_i \geq 0$ for $i = 1, 2, 3$. Without loss of generality, let $x_1 = 1 - t$, $x_2 = tv$, $x_3 = tw$ for $t, v \in [0, 1]$ and $w = 1 - v$. Then

$$\mathcal{A}x^4 = \sum_{i,j,k,l=1}^{3} a_{ijkl}x_ix_jx_kx_l$$

$$= a_{1111}(1-t)^4 + a_{2222}(tv)^4 + a_{3333}(tw)^4 + 4a_{1222}(1-t)(tv)^3 + 4a_{1333}(1-t)(tw)^3$$

$$+4a_{2111}(1-t)^3(tv) + 4a_{2333}(tv)(tw)^3 + 4a_{3111}(1-t)^3(tw) + 4a_{3222}(tv)^3(tw)$$

$$+6a_{1122}(1-t)^2(tv)^2 + 6a_{1133}(1-t)^2(tw)^2 + 6a_{2233}(tv)^2(tw)^2$$

$$+12a_{1231}(1-t)^2(tv)(tw) + 12a_{1232}(1-t)(tv)^2(tw) + 12a_{1233}(1-t)(tv)(tw)^2. \quad (3.3)$$

Simple calculation yields

$$\mathcal{A}x^4 = [a_{1111}(1-t)^2 + 2(2a_{3111}w + 2a_{2111}v)t(1-t) + (3a_{1122}v^2 + 12a_{1231}vw + 3a_{1133}w^2)t^2](1-t)^2$$

$$+[3a_{1122}(1-t)^2 + 2(6a_{1232}w + 2a_{1222}v)t(1-t) + (a_{2222}v^2 + 4a_{3222}vw + 3a_{2233}w^2)t^2](tv)^2$$

$$+[3a_{1133}(1-t)^2 + 2(2a_{1333}w + 6a_{1233}v)t(1-t) + (3a_{2233}v^2 + 4a_{2333}vw + a_{3333}w^2)t^2](tw)^2.$$

Define

$$F_1(1-t,tv,tw) = a_{1111}(1-t)^2+2(2a_{3111}w+2a_{2111}v)t(1-t)+(3a_{1122}v^2+12a_{1231}vw+3a_{1133}w^2)t^2,$$

$$F_2(1-t,tv,tw) = 3a_{1122}(1-t)^2+2(6a_{1232}w+2a_{1222}v)t(1-t)+(a_{2222}v^2+4a_{3222}vw+3a_{2233}w^2)t^2,$$

$$F_3(1-t,tv,tw) = 3a_{1133}(1-t)^2+2(2a_{1333}w+6a_{1233}v)t(1-t)+(3a_{2233}v^2+4a_{2333}vw+a_{3333}w^2)t^2.$$

For $F_1(1-t,tv,tw)$, with assumptions $a_{1111} \geq 0$, $a_{1122} \geq 0$, $a_{1133} \geq 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$, $\gamma_1 \geq 0$, $\tau_1 \geq 0$, Lemma 2.5 gives $F_1(1-t,tv,tw) \geq 0$ for all $t,v \in [0,1]$ and $w = 1-v$. Similarly, $F_2(1-t,tv,tw) \geq 0$ and $F_3(1-t,tv,tw) \geq 0$ for all $t,v \in [0,1]$ and $w = 1-v$. Therefore,

$$\mathcal{A}x^4 = F_1(1-t,tv,tw)(1-t)^2 + F_2(1-t,tv,tw)(tv)^2 + F_3(1-t,tv,tw)(tw)^2 \geq 0.$$

Thus $\mathcal{A}x^4 \geq 0$ for all $x \geq 0$ with $\|x\| = 1$, i.e., $\mathcal{A}$ is copositive.

If "$\geq$" is replaced by "$>$" in all conditions of Theorem 3.2, strict copositivity of $\mathcal{A}$ follows easily.

Theorem 3.3. Let $\mathcal{A}$ be a symmetric tensor of order 4 and dimension 3. Assume:

$$a_{1111} > 0, \quad a_{2222} > 0, \quad a_{3333} > 0, \quad a_{1122} > 0, \quad a_{1133} > 0, \quad a_{2233} > 0,$$

$$\alpha_1 = 6a_{1231}+3\sqrt{a_{1122}a_{1133}} > 0, \quad \beta_1 = 2a_{1113}+3\sqrt{a_{1111}a_{1122}} > 0, \quad \gamma_1 = 2a_{2111}+\sqrt{a_{1122}a_{2233}} > 0,$$

$$\alpha_2 = 2a_{3222}+3\sqrt{a_{3333}a_{2233}} > 0, \quad \beta_2 = 6a_{1223}+3\sqrt{a_{1122}a_{2222}} > 0, \quad \gamma_2 = 2a_{1222}+\sqrt{a_{1122}a_{2233}} > 0,$$

$$\alpha_3 = 2a_{2333}+3\sqrt{a_{1133}a_{2233}} > 0, \quad \beta_3 = 2a_{1333}+3\sqrt{a_{1111}a_{1133}} > 0, \quad \gamma_3 = 6a_{1233}+3\sqrt{a_{2222}a_{2233}} > 0,$$

$$\tau_1 = 3\sqrt{a_{1111}a_{1122}a_{1133}} + 6a_{1231} + \sqrt{2\alpha_1\beta_1\gamma_1} > 0,$$

$$\tau_2 = 3\sqrt{a_{1122}a_{2222}a_{2233}} + 2a_{3222} + \sqrt{2\alpha_2\beta_2\gamma_2} > 0,$$

$$\tau_3 = 3\sqrt{a_{1133}a_{3333}a_{2233}} + 2a_{2333} + \sqrt{2\alpha_3\beta_3\gamma_3} > 0.$$

Then $\mathcal{A}$ is strictly copositive.

Remark 3.2. From the proofs of Theorems 3.2 and 3.3, we see the results are obtained by reducing tensor order. That is, a 4th order 3-dimensional tensor is decomposed into three 2nd order 3-dimensional tensors, and copositivity of these 2nd order tensors is analyzed to obtain sufficient conditions. This may be an important method for studying higher order tensors.

Theorem 3.4. Let $\mathcal{A}$ be a symmetric tensor of order 4 and dimension 2 with $a_{1111} > 0$ and $a_{2222} > 0$. Assume:

(1) $a_{1221} \le \sqrt{a_{1111}a_{2222}}$, $\frac{4a_{1211}}{a_{2222}} + 4\sqrt{\frac{a_{1111}}{a_{2222}}(3a_{1221} + \sqrt{a_{1111}a_{2222}})} > 0$, and $\frac{4a_{1222}}{a_{1111}} + 4\sqrt{\frac{a_{2222}}{a_{1111}}(3a_{1221} + \sqrt{a_{1111}a_{2222}})} > 0$;

(2) $a_{1221} > \sqrt{a_{1111}a_{2222}}$, $2a_{1211} + \sqrt{6a_{1221}a_{1111} - 2a_{1111}\sqrt{a_{1111}a_{2222}}} > 0$, and $2a_{1222} + \sqrt{6a_{1221}a_{2222} - 2a_{2222}\sqrt{a_{1111}a_{2222}}} > 0$.

Then $\mathcal{A}$ is copositive.

Proof. By Lemma 2.1, restrict $x = (x_1, x_2)^\top$ to $x_1 \ge 0$, $x_2 \ge 0$, $\|x\| = x_1 + x_2 = 1$. Consider $\mathcal{A}x^4$ with $a_{1111} > 0$ and $a_{2222} > 0$ in three cases.

Case 1: $x_1 = 0$ and $x_2 \ne 0$. Then $x_2 = 1$, so $\mathcal{A}x^4 = a_{2222} > 0$.

Case 2: $x_1 \ne 0$ and $x_2 = 0$. Then $x_1 = 1$, so $\mathcal{A}x^4 = a_{1111} > 0$.

Case 3: $x_1 \ne 0$ and $x_2 \ne 0$. Divide $\mathcal{A}x^4$ by x_2^4:

$$\frac{\mathcal{A}x^4}{x_2^4} = a_{1111}\left(\frac{x_1}{x_2}\right)^4 + 4a_{1211}\left(\frac{x_1}{x_2}\right)^3 + 6a_{1221}\left(\frac{x_1}{x_2}\right)^2 + 4a_{1222}\left(\frac{x_1}{x_2}\right) + a_{2222}.$$

Let $t = \frac{x_1}{x_2}$ and $g(t) = \frac{\mathcal{A}x^4}{x_2^4}$:

$$g(t) = a_{1111}t^4 + 4a_{1211}t^3 + 6a_{1221}t^2 + 4a_{1222}t + a_{2222}. \tag{3.4}$$

Clearly, $g(t) \ge 0$ iff $\mathcal{A}x^4 \ge 0$. Let

$$\alpha = \frac{4a_{1211}}{a_{1111}^{1/4}a_{2222}^{3/4}}, \quad \beta = \frac{6a_{1221}}{\sqrt{a_{1111}a_{2222}}}, \quad \gamma = \frac{4a_{1222}}{a_{1111}^{3/4}a_{2222}^{1/4}}.$$

For assumption (1), $a_{1221} \leq \sqrt{a_{1111}a_{2222}}$ means $\beta \leq 6$. Multiply $\frac{4a_{1211}}{a_{2222}} + 4\sqrt{\frac{a_{1111}}{a_{2222}}(3a_{1221} + \sqrt{a_{1111}a_{2222}})} > 0$ by $a_{1111}^{1/4}a_{2222}^{-1/4}$ to get $\alpha > -\frac{\beta}{2} + 2$. Similarly, multiply $\frac{4a_{1222}}{a_{1111}} + 4\sqrt{\frac{a_{2222}}{a_{1111}}(3a_{1221} + \sqrt{a_{1111}a_{2222}})} > 0$ by $a_{1111}^{-1/4}a_{2222}^{1/4}$ to get $\gamma > -\frac{\beta}{2} + 2$ for $\beta \leq 6$.

Assumption (2) similarly gives $\alpha > -2\sqrt{\beta - 2}$ and $\gamma > -2\sqrt{\beta - 2}$ for $\beta > 6$. The conclusions follow from Lemma 2.4(ii).

Using Lemma 2.4(i), we obtain:

Theorem 3.5. Let $\mathcal{A}$ be a symmetric tensor of order 4 and dimension 2 with $a_{1111} > 0$ and $a_{2222} > 0$. Then $\mathcal{A}$ is copositive iff:

(1) $a_{1221} < -\frac{1}{2}\sqrt{a_{1111}a_{2222}}$, $a_{1211} > 0$, and $(a_{1111}a_{2222} - 4a_{1211}a_{1222} + a_{1111}a_{2222})^3 \leq 27(a_{1111}a_{1221}a_{2222} + 2a_{1211}a_{1221}a_{1222} - a_{1111}a_{1222}^2 - a_{1211}^2a_{2222})^2$;

(2) $-\frac{1}{2}\sqrt{a_{1111}a_{2222}} \leq a_{1221} \leq \sqrt{a_{1111}a_{2222}}$, $a_{1211} > 0$, $a_{1222} > 0$, and $(a_{1111}a_{2222} - 4a_{1211}a_{1222} + a_{1111}a_{2222})^3 \geq 27(a_{1111}a_{1221}a_{2222} + 2a_{1211}a_{1221}a_{1222} - a_{1111}a_{1222}^2 - a_{1211}^2a_{2222})^2$;

(3) $a_{1221} > \sqrt{a_{1111}a_{2222}}$, $a_{1211} > 0$, $a_{1222} > 0$, and $(a_{1111}a_{2222} - 4a_{1211}a_{1222} + a_{1111}a_{2222})^3 \leq 27(a_{1111}a_{1221}a_{2222} + 2a_{1211}a_{1221}a_{1222} - a_{1111}a_{1222}^2 - a_{1211}^2a_{2222})^2$, with additional conditions on a_{1211} and a_{1222}.

Proof. Using the technique of Theorem 3.4, we need only consider nonnegativity of

$$g(t) = a_{1111}t^4 + 4a_{1211}t^3 + 6a_{1221}t^2 + 4a_{1222}t + a_{2222},$$

where $t = \frac{x_1}{x_2}$. Let

$$\alpha = \frac{4a_{1211}}{a_{1111}^{1/4}a_{2222}^{3/4}}, \quad \beta = \frac{6a_{1221}}{\sqrt{a_{1111}a_{2222}}}, \quad \gamma = \frac{4a_{1222}}{a_{1111}^{3/4}a_{2222}^{1/4}},$$

$$\Delta = 4(\beta^2 - 3\alpha\gamma + 12)^3 - (72\beta + 9\alpha\beta\gamma - 2\beta^3 - 27\alpha^2 - 27\gamma^2)^2,$$

$$\mu = (\alpha - \gamma)^2 - 16(\alpha + \beta + \gamma + 2),$$

$$\eta = (\alpha - \gamma)^2 - 4(\beta + 2)\sqrt{\beta - 2}(\alpha + \gamma + 4\sqrt{\beta - 2}).$$

Assumption (1) means $\beta \leq -2$, $\Delta \leq 0$, and $\alpha + \gamma > 0$. Similar calculations give the conditions for cases (2) and (3). The conclusions follow from Lemma 2.4(i).

Now we give simpler sufficient conditions for (strict) copositivity.

Theorem 3.6. Let $\mathcal{A}$ be a symmetric tensor of order 4 and dimension 2. Assume:

$$a_{1111} \geq 0(>0), \quad a_{2222} \geq 0(>0), \quad a_{1112} \geq 0(>0), \quad a_{2221} \geq 0(>0),$$

$$3a_{1221} + \sqrt{a_{1111}a_{2222}} + 4\sqrt{a_{1112}a_{2221}} \geq 0(>0).$$

Then $\mathcal{A}$ is (strictly) copositive.

Proof. By Lemma 2.1, restrict $x = (x_1, x_2)^\top$ to $x_1 \geq 0$, $x_2 \geq 0$, $\|x\| = x_1 + x_2 = 1$. Let $x_1 = t$, $x_2 = 1 - t$ for $t \in [0, 1]$. Then

$$\mathcal{A}x^4 = a_{1111}t^4 + 4a_{1112}t^3(1-t) + 6a_{1221}t^2(1-t)^2 + 4a_{2221}t(1-t)^3 + a_{2222}(1-t)^4. \tag{3.5}$$

For $t \in (0, 1)$, rewrite (3.5) as

$$\mathcal{A}x^4 = (\sqrt{a_{1111}}t^2 - \sqrt{a_{2222}}(1-t)^2)^2 + 2t(1-t)p(t),$$

where

$$p(t) = 2a_{1112}t^2 + (3a_{1221} + \sqrt{a_{1111}a_{2222}})t(1-t) + 2a_{2221}(1-t)^2.$$

The inequality $3a_{1221} + \sqrt{a_{1111}a_{2222}} + 4\sqrt{a_{1112}a_{2221}} \geq 0(>0)$ implies $3a_{1221} + \sqrt{a_{1111}a_{2222}} + 2\sqrt{a_{1112}a_{2221}} \geq 0(>0)$. By Lemma 2.2, $p(t) \geq 0(>0)$, so

$$\mathcal{A}x^4 = (\sqrt{a_{1111}}t^2 - \sqrt{a_{2222}}(1-t)^2)^2 + 2t(1-t)p(t) \geq 0(>0).$$

If $t = 0$ or $t = 1$, $\mathcal{A}x^4 = a_{2222}$ or a_{1111}. Thus $\mathcal{A}x^4 \geq 0(>0)$ for all $x \geq 0$ with $\|x\| = 1$. The conclusions follow from Lemma 2.1.

Theorem 3.7. Let $\mathcal{A}$ be a symmetric tensor of order 4 and dimension 2. Assume:

$$a_{1111} \geq 0(>0), \quad a_{2222} \geq 0(>0),$$

$$\sqrt{a_{1112}} + \sqrt[4]{a_{1111}a_{2222}} \geq 0(>0), \quad \sqrt{a_{2221}} + \sqrt[4]{a_{1111}^3 a_{2222}} \geq 0(>0),$$

$$(\sqrt{a_{1112}} + \sqrt[4]{a_{1111}a_{2222}})\left(\sqrt{a_{2221}} + \sqrt[4]{a_{1111}^3 a_{2222}}\right) \geq 0(>0).$$

Then $\mathcal{A}$ is (strictly) copositive.

Proof. Using the same technique, for $t \in (0,1)$ rewrite (3.5) as

$$\mathcal{A}x^4 = (\sqrt[4]{a_{1111}}t - \sqrt[4]{a_{2222}}(1-t))^4 + 4\left(\sqrt{a_{1112}} + \sqrt[4]{a_{1111}a_{2222}}\right)t^3(1-t)$$

$$+6(a_{1221} - \sqrt{a_{1111}a_{2222}})t^2(1-t)^2 + 4\left(\sqrt{a_{2221}} + \sqrt[4]{a_{1111}^3 a_{2222}}\right)t(1-t)^3$$

$$= (\sqrt[4]{a_{1111}}t - \sqrt[4]{a_{2222}}(1-t))^4 + t(1-t)p(t),$$

where

$$p(t) = 4\left(\sqrt{a_{1112}} + \sqrt[4]{a_{1111}a_{2222}}\right)t^2 + 6(a_{1221} - \sqrt{a_{1111}a_{2222}})t(1-t) + 4\left(\sqrt{a_{2221}} + \sqrt[4]{a_{1111}^3 a_{2222}}\right)(1-t)^2.$$

The assumptions ensure $p(t) \geq 0 (> 0)$ for all $t \in (0,1)$ by Lemma 2.2. For $t = 0$ or $t = 1$, $\mathcal{A}x^4 = a_{2222}$ or a_{1111}. Thus $\mathcal{A}x^4 \geq 0 (> 0)$ for all $x \geq 0$ with $\|x\| = 1$. Therefore, $\mathcal{A}$ is (strictly) copositive.

From the proofs of Theorems 3.6 and 3.7, we obtain:

Corollary 3.8. Let $\mathcal{A}$ be a symmetric and strictly copositive tensor of order 4 and dimension 2. Then

$$2a_{1112}\sqrt{a_{2222}} + (3a_{1221} + \sqrt{a_{1111}a_{2222}})\sqrt[4]{a_{1111}a_{2222}} + 2a_{1222}\sqrt{a_{1111}} > 0. \quad (3.6)$$

Proof. Strict copositivity implies $a_{1111} > 0$ and $a_{2222} > 0$. For $t \in (0,1)$ and $x = (t, 1-t)^\top$,

$$\mathcal{A}x^4 = (\sqrt{a_{1111}}t^2 - \sqrt{a_{2222}}(1-t)^2)^2 + 2t(1-t)\left(2a_{1112}t^2 + (3a_{1221} + \sqrt{a_{1111}a_{2222}})t(1-t) + 2a_{2221}(1-t)^2\right).$$

Take $t_0 = \frac{\sqrt{a_{2222}}}{\sqrt{a_{1111}} + \sqrt[4]{a_{1111}a_{2222}}}$. Then $x_0 = (t_0, 1-t_0)^\top$ satisfies

$$\mathcal{A}(x_0)^4 = 2t_0(1-t_0)\left(2a_{1112}t_0^2 + (3a_{1221} + \sqrt{a_{1111}a_{2222}})t_0(1-t_0) + 2a_{2221}(1-t_0)^2\right) > 0,$$

which yields the desired inequality.

Now we give simpler sufficient conditions for (strict) copositivity of 4th order 3-dimensional tensors by reducing dimensions.

Theorem 3.9. Let $\mathcal{A}$ be a symmetric tensor of order 4 and dimension 3. Assume:

$$a_{1111} \geq 0(>0), \quad a_{2222} \geq 0(>0), \quad a_{3333} \geq 0(>0),$$

$$a_{1123} \geq 0(>0), \quad a_{1223} \geq 0(>0), \quad a_{1233} \geq 0(>0),$$

$$\eta_1 = 2a_{1112} + 4\sqrt[4]{a_{1111}a_{2222}} \geq 0(>0), \quad \eta_2 = 2a_{1113} + 4\sqrt[4]{a_{1111}a_{3333}} \geq 0(>0),$$

$$\eta_3 = 2a_{2223} + 4\sqrt[4]{a_{2222}a_{3333}} \geq 0(>0),$$

$$\mu_1 = 2a_{1222} + 4\sqrt[4]{a_{1111}^3a_{2222}} \geq 0(>0), \quad \mu_2 = 2a_{1333} + 4\sqrt[4]{a_{1111}^3a_{3333}} \geq 0(>0),$$

$$\mu_3 = 2a_{2333} + 4\sqrt[4]{a_{2222}^3a_{3333}} \geq 0(>0),$$

$$\theta_1 = 3(2a_{1122} - \sqrt{a_{1111}a_{2222}}) + 4\sqrt{\eta_1\mu_1} \geq 0(>0),$$

$$\theta_2 = 3(2a_{1133} - \sqrt{a_{1111}a_{3333}}) + 4\sqrt{\eta_2\mu_2} \geq 0(>0),$$

$$\theta_3 = 3(2a_{2233} - \sqrt{a_{2222}a_{3333}}) + 4\sqrt{\eta_3\mu_3} \geq 0(>0).$$

Then $\mathcal{A}$ is (strictly) copositive.

Proof. For $x = (x_1, x_2, x_3)^\top$,

$$\mathcal{A}x^4 = a_{1111}x_1^4 + a_{2222}x_2^4 + a_{3333}x_3^4 + 4a_{1112}x_1^3x_2 + 6a_{1122}x_1^2x_2^2 + 4a_{1222}x_1x_2^3$$

$$+4a_{1113}x_1^3x_3 + 6a_{1133}x_1^2x_3^2 + 4a_{1333}x_1x_3^3 + 4a_{2223}x_2^3x_3 + 6a_{2233}x_2^2x_3^2 + 4a_{2333}x_2x_3^3$$

$$+12a_{1123}x_1^2x_2x_3 + 12a_{1223}x_1x_2^2x_3 + 12a_{1233}x_1x_2x_3^2.$$

Define

$$g_1(x_1,x_2) = a_{1111}x_1^4 + 4a_{1112}x_1^3x_2 + 6a_{1122}x_1^2x_2^2 + 4a_{1222}x_1x_2^3 + a_{2222}x_2^4.$$

Then $g_1(x_1,x_2)$ is a homogeneous polynomial defined by a 4th order 2-dimensional symmetric tensor $\mathcal{B} = (b_{ijkl})$ with entries $b_{1111} = a_{1111}$, $b_{1112} = a_{1112}$, $b_{1122} = a_{1122}$, $b_{1222} = a_{1222}$, $b_{2222} = a_{2222}$. That is,

$$g_1(x_1,x_2) = \mathcal{B}y^4 = \sum_{i,j,k,l=1}^{2} b_{ijkl}x_ix_jx_kx_l, \quad \text{for } y = (x_1,x_2)^\top.$$

The assumptions imply

$$b_{1111} = a_{1111} \geq 0, \quad b_{2222} = a_{2222} \geq 0,$$

$$\sqrt{b_{1112}} + \sqrt[4]{b_{1111}b_{2222}} = \sqrt{a_{1112}} + \sqrt[4]{a_{1111}a_{2222}} = \eta_1 \geq 0,$$

$$\sqrt{b_{1222}} + \sqrt[4]{b_{1111}^3b_{2222}} = \sqrt{a_{1222}} + \sqrt[4]{a_{1111}^3a_{2222}} = \mu_1 \geq 0,$$

$$3(b_{1122}-\sqrt{b_{1111}b_{2222}})+4\sqrt{(\sqrt{b_{1112}} + \sqrt[4]{b_{1111}b_{2222}})(\sqrt{b_{1222}} + \sqrt[4]{b_{1111}^3b_{2222}})} = \theta_1 \geq 0.$$

By Theorem 3.7, $\mathcal{B}$ is copositive, so $g_1(x_1,x_2) = \mathcal{B}y^4 \geq 0$ for all $y = (x_1,x_2)^\top \geq 0$.

Similarly,

$$g_2(x_1,x_3) = a_{1111}x_1^4 + 4a_{1113}x_1^3x_3 + 6a_{1133}x_1^2x_3^2 + 4a_{1333}x_1x_3^3 + a_{3333}x_3^4 \geq 0,$$

$$g_3(x_2,x_3) = a_{2222}x_2^4 + 4a_{2223}x_2^3x_3 + 6a_{2233}x_2^2x_3^2 + 4a_{2333}x_2x_3^3 + a_{3333}x_3^4 \geq 0.$$

Thus for all $x = (x_1,x_2,x_3)^\top \geq 0$,

$$\mathcal{A}x^4 = g_1(x_1,x_2)+g_2(x_1,x_3)+g_3(x_2,x_3)+12a_{1123}x_1^2x_2x_3+12a_{1223}x_1x_2^2x_3+12a_{1233}x_1x_2x_3^2 \geq 0,$$

so $\mathcal{A}$ is copositive. The proof for strict copositivity is identical.

Remark 3.3. Theorem 3.9 is proved by reducing tensor dimensions. A 4th order 3-dimensional tensor is decomposed into three 4th order 2-dimensional tensors, and copositivity of these 2-dimensional tensors is analyzed using Theorem 3.7 to obtain sufficient conditions. Distinct sufficient conditions can be established by applying Theorems 3.4, 3.5, or 3.6.

4 Checking Vacuum Stability of Scalar Potentials

4.1 Vacuum Stability of the Scalar Potential of Two Real Scalars and the Higgs Boson

Recently, Kannike [25, 26] studied vacuum stability of general scalar potentials. The most general scalar potential of two real scalar fields ϕ_1 and ϕ_2 is

$$V(\phi_1,\phi_2) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4 = \Lambda\phi^4, \quad (4.1)$$

where $\Lambda = (\lambda_{ijkl})$ is the symmetric tensor of scalar couplings and $\phi = (\phi_1,\phi_2)^\top$ is the field vector. The coupling tensor is defined by

$$\lambda_{1111} = \lambda_{40}, \quad \lambda_{2222} = \lambda_{04}, \quad \lambda_{1112} = \lambda_{1121} = \lambda_{1211} = \lambda_{2111} = \frac{\lambda_{31}}{4},$$

$$\lambda_{1122} = \lambda_{1212} = \lambda_{1221} = \lambda_{2112} = \lambda_{2121} = \lambda_{2211} = \frac{\lambda_{22}}{6},$$

$$\lambda_{1222} = \lambda_{2122} = \lambda_{2212} = \lambda_{2221} = \frac{\lambda_{13}}{4}.$$

Vacuum stability of this potential is equivalent to positivity of polynomial (4.1) [25], i.e., positive definiteness of $\Lambda = (\lambda_{ijkl})$. By Theorem 3.1, with $\lambda_{1111} = \lambda_{40} > 0$ and $\lambda_{2222} = \lambda_{04} > 0$, Λ is positive definite iff (after clearing denominators):

(1) $8\lambda_{40}\lambda_{22} - 3\lambda_{31}^2 \geq 0$, and $(12\lambda_{40}\lambda_{04} - 3\lambda_{31}\lambda_{13} + \lambda_{22}^2)^3 > (72\lambda_{40}\lambda_{22}\lambda_{04} + 9\lambda_{31}\lambda_{22}\lambda_{31} - 2\lambda_{31}^3 - 27\lambda_{40}\lambda_{31}^2 - 27\lambda_{31}^2\lambda_{04})^2$;

(2) $8\lambda_{40}\lambda_{22} - 3\lambda_{31}^2 < 0$, with additional inequalities;

(3) $8\lambda_{40}\lambda_{22} - 3\lambda_{31}^2 > 0$, with equality conditions.

One of these cases guarantees vacuum stability of $V(\phi_1, \phi_2)$.

The most general potential of two real scalars ϕ_1, ϕ_2 and the Higgs doublet H [25, 26] is

$$V(\phi_1, \phi_2, |H|) = \lambda_H |H|^4 + \lambda_{H20}|H|^2\phi_1^2 + \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{H11}|H|^2\phi_1\phi_2$$

$$+\lambda_{H02}|H|^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4 + \lambda_{22}\phi_1^2\phi_2^2$$

$$= \lambda_H|H|^4 + M^2(\phi_1, \phi_2)|H|^2 + \bar{V}(\phi_1, \phi_2),$$

where

$$M^2(\phi_1, \phi_2) = \lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2,$$

$$\bar{V}(\phi_1, \phi_2) = V(\phi_1, \phi_2, 0) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4.$$

Let $x = (\phi_1, \phi_2, |H|)^\top$. Then $V(\phi_1, \phi_2, |H|) = Vx^4$, where $V = (v_{ijkl})$ is a 4th order 3-dimensional symmetric tensor with entries:

$$v_{1111} = \lambda_{40}, \quad v_{2222} = \lambda_{04}, \quad v_{3333} = \lambda_H, \quad v_{1112} = \frac{\lambda_{31}}{4},$$

$$v_{1133} = \frac{\lambda_{H20}}{2}, \quad v_{2233} = \frac{\lambda_{H02}}{2}, \quad v_{1233} = \frac{\lambda_{H11}}{4}, \quad v_{ijkl} = 0 \text{ otherwise.}$$

$\bar{V}(\phi_1, \phi_2)$ is a 4th order 2-dimensional tensor. Let $\phi = (\phi_1, \phi_2)^\top$. Then $\bar{V}(\phi_1, \phi_2) = \Lambda\phi^4$, where Λ is given by (4.2), a principal subtensor of V. Conditions (a)-(c) ensure positive definiteness of Λ, i.e., $\bar{V}(\phi_1, \phi_2) = \Lambda\phi^4 > 0$.

Moreover, $M^2(\phi_1, \phi_2) = \phi^\top M\phi$, where M is the symmetric matrix

$$M = \begin{pmatrix} \lambda_{H20} & \frac{\lambda_{H11}}{2} \\ \frac{\lambda_{H11}}{2} & \lambda_{H02} \end{pmatrix}.$$

M is positive definite iff

$$\lambda_{H20} > 0, \quad \lambda_{H02} > 0, \quad 4\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 > 0. \tag{4.4}$$

Positivity of $V(\phi_1, \phi_2, |H|)$ is ensured by $\lambda_H > 0$, (4.4), and conditions (a) or (b) or (c). Therefore, vacuum stability conditions for $V(\phi_1, \phi_2, |H|)$ are:

$$\lambda_{40} > 0, \quad \lambda_{04} > 0, \quad \lambda_H > 0, \quad \lambda_{H20} > 0, \quad \lambda_{H02} > 0, \quad 4\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 > 0,$$

plus inequality system (a) or (b) or (c).

4.2 Vacuum Stability for Z_3 Scalar Dark Matter

Kannike [25,26] gave a physical example of scalar dark matter stable under a Z_3 discrete group. The most general scalar quartic potential of the SM Higgs H_1, an inert doublet H_2, and a complex singlet S, symmetric under Z_3, is

$$V(h_1, h_2, s) = \lambda_1 |H_1|^4 + \lambda_2 |H_2|^4 + \lambda_3 |H_1|^2 |H_2|^2 + \lambda_4 (H_1^\dagger H_2)(H_2^\dagger H_1)$$

$$+\lambda_S |S|^4 + \lambda_{S1} |S|^2 |H_1|^2 + \lambda_{S2} |S|^2 |H_2|^2 + (\lambda_{S12} S^2 H_1^\dagger H_2 + \text{h.c.})$$

$$= \lambda_1 h_1^4 + \lambda_2 h_2^4 + \lambda_3 h_1^2 h_2^2 + \lambda_4 \rho^2 h_1^2 h_2^2 + \lambda_S s^4 + \lambda_{S1} s^2 h_1^2 + \lambda_{S2} s^2 h_2^2 - |\lambda_{S12}| \rho s^2 h_1 h_2$$

$$\equiv \lambda_S s^4 + M^2(h_1, h_2) s^2 + \hat{V}(h_1, h_2),$$

where

$$M^2(h_1, h_2) = \lambda_{S1} h_1^2 + \lambda_{S2} h_2^2 - |\lambda_{S12}| \rho h_1 h_2,$$

$$\hat{V}(h_1, h_2) = V(h_1, h_2, 0) = \lambda_1 h_1^4 + \lambda_2 h_2^4 + \lambda_3 h_1^2 h_2^2 + \lambda_4 \rho^2 h_1^2 h_2^2,$$

with $h_1 = |H_1|$, $h_2 = |H_2|$, $H_2^\dagger H_1 = h_1 h_2 \rho e^{i\phi}$, $S = s e^{i\phi_S}$, $\lambda_{S12} = -|\lambda_{S12}|$, and orbit space parameter $\rho \in [0, 1]$ from Cauchy inequality $0 \leq |H_2^\dagger H_1| \leq |H_1||H_2|$.

Let $x = (h_1, h_2, s)^\top$. Then $V(h_1, h_2, s) = V x^4$, where $V = (v_{ijkl})$ is a 4th order 3-dimensional real symmetric tensor with entries:

$$v_{1111} = \lambda_1, \quad v_{2222} = \lambda_2, \quad v_{3333} = \lambda_S, \quad v_{1122} = \frac{1}{6}(\lambda_3 + \lambda_4 \rho^2),$$

$$v_{1133} = \frac{\lambda_{S1}}{6}, \quad v_{2233} = \frac{\lambda_{S2}}{6}, \quad v_{1233} = -\frac{|\lambda_{S12}|\rho}{12}, \quad v_{ijkl} = 0 \text{ otherwise.}$$

By Theorem 3.3, strict copositivity conditions for V (i.e., $V(h_1, h_2, s) = Vx^4 > 0$) are:

$$\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_S > 0, \quad \lambda_3 + \lambda_4\rho^2 > 0, \quad \lambda_{S1} > 0, \quad \lambda_{S2} > 0,$$

$$\sqrt{\lambda_{S1}\lambda_{S2}} - |\lambda_{S12}|\rho > 0,$$

$$\sqrt{\lambda_S\lambda_{S1}\lambda_{S2}} - |\lambda_{S12}|\rho\sqrt{\sqrt{\lambda_{S1}\lambda_{S2}}(\sqrt{\lambda_{S1}\lambda_{S2}} - |\lambda_{S12}|\rho)} > 0.$$

These conditions ensure the Z_3-symmetric potential $V(h_1, h_2, s)$ is bounded from below. They differ from those of Kannike [25, 26] and Chen-Huang-Qi [12].

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