Exact Solution of Quantum Rabi Model

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Date: 2019-07-31T00:00:00+00:00

Abstract

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Full Text

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The quantum Rabi model is exactly solved by employing the parameter-dependent unitary transformation method in the Bargmann space. The complete energy spectrum consists of two double-fold sub-energy spectra. The eigenvalue is determined by the parameter in the unitary transformation, which satisfies a highly nonlinear equation. Such the energy spectrum completely coincides with that obtained in the occupation number representation [D. Zhang, chinaXiv:201708.00168].

PACS numbers: 03.65.Ge, 02.30.Ik, 42.50.Pq

The quantum Rabi model describes the response of a two-level atom to an applied bosonic field [1]. This simplest light-matter interacting model has been widely applied in atomic physics [2], quantum optics [3], quantum information and quantum technology [4], among other fields. Its Hamiltonian reads

$$H = \omega a^\dagger a + g(a^\dagger + a)\sigma_x + \lambda\sigma_z + \epsilon\sigma_x,$$

where σ_x and σ_z are the Pauli matrices, $a^\dagger$ and a are the creation and annihilation operators for the single bosonic mode with frequency ω, respectively, 2λ is the level splitting for the two-level system, g is the light-matter interaction strength, and the last term $\epsilon\sigma_x$ is the driving term, which leads to tunneling between the two levels.

The ratio between g and ω divides three experimental regimes: (a) the weak coupling regime ($g/\omega \lesssim 0.1$), (b) the ultrastrong coupling regime ($\sim 0.1 < g/\omega \lesssim 1.0$), and (c) the deep strong coupling regime ($g/\omega \gtrsim 1.0$). Most experiments have been performed in the weak coupling regime, where the quantum Rabi model with $\epsilon = 0$ is equivalent to the exactly solvable Jaynes-Cummings model in the rotating-wave approximation [5]. Recently, more and more attention has been focused on the strong coupling regimes due to their fundamental characteristics and potential applications in quantum devices [4].

Exact solution of the quantum Rabi model has been a long-standing problem in physics. Such an analytical solution is expected to accurately explore light-matter interaction from weak to extremely strong regimes. In Ref. [6], Braak presented an analytical solution of the Rabi model by employing the representation of bosonic operators in the Bargmann space of analytical functions. The energy spectrum consists of two parts, i.e., the regular and the exceptional spectrum. However, it has been proved that Braak' s analytical solution is incorrect due to derivation errors in solving the time-independent Schrödinger equation in the positive and negative parity parts [7]. In Ref. [8], we obtained the exact solution of the Hamiltonian in the whole range of physical parameters by using the parameter-dependent unitary transformation technique in the occupation number representation. This direct and powerful approach has also been used to successfully solve complex two-dimensional electron gas systems in the presence of both Rashba and Dresselhaus spin-orbit interactions under a perpendicular magnetic field [9,10]. The complete energy spectrum of the quantum Rabi model consists of two double-fold sub-energy spectra.

In this work, we reinvestigate the eigenvalue problem for the Hamiltonian in the Bargmann space, where the bosonic creation and annihilation operators in terms of a complex variable z can be transformed as $a^\dagger \to z$ and $a \to \partial/\partial z$, respectively. In this representation, the state $\Psi(z)$ can be normalized according to

$$\langle\Psi|\Psi\rangle = \int dz\, d\bar{z}\, e^{-z\bar{z}}\Psi^\dagger(z)\Psi(z) \equiv 1.$$

We shall see below that the energy spectrum is completely consistent with that in the occupation number representation [8].

In the Bargmann space, the two-component eigenstate of the Hamiltonian for the nth energy level with quantum number s has the general form

$$\Psi_{ns} = \begin{pmatrix} \sum_{i=0}^{\infty} A_{ns}^i z^i \\ \sum_{i=0}^{\infty} B_{ns}^i z^i \end{pmatrix},$$

where $s = \pm 1$, Δ_{ns} is a real parameter in the unitary matrix to be determined below by requiring the coefficients to be nonzero. When $i \to +\infty$, $A_{ns}^i \to 0$. Substituting the eigenfunction into the eigen-equation $H\Psi_{ns} = E_{ns}\Psi_{ns}$ and requiring the coefficients of z^i to be zero, we obtain the infinite system of homogeneous linear equations with the variables A_{ns}^i and B_{ns}^i:

$$\begin{aligned}&[E_{ns} - i\omega - \lambda(1-\Delta_{ns}^2) - 2\epsilon\Delta_{ns}]A_{ns}^i + g(1-\Delta_{ns}^2)(i+1)A_{ns}^{i+1} \\ &\quad + 2g\Delta_{ns}(i+1)B_{ns}^{i+1} - g(1-\Delta_{ns}^2)B_{ns}^{i-1} + 2\lambda\Delta_{ns}B_{ns}^i + \epsilon(1-\Delta_{ns}^2)B_{ns}^i = 0,\end{aligned}$$

and

$$\begin{aligned}&[E_{ns} - i\omega + \lambda(1-\Delta_{ns}^2) + 2\epsilon\Delta_{ns}]B_{ns}^i + g(1-\Delta_{ns}^2)(i+1)B_{ns}^{i+1} \\ &\quad + 2g\Delta_{ns}(i+1)A_{ns}^{i+1} - g(1-\Delta_{ns}^2)A_{ns}^{i-1} - 2\lambda\Delta_{ns}A_{ns}^i - \epsilon(1-\Delta_{ns}^2)A_{ns}^i = 0,\end{aligned}$$

where $i = 0, 1, 2, \cdots, \infty$, and $A_{ns}^m = B_{ns}^m = 0$ for $m < 0$.

It seems very hard to solve these equations directly. However, we can acquire all the physical eigenvalues and the corresponding eigenfunctions by using a special trick.

Sub-energy Spectrum I

In order to obtain the analytical solution of the Hamiltonian in the whole parameter space, we first choose

$$[E_{ns} - \omega(n+1) - \lambda(1-\Delta_{ns}^2) - 2\epsilon\Delta_{ns}]A_{ns}^n + g(1-\Delta_{ns}^2)(n+1)B_{ns}^{n+1} = 0,$$

and

$$[E_{ns} - \omega n + \lambda(1-\Delta_{ns}^2) + 2\epsilon\Delta_{ns}]B_{ns}^{n+1} + g(1-\Delta_{ns}^2)(n+1)A_{ns}^n = 0,$$

in the eigen-equations. These equations originate from the vanishing of the two terms about A_{ns}^n and B_{ns}^{n+1} in the equations with $i = n+1$ and $i = n$, respectively. Such a choice is based on the observation of the exact solution of the Hamiltonian for the nth energy level with quantum number s when $g = 0$. The non-zero eigenfunction associated with the eigenvalue E_{ns} is then uniquely fixed by requiring

$$[2\lambda\Delta_{ns} + \epsilon(1 - \Delta_{ns}^2)]B_{ns}^{n+1} - 2g\Delta_{ns}(n+1)A_{ns}^{n+1} = 0,$$

and

$$2g\Delta_{ns}B_{ns}^{n} + [2\lambda\Delta_{ns} + \epsilon(1 - \Delta_{ns}^2)]A_{ns}^{n+1} = 0.$$

Solving the homogeneous linear equations about A_{ns}^{n+1} and B_{ns}^{n+1} by vanishing the coefficient determinant yields the eigenvalue for the nth eigenstate with quantum number s:

$$E_{ns} = \left(n + \frac{1}{2}\right)\omega + s\Xi_{ns},$$

where

$$\Xi_{ns} = \sqrt{\left[\lambda(1 - \Delta_{ns}^2) - 2\epsilon\Delta_{ns}\right]^2 + (n+1)g^2(1 - \Delta_{ns}^2)^2}.$$

Note that the quasiparticle energy E_{ns} must be larger than zero. Combining these equations, the parameter Δ_{ns} is determined by the constraint

$$\epsilon(1 + \Delta_{ns}^2) - 2\Delta_{ns}(E_{ns} - \omega n) = 0,$$

and

$$\epsilon(1 + \Delta_{ns}^2) + 2\Delta_{ns}[E_{ns} - \omega(n+1)] = 0.$$

After careful analysis, we find that the first equation with $s = -1$ coincides with the second equation with $s = 1$, and vice versa. Therefore, we obtain

$$\epsilon(1 + \Delta_{ns}^2) + \Delta_{ns}(2\sigma\Xi_{ns} - \omega) = 0,$$

where $\sigma = \pm 1$. It is absolutely surprising that this analytical expression of E_{ns} with the parameter constraint has been obtained in the occupation number representation [8].

Obviously, Δ_{ns} is independent of s in the constraint equation. When $\epsilon = 0$, then $\Delta_{ns} = 0$. Therefore, the eigenvalue E_{ns} has a simple form

$$E_{ns} = \left(n + \frac{1}{2}\right)\omega + s\sqrt{\lambda^2 + (n+1)g^2}$$

in the absence of the driving term. It is easy to conclude that the analytical solution for E_{ns} is physical if and only if it satisfies the limit $\Delta_{ns} \to 0$ when $\epsilon \to 0$.

To facilitate comparison with the energy spectrum presented by Braak, we use the physical parameters from Ref. [6]. FIG. 1 [Figure 1: see original paper] plots the low-lying energy levels as a function of g at $\lambda = 0.4\omega$ and $\epsilon = 0$. We can see that this energy spectrum possesses level crossings between neighboring eigenstates, which is dramatically different from that in Ref. [6].

When $\epsilon \neq 0$, Δ_{ns} in the constraint equation with $\sigma = 1$ has an ω-dependent solution. The corresponding eigenvalues E_{ns} ($n = 0, 1, 2, \cdots, \infty$, $s = \pm 1$) form the sub-energy spectrum I. In FIG. 2 [Figure 2: see original paper], we show the low-lying energy levels and the corresponding parameter Δ_{ns} of the sub-energy spectrum I as a function of g at $\lambda = 0.7\omega$ and $\epsilon = 0.2\omega$. When $g = 0$, the sub-energy spectrum I recovers the exact eigenvalues for the interactionless case, i.e., $E_{ns} = \omega n + s\sqrt{\lambda^2 + \epsilon^2}/\epsilon$. Obviously, if $\epsilon \to 0$, then $\Delta_{ns} \to 0$. We note that another ω-dependent solution Δ_{ns} with $\sigma = -1$ corresponds to the sub-energy spectrum II (see below).

For the eigenstate associated with the eigenvalue E_{ns}, from the earlier equations we have

$$B_{ns}^{n+1} = \frac{(1+\Delta_{ns}^2)(E_{ns} - n\omega) + \lambda(1-\Delta_{ns}^2) - 2\epsilon\Delta_{ns}}{g(1-\Delta_{ns}^2)(n+1)} A_{ns}^n,$$

where A_{ns}^n is a constant to be determined by the normalization condition. The coefficients are uniquely obtained by the recursion relations

$$\begin{pmatrix} A_{ns}^i \\ B_{ns}^i \end{pmatrix} = -M_{ns}^{-1} \begin{pmatrix} g(1-\Delta_{ns}^2) & 2g\Delta_{ns}(i+1) \\ 2g\Delta_{ns}(i+1) & g(1-\Delta_{ns}^2) \end{pmatrix} \begin{pmatrix} A_{ns}^{i+1} \\ B_{ns}^{i+1} \end{pmatrix}$$

for $i = 0, 1, 2, \cdots, n$, and

$$\begin{pmatrix} A_{ns}^{i+1} \\ B_{ns}^{i+1} \end{pmatrix} = -M_{ns}^{-1} \begin{pmatrix} g(1-\Delta_{ns}^2) & 2g\Delta_{ns}(i+1) \\ 2g\Delta_{ns}(i+1) & g(1-\Delta_{ns}^2) \end{pmatrix} \begin{pmatrix} A_{ns}^{i} \\ B_{ns}^{i} \end{pmatrix}$$

for $i = n+1, n+2, \cdots, +\infty$. Here we have defined

$$M_{ns} = (\omega i - E_{ns})I + \lambda(1-\Delta_{ns}^2)\sigma_z + 2\lambda\Delta_{ns}\sigma_x + \epsilon(1-\Delta_{ns}^2)\sigma_x - 2\epsilon\Delta_{ns}\sigma_z,$$

where I is the 2×2 unit matrix.

Sub-energy Spectrum II

Now we use another choice:

$$[E_{ns} - \omega n + \lambda(1-\Delta_{ns}^2) + 2\epsilon\Delta_{ns}]A_{ns}^{n+1} + g(1-\Delta_{ns}^2)(n+1)B_{ns}^n = 0,$$

and

$$[E_{ns} - \omega(n+1) - \lambda(1-\Delta_{ns}^2) - 2\epsilon\Delta_{ns}]B_{ns}^n + g(1-\Delta_{ns}^2)(n+1)A_{ns}^{n+1} = 0,$$

in the eigen-equations. These equations originate from the vanishing of the two terms about A_{ns}^{n+1} and B_{ns}^n in the equations with $i = n$ and $i = n+1$, respectively. The corresponding eigenfunction is uniquely determined by letting

$$[2\lambda\Delta_{ns} + \epsilon(1-\Delta_{ns}^2)]A_{ns}^n + 2g\Delta_{ns}(n+1)B_{ns}^{n+1} = 0,$$

and

$$-2g\Delta_{ns}A_{ns}^n + [2\lambda\Delta_{ns} + \epsilon(1-\Delta_{ns}^2)]B_{ns}^{n+1} = 0.$$

Solving these equations yields the eigenvalue for the nth eigenstate with quantum number s:

$$E_{ns} = \left(n + \frac{1}{2}\right)\omega + s\Theta_{ns},$$

where

$$\Theta_{ns} = \sqrt{[-\lambda(1-\Delta_{ns}^2) - 2\epsilon\Delta_{ns}]^2 + (n+1)g^2(1-\Delta_{ns}^2)^2}.$$

Here Δ_{ns} satisfies the nonlinear equation

$$\epsilon(1+\Delta_{ns}^2) + 2\Delta_{ns}(E_{ns} - \omega n) = 0,$$

which is derived from the previous equations. Similar to the earlier case, this equation with $s = -1$ is consistent with the corresponding equation with $s = 1$, and vice versa. Therefore, we obtain

$$\epsilon(1+\Delta_{ns}^2) + \Delta_{ns}(2\tau\Theta_{ns} + \omega) = 0,$$

where $\tau = \pm 1$. We note that the analytical expression for E_{ns} and the parameter equation are also fully consistent with those presented in the occupation number representation [8].

If $\epsilon = 0$, then $\Delta_{ns} = 0$ from the constraint equation. So the eigenvalue also has an explicit expression

$$E_{ns} = \left(n + \frac{1}{2}\right)\omega + s\sqrt{(-\lambda)^2 + (n+1)g^2}.$$

We depict the low-lying energy levels as a function of g at $\lambda = 0.4\omega$ and $\epsilon = 0$ in FIG. 3 [Figure 3: see original paper].

When $\epsilon \neq 0$, Δ_{ns} in the constraint equation with $\tau = 1$ also has an ω-dependent solution. The corresponding eigenvalues constitute the sub-energy spectrum II. FIG. 4 [Figure 4: see original paper] exhibits the low-lying energy levels of the sub-energy spectrum II as a function of g at $\lambda = 0.7\omega$ and $\epsilon = 0.2\omega$, along with the corresponding parameter Δ_{ns}. When $g = 0$, the sub-energy spectrum II also recovers the exact eigenvalues for the interactionless two-level system, i.e., $E_{ns} = \omega n + s\sqrt{\lambda^2 + \epsilon^2}$ with $\Delta_{ns} = (\lambda - \sqrt{\lambda^2 + \epsilon^2})/\epsilon$. We note that after taking the transformation $\Delta_{ns} \to -1/\Delta_{ns}$, the eigenvalue with $\tau = 1$ becomes the eigenvalue with $\sigma = -1$. Therefore, both the sub-energy spectrum I and II are doubly degenerate.

For the nth eigenstate with quantum number s in the sub-energy spectrum II, from the recursion relations we have

$$A_{ns}^{n+1} = \frac{(1 + \Delta_{ns}^2)(E_{ns} - n\omega) - \lambda(1 - \Delta_{ns}^2) + 2\epsilon\Delta_{ns}}{g(1 - \Delta_{ns}^2)(n+1)} B_{ns}^n,$$

where A_{ns}^n is the normalization constant. The coefficients obey the same recursion relations as in the sub-energy spectrum I.

Conclusion

In summary, we have exactly solved the quantum Rabi model in the Bargmann space. The complete energy spectrum is comprised of two doubly degenerate sub-energy spectra I and II. Such an exact solution can help us deeply understand light-matter interaction, especially in strong coupling regimes. Because the eigenvalue E_{ns} has the same analytical expression in both the Bargmann space and the occupation number representation, this guarantees the correctness of the exact solution for the quantum Rabi model.

This work was supported by the Sichuan Normal University, the "Thousand Talents Program" of Sichuan Province, China, the Texas Center for Superconductivity at the University of Houston, and the Robert A. Welch Foundation under grant No. E-1146.

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Note: Figure translations are in progress. See original paper for figures.