Entanglement Criteria and Necessary and Sufficient Conditions for Higher-Order Tensor Separability

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Abstract

[Purpose] To find the necessary and sufficient conditions for higher-order tensors and pure states to be expressed as products of multiple lower-order tensors.

[Method] Through analogical and inductive reasoning, starting from the necessary and sufficient conditions for variable separability in multivariate functions and utilizing properties of matrix cofactors, general conclusions are obtained.

[Results] The theorem and its multiple equivalent forms are proven.

[Limitations] Not applicable to mixed states.

[Conclusion] The necessary and sufficient condition for determining whether a pure state is entangled.

Full Text

Criteria for Entanglement of Pure States and Separability of Higher-Order Tensors

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Abstract:
This work establishes necessary and sufficient conditions for determining whether a pure quantum state or a higher-order tensor can be expressed as a product of lower-order tensors. Using analogical and inductive reasoning, we begin from the separability conditions for multivariate functions and leverage properties of matrix algebraic cofactors to derive general results.

The main outcomes include the proof of a central theorem and its multiple equivalent formulations. A key limitation is that these criteria do not apply to mixed quantum states. The fundamental conclusion provides a necessary and sufficient condition for identifying whether a pure state is entangled.

Keywords: Separation of variables, Algebraic cofactor, Rank of tensors

1. Separability of Second-Order Tensors

A bipartite quantum state can be completely represented by a second-order tensor a_{ij}. In an orthonormal basis $|ij\rangle$, any quantum state can be written as $\psi = a_{ij}|ij\rangle$ (with the summation convention implied). For continuous spectra, this becomes $\psi = \int f(x,y)|xy\rangle dxdy$. A quantum state is a product state if and only if $a_{ij} = \alpha_i\beta_j$, or for continuous spectra, $f(x,y) = g(x)h(y)$. This corresponds to the separability of tensor indices or functional variables. Reference [1] provides the following conditions for separable functions:

Proposition 1: If there exist differentiable functions $\varphi_1(x)$ and $\varphi_2(y)$ such that $f(x,y) = \varphi_1(x)\varphi_2(y)$ for all $(x,y) \in D \subseteq \mathbb{R}^2$, then

$$f(x,y)f''_{xy}(x,y) = f'_x(x,y)f'_y(x,y) \tag{1.1}$$

holds for all $(x,y) \in D \subseteq \mathbb{R}^2$.

Proposition 2: If $f(x,y)$, $f'_x(x,y)$, $f'_y(x,y)$, and $f''_{xy}(x,y)$ are continuous in $D \subseteq \mathbb{R}^2$, $f(x,y)$ has no zeros in D, and $f(x,y)f''_{xy}(x,y) = f'_x(x,y)f'_y(x,y)$ holds for all $(x,y) \in D \subseteq \mathbb{R}^2$, then there exist continuously differentiable functions $\varphi_1(x)$ and $\varphi_2(y)$ such that $f(x,y) = \varphi_1(x)\varphi_2(y)$ for all $(x,y) \in D \subseteq \mathbb{R}^2$.

Under relaxed conditions, equation (1.1) can be regarded as a necessary and sufficient condition for a quantum state to be a product state. Does this theorem have a corresponding discrete spectrum version? The answer is affirmative.

Theorem 1: The following statements are equivalent: 1. The non-zero tensor a_{ij} is separable; 2. $R(\mathbf{a}) = 1$ (the rank of tensor $\mathbf{a}$ equals 1); 3. The tensor equation

$$a_{i+1,j}a_{k,l+1}-a_{i+1,j}a_{kl}-a_{ij}a_{k,l+1}+a_{ij}a_{kl} = a_{i+1,l+1}a_{kj}-a_{i,l+1}a_{kj}-a_{i+1,l}a_{kj}+a_{il}a_{kj} \tag{1.2}$$

holds.

Proof: The equivalence $(i) \Leftrightarrow (ii)$ follows directly from reference [2]. For $(i) \Rightarrow (iii)$, substituting $a_{ij} = \alpha_i\beta_j$ yields $\alpha_{i+1}\beta_j\alpha_k\beta_{l+1} - \alpha_{i+1}\beta_j\alpha_k\beta_l - \alpha_i\beta_j\alpha_k\beta_{l+1} + \alpha_i\beta_j\alpha_k\beta_l = \alpha_{i+1}\beta_{l+1}\alpha_k\beta_j - \alpha_i\beta_{l+1}\alpha_k\beta_j - \alpha_{i+1}\beta_l\alpha_k\beta_j + \alpha_i\beta_l\alpha_k\beta_j$, which holds identically.

For $(iii) \Rightarrow (ii)$, we first explain the origin of equation (1.2). By analogy, we treat partial derivatives in Propositions 1 and 2 as differences between indices, with $a_{i+1,j} - a_{ij}$ analogous to $f'_x(x, y)$, and tensor products analogous to function multiplication, yielding the equation in this theorem. We can express tensor a_{ij} in matrix form, where D is the dimension of the Hilbert space containing the quantum state under investigation. We need only consider the rows and columns appearing in the equation.

Rewriting (1.2) reveals relationships between determinants. We must prove that all second-order minors of matrix $\mathbf{a}$ equal zero, which under the premise that $\mathbf{a}$ is non-zero establishes that its rank is 1.

We analyze this through case distinctions:

Case 1: When $i = k$, we obtain a relation among second-order minors of adjacent rows in the large matrix $\mathbf{a}$, which also serves as a recurrence formula with respect to index l.

Case 2: When $j = l$, we obtain a relation among second-order minors of adjacent columns, which serves as a recurrence formula with respect to index i.

Case 3: When $i = k$ and $j = l$, we find that any minimal second-order minor (with adjacent rows and columns) equals zero. Substituting this into the recurrence formula from Case 1 shows that all second-order minors of adjacent rows are zero. Finally, substituting these into the recurrence formula from Case 2 proves that any second-order minor equals zero. Since matrix $\mathbf{a} \neq 0$, we conclude $R(\mathbf{a}) = 1$ [2].

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2. Separability of Higher-Order Tensors

The approach for general tensors is similar to the bipartite case, though the complexity of multi-partite systems lies in determining which indices are separable and which are not. We begin with third-order tensor separability and then generalize further.

Lemma: The following statements are equivalent for a non-zero third-order tensor A: 1. A can be expressed as $A_{ijk} = \alpha_i a_{jk}$, where α is a vector and a is a second-order tensor; 2. The generalized rank between index i and the combined indices jk equals 1 (denoted $R(i, jk) = 1$); 3. The partial ranks between indices (i, j) and (i, k) both equal 1 (denoted $R(i, j) = R(i, k) = 1$); 4. Indices i and j, as well as i and k, satisfy equation (1.2) respectively:

$$A_{i+1,j,k}A_{i',j'+1,k} - A_{i+1,j,k}A_{i'j',k} - A_{ijk}A_{k,j'+1,k} + A_{ij,k}A_{i'j'k} = A_{i+1,j'+1,k}A_{i'jk} - A_{i,j'+1,k}A_{i'jk} - A_{i+1,j',k}A_{i'jk} + A_{i}$$

with the third index held fixed, applying the equation to the first two indices. The case for i and k is analogous, holding the second index fixed.

Proof: For $(i) \Rightarrow (ii)$, let the dimensions of the spaces for indices i, j, k be N_1, N_2, N_3 respectively. Define $l = jN_2 + k$, so $k = l \bmod N_2$ and $j = \lfloor l/N_2 \rfloor$. This allows us to treat matrix a as a vector with $N_2 \cdot N_3$ components, yielding

$A_{il} = \alpha_i a_l$, which has rank 1 [2] (the generalized rank between indices i and jk).

The implication $(ii) \Rightarrow (i)$ is immediate. For $(i) \Rightarrow (iii)$, the partial rank between indices i and j is defined by fixing index k and computing the rank of the remaining second-order tensor. Fixing $k = k'$, we have $a_{jk'}$ as a vector and $A_{ijk'} = \alpha_i a_{jk'}$, so the resulting second-order tensor clearly has rank 1. The case for i and k follows similarly.

For $(iii) \Rightarrow (i)$, we use the tensor rank decomposition from reference [3], which guarantees a unique decomposition:

$$A_{ijk} = \sum \alpha_i^{(1)} \cdots \alpha_i^{(r)} \tag{2.1}$$

where r denotes the rank of tensor A, consisting of r terms. This can be rewritten as a tensor product:

$$A_{ijk} = a_{il} b_{jl} c_{kl} \tag{2.1'}$$

where three matrices contract simultaneously on index $l \in \{1, 2, \dots, r\}$.

We analyze this through case distinctions:

Case 1: Given k, if the partial rank between i and j equals 1, then either all $\alpha_i^{(p)}$ are multiples of $\alpha_i^{(1)}$ (in which case the conclusion holds), or all $\beta_j^{(p)}$ are multiples of $\beta_j^{(1)}$, yielding $R(A) = 1$ and the conclusion holds.

Case 2: Given j, if the partial rank between i and k equals 1, similar possibilities arise. If both Case 1 and Case 2 hold simultaneously, then the tensor can be expressed as a product of vectors.

The equivalence $(iii) \Leftrightarrow (iv)$ follows directly from Theorem 1.

This lemma readily extends to determining whether an n-order tensor can be separated into a vector and an $(n-1)$-order tensor. Using this lemma, we obtain:

Theorem 2: The following statements are equivalent for a non-zero n-order tensor A: 1. A can be expressed as $A_{i_1 i_2 \cdots i_n} = \alpha^1_{i_1} \alpha^2_{i_2} \cdots \alpha^n_{i_n}$, where each α is a vector; 2. The generalized rank between each index and the remaining indices equals 1: $\forall i \in \{i_1, i_2, \dots, i_n\}, R(\{i\}, \{i_1, i_2, \dots, i_n\} \setminus \{i\}) = 1$; 3. The partial rank between any two indices equals 1: $\forall i, j \in \{i_1, i_2, \dots, i_n\}, i \neq j, R(i, j) = 1$; 4. Any two indices of A satisfy equation (1.2).

Proof: For $(i) \Leftrightarrow (ii)$, consider index i_1 as an example. Treating $\alpha^2_{i_2} \cdots \alpha^n_{i_n}$ as tensor $a_{i_2 \cdots i_n}$ and applying the lemma yields the result; the same logic applies to other indices. The implication $(i) \Rightarrow (iii)$ is immediate. For $(iii) \Rightarrow (i)$, using the lemma with index i_1 shows it separates from $i_2 \dots i_n$. By the same reasoning, each index separates from the rest, so tensor A must be expressible as a product of n vectors. The equivalence $(iii) \Leftrightarrow (iv)$ follows from Theorem 1 and the lemma.

If we relax the constraint of expressing a tensor as a product of vectors and instead consider decomposition into several lower-order tensors, we have:

Theorem 3: The following statements are equivalent for a non-zero M-order tensor A: 1. A can be expressed as $A_{i_1^1 \dots i_{N_1}^1 i_1^2 \dots i_{N_2}^2 \dots i_1^n \dots i_{N_n}^n} = a_{i_1^1 \dots i_{N_1}^1} a_{i_1^2 \dots i_{N_2}^2} \cdots a_{i_1^n \dots i_{N_n}^n}$, where each a is a tensor and $N_1 + N_2 + \cdots + N_n = M$; 2. Partitioning the indices into n groups $\{i_1^1, i_2^1, \dots, i_{N_1}^1\}, \{i_1^2, \dots, i_{N_2}^2\}, \dots, \{i_1^n, \dots, i_{N_n}^n\}$, the generalized rank between any index in a group and all indices outside that group equals 1: $\forall I \in \{\{i_1^1, i_2^1, \dots, i_{N_1}^1\}, \{i_1^2, \dots, i_{N_2}^2\}, \dots, \{i_1^n, \dots, i_{N_n}^n\}\}, \forall i \in I, R(\{i\}, \{i_1^1, i_2^1, \dots, i_{N_1}^1, i_1^2, \dots, i_{N_2}^2, \dots, i_1^n, \dots, i_{N_n}^n\} \setminus I) = 1$; 3. With the same index grouping, selecting any two groups and taking one index from each yields partial rank 1: $\forall I, J \in \{\{i_1^1, i_2^1, \dots, i_{N_1}^1\}, \{i_1^2, \dots, i_{N_2}^2\}, \dots, \{i_1^n, \dots, i_{N_n}^n\}\}, I \neq J, \forall i \in I, \forall j \in J, R(i, j) = 1$; 4. With the same index grouping, any two indices from different groups satisfy equation (1.2).

To determine whether an n-order tensor $A_{i_1 i_2 \dots i_n}$ is entangled and understand its entanglement structure, we can apply the above theorems iteratively:

I. Sequentially test whether i_1 separates from the remaining indices, then i_2, and so on through i_n, for a total of n tests. If each index separates from the rest, tensor A is a complete product state.

II. Treat i_1, i_2 as a combined entity and test whether i_1 separates from the remaining $n-2$ indices (excluding i_2), then test whether i_2 separates from the remaining $n-2$ indices (excluding i_1). Continue this process for all $\binom{n}{2}$ pairs of indices, eliminating those that are separable.

III. For the remaining indices, take any three indices as a combined entity and test separability from the rest.

IV. If n is even, continue testing until checking separability of any $n/2$ indices from the rest; if n is odd, test up to $(n-1)/2$ indices. The complete separability structure of tensor A will then be fully determined (computing in reverse order may be more efficient for reducing computational cost).

Note that Theorem 3 provides three 判别 conditions. For finite-dimensional cases, any of the three can be used. However, for infinite but countable dimensions with known analytical expressions for the tensor, only the last method is applicable.

The above discussion concerns discrete spectra. The continuous spectrum case is similar, with the following theorems:

Theorem 4: If there exist differentiable functions $\varphi_1(\vec{x}_1), \varphi_2(\vec{x}_2), \dots, \varphi_n(\vec{x}_n)$ such that $f(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) = \varphi_1(\vec{x}_1)\varphi_2(\vec{x}_2)\cdots\varphi_n(\vec{x}_n)$, then the following system of equations holds:

$$f(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) f''_{\vec{x}_i \vec{x}_j}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) = f'_{\vec{x}_i}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) f'_{\vec{x}_j}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) \tag{2.2}$$

for all $i, j \in \{1, 2, \dots, n\}$ and any $(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) \in D \subseteq \mathbb{R}^{N_1 + N_2 + \cdots + N_n}$. Here $N_1, N_2, \dots, N_n$ are the dimensions of vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ respectively.

Theorem 5: If $f(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$, $f'_{\vec{x}_i}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$, $f'_{\vec{x}_j}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$, and $f''_{\vec{x}_i\vec{x}_j}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$ are continuous in $D \subseteq \mathbb{R}^{N_1+N_2+\dots+N_n}$, $f(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$ has no zeros in D, and the system of equations $f(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) f''_{\vec{x}_i\vec{x}_j}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) = f'_{\vec{x}_i}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) f'_{\vec{x}_j}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$ holds for all $(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) \in D \subseteq \mathbb{R}^{N_1+N_2+\dots+N_n}$, then there exist continuously differentiable functions $\varphi_1(\vec{x}_1), \varphi_2(\vec{x}_2), \dots, \varphi_n(\vec{x}_n)$ such that $f(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) = \varphi_1(\vec{x}_1)\varphi_2(\vec{x}_2)\cdots\varphi_n(\vec{x}_n)$ for all $(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n) \in D \subseteq \mathbb{R}^{N_1+N_2+\dots+N_n}$.

Proof: The proof follows essentially the same approach as Theorems 1 and 2 in reference [4]. Let $N_1 + N_2 + \dots + N_n = N$, treating the theorem as separating N one-dimensional variables into n groups. From Propositions 2 and 3, we need only eliminate the equations satisfied within each group of variables (for instance, if the first group contains N_1 variables, the internal separability partial differential equations need not be considered). The remaining equations, when arranged in order, yield the desired result.

Theorems 4 and 5 suggest a symbolic representation approach using vectors as indices [5], leading to an equivalent formulation of Theorem 3:

Theorem 3' : The following statements are equivalent: 1. A non-zero M-order tensor A can be expressed as $A_{\vec{i}_1\vec{i}_2\dots\vec{i}_n} = A_{i_1^1\dots i_{N_1}^1 i_1^2\dots i_{N_2}^2\dots i_1^n\dots i_{N_n}^n} = \alpha_{\vec{i}_1}\cdots\alpha_{\vec{i}_n}$, where the first equality indicates equivalence between vector indices and scalar component indices, with $\vec{i}_1 \in \{1, 2, \dots, D\}^{N_1}$, $\vec{i}_n \in \{1, 2, \dots, D\}^{N_n}$, and similarly for others, where $N_1 + N_2 + \dots + N_n = M$; 2. In tensor A, the generalized rank between each vector index and the remaining indices equals 1 (e.g., between $\vec{i}_1$ and $\vec{i}_2\vec{i}_3\dots\vec{i}_n$, and similarly for others); 3. The partial rank between any two vector indices of A equals 1; 4. For any two vector indices, the following system of equations holds:

$$A_{\vec{i}+\hat{1},\vec{j}}A_{\vec{k},\vec{l}+\hat{1}} - A_{\vec{i}+\hat{1},\vec{j}}A_{\vec{k}\vec{l}} - A_{\vec{i}\vec{j}}A_{\vec{k},\vec{l}+\hat{1}} + A_{\vec{i}\vec{j}}a_{\vec{k}\vec{l}} = A_{\vec{i}+\hat{1},\vec{l}+1}A_{\vec{k}\vec{j}} - A_{\vec{i},\vec{l}+\hat{1}}A_{\vec{k}\vec{j}} - A_{\vec{i}+\hat{1},\vec{l}}A_{\vec{k}\vec{j}} + A_{\vec{i}\vec{l}}A_{\vec{k}\vec{j}}$$

where $\hat{1} \in \{(1, 0, \dots, 0)\}$.

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Note: Figure translations are in progress. See original paper for figures.