Postprint of Wavelet Transform Image Denoising Algorithm Based on Improved Threshold Function

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Abstract

To address the issues of discontinuity in traditional hard threshold functions at the threshold and the constant bias between original wavelet coefficients and wavelet estimated coefficients in soft threshold functions, an image denoising algorithm based on an improved threshold function is proposed. This algorithm leverages the advantages of improved threshold functions, dynamically selects a fixed threshold by configuring appropriate adjustment parameters, and introduces adjustment factors to mitigate the constant bias between original and estimated wavelet coefficients, thereby enhancing the approximation degree between the reconstructed image and the original image. The improved threshold function satisfies continuity at the threshold, while also fulfilling the asymptotic property and high-order differentiability. Simulation results indicate that employing the improved threshold function for image denoising achieves favorable visual quality. Furthermore, when comparing denoising performance metrics such as Mean Square Error (MSE), Peak Signal-to-Noise Ratio (PSNR), and Signal-to-Noise Ratio (SNR), both PSNR and SNR are enhanced while MSE is reduced, demonstrating optimized denoising performance.

Full Text

Abstract

This paper proposes an image denoising algorithm based on an improved threshold function to address the discontinuity of traditional hard threshold functions at the threshold point and the constant deviation between original wavelet coefficients and estimated wavelet coefficients in soft threshold functions. The algorithm combines the advantages of existing improved threshold functions by dynamically selecting fixed thresholds through appropriate adjustment parameters and introducing adjustment factors to reduce the constant deviation between original and estimated wavelet coefficients, thereby improving the approximation between reconstructed and original images. The improved threshold function satisfies continuity at the threshold point while also meeting asymptotic and higher-order differentiability requirements. Simulation results demonstrate that the proposed threshold function yields excellent visual denoising effects. Comparative analysis of denoising performance metrics—including Mean Squared Error (MSE), Peak Signal-to-Noise Ratio (PSNR), and Signal-to-Noise Ratio (SNR)—shows that both PSNR and SNR improve while MSE decreases, confirming the optimization of denoising performance.

Keywords: wavelet transform; threshold function; threshold image denoising; mean-square error; peak signal-to-noise ratio; signal-to-noise ratio

0 Introduction

During transmission, images are inevitably contaminated by noise to varying degrees, causing blurring and other issues that degrade image quality and hinder subsequent processing. Effective image denoising schemes are therefore essential. Wavelet transform offers multi-resolution analysis capabilities with adjustable sampling lengths for different frequencies in the time domain. In 1994, Donoho and Johnstone proposed the wavelet threshold denoising algorithm, which has since matured for image denoising applications [?]. However, typical hard and soft threshold functions suffer from notable drawbacks: hard threshold functions exhibit discontinuity at the threshold point, while soft threshold functions introduce a constant bias between original and estimated wavelet coefficients. Numerous scholars have investigated these issues and proposed improvements to classical threshold functions [?].

Reference [?] introduced a new threshold function with adjustment factors for dynamic tuning, though the denoising performance remained suboptimal. Reference [?] proposed an improved threshold function with adjustment parameter m to optimize classical hard and soft threshold functions, achieving certain improvements. Reference [?] employed multi-layer threshold functions for wavelet image denoising, determining adjustment factors based on different sampling lengths but without analyzing the impact of decomposition levels. Reference [?] presented an adaptive threshold denoising function that reduces the bias between original and estimated coefficients through appropriate parameters, yet the results were unsatisfactory. Reference [?] introduced a novel sign-function-based wavelet threshold denoising function, comparing PSNR variations across different adjustment factors, though the constant bias issue persisted. Reference [?] combined traditional hard and soft threshold functions to propose a new denoising function, showing improved PSNR but with limited significance.

Addressing these limitations, this paper synthesizes the advantages of classical hard and soft threshold functions with existing improved approaches to propose a novel threshold function. By setting different adjustment factors and comparing denoising performance metrics, optimal parameters are selected. Similarly, different wavelet decomposition levels are evaluated to determine the best total decomposition number n, thereby establishing the relevant adjustment parameters for the improved threshold function.

1.1 Wavelet Threshold Denoising Principle

The selection of threshold functions directly determines the processing strategy for wavelet coefficients and consequently affects final denoising performance. Widely used methods include the hard and soft threshold approaches proposed by Donoho et al., with subsequent improvements such as the hard-soft compromise threshold function [**?**]. The traditional threshold functions are expressed as follows:

Hard threshold function:

$$\hat{w}_{j,l} = \begin{cases} w_{j,l}, & |w_{j,l}| \geq \lambda \\ 0, & |w_{j,l}| < \lambda \end{cases}$$

Soft threshold function:

$$\hat{w}_{j,l} = \begin{cases} \operatorname{sgn}(w_{j,l})(|w_{j,l}| - \lambda), & |w_{j,l}| \geq \lambda \\ 0, & |w_{j,l}| < \lambda \end{cases}$$

Hard-soft compromise threshold function:

$$\hat{w}_{j,l} = \begin{cases} \operatorname{sgn}(w_{j,l})(|w_{j,l}| - \alpha\lambda), & |w_{j,l}| \geq \lambda \\ 0, & |w_{j,l}| < \lambda \end{cases}$$

where $\alpha = 0.5$.

Figure 2 [Figure 2: see original paper] illustrates the curves of typical hard and soft threshold functions. The hard threshold function yields poorly continuous estimated wavelet coefficients, with discontinuity at $|w_{j,l}| = \lambda$, causing oscillations and truncation effects in the reconstructed image. While the soft threshold function offers better continuity, the constant deviation between original and estimated coefficients reduces the approximation to the original image, yielding suboptimal reconstruction and denoising results.

Wavelet threshold denoising is a common approach that processes high-frequency noise components through thresholding before reconstructing the denoised image from the modified coefficients. Threshold estimation and function selection are critical factors: thresholds set too low leave residual noise, while thresholds set too high eliminate important image features, causing blurring. Therefore, careful selection of threshold functions and decision criteria is essential.

The mathematical model for an image contaminated with Gaussian white noise is described as:

$$f(j,l) = g(j,l) + n(j,l)$$

where $f(j,l)$ represents the noisy image, $g(j,l)$ denotes the noise-free image, $n(j,l)$ is Gaussian white noise following a normal distribution $N(0,\delta^2)$, and l indicates pixel position.

The main steps of wavelet threshold denoising (Figure 1 [Figure 1: see original paper]) are: a) Perform multi-level orthogonal wavelet transform on the noisy image to obtain wavelet decomposition coefficients $w_{j,l}$; b) Set thresholds for each decomposition level and process coefficients to obtain estimated coefficients $\hat{w}_{j,l}$; c) Reconstruct the image using the estimated coefficients $\hat{w}_{j,l}$ to produce the denoised image $\hat{f}_{j,l}$.

2.1 Common Wavelet Threshold Functions

To address defects in traditional hard and soft threshold functions, references [?, ?] propose improved threshold functions:

Reference [?]:

$$\hat{w}_{j,l} = \begin{cases} \operatorname{sgn}(w_{j,l}) \frac{(|w_{j,l}|-\lambda)}{1+e^{|w_{j,l}|-\lambda}}, & |w_{j,l}| \geq \lambda \\ 0, & |w_{j,l}| < \lambda \end{cases}$$

Reference [?]:

$$\hat{w}, l = \begin{cases} 2\operatorname{sgn}(w_{j,l}) \left(\frac{|w_{j,l}|+k}{2k+1}\right), & |w_{j,l}| \geq \lambda \\ \frac{|w_{j,l}|^2}{(2k+1)\lambda}, & |w_{j,l}| < \lambda \end{cases}$$

Equation (7) effectively addresses the constant deviation between noisy and estimated coefficients but lacks adjustment factors for dynamic tuning. Equation (8) includes adjustment factor k (with $k = 1$) and employs a nonlinear function for gradual threshold compression rather than direct truncation, ensuring continuity while avoiding oscillation effects. This approach effectively resolves the constant deviation problem and demonstrates good denoising performance.

2.2 Improved Wavelet Threshold Selection

In wavelet threshold denoising, threshold selection directly impacts performance. Common methods include Stein' s Unbiased Risk Estimate (rigrsure), Heuristic SURE (heursure), Minimax, and Fixed Threshold (sqtwolog). Different criteria yield different results: fixed and heuristic thresholds tend to be overly large, causing loss of detail coefficients, while rigrsure and minimax thresholds are often too small, producing insufficient denoising. The most commonly used threshold, derived by Donoho et al. for Gaussian noise models through decision theory for independent normal variables, is expressed as:

$$\lambda = \sqrt{2\ln(MN)} \times \delta$$

where λ is the threshold, $M \times N$ is the image size, and δ is the Gaussian noise variance.

Since the actual noise variance is unknown, it must be estimated beforehand. To avoid fixing the threshold at a single value, this paper adopts the improved threshold selection method from reference [**?**]:

$$\lambda_{\text{new}} = \sqrt{2\ln(MN)} \times e^{(1-1/n)} \times \delta$$

where $e^{(1-1/n)}$ is a shrinkage factor and n is the total number of wavelet decomposition levels. All fixed thresholds in subsequent discussions are updated using this formula.

2.3 A Novel Improved Wavelet Threshold Function

This paper combines the advantages of traditional hard and soft threshold functions with current improved approaches to propose a new threshold function. By introducing adjustment factors to reduce the constant deviation between original and estimated coefficients, the improved denoising function is expressed as:

$$\hat{w}_{j,l} = \begin{cases} (1 + ae^{-m|w_{j,l}|^t})\,\mathrm{sgn}(w_{j,l})(|w_{j,l}| - b\lambda), & |w_{j,l}| \geq \lambda \\ \mathrm{sgn}(w_{j,l})\frac{(|w_{j,l}|-\lambda)}{2}e^{(1/2-|w_{j,l}|/\lambda)}, & |w_{j,l}| < \lambda \end{cases}$$

where a, b, m, and t are adjustable positive parameters that can be tuned to enhance denoising performance.

In the interval $|w_{j,l}| < \lambda$, the function is not simply set to zero but gradually compresses the threshold through a nonlinear function, avoiding truncation-induced oscillations while further addressing constant deviation issues.

Mathematical Analysis of the Improved Threshold Function:

1) **Continuity Analysis:**

$$\lim_{w_{j,l}\to\lambda^+} \hat{w}_{j,l} = \lim_{w_{j,l}\to\lambda^+} [(1 + ae^{-m|w_{j,l}|^t})\,\mathrm{sgn}(w_{j,l})(|w_{j,l}| - b\lambda)] = 0$$

$$\lim_{w_{j,l}\to\lambda^-} \hat{w}_{j,l} = \lim_{w_{j,l}\to\lambda^-} \left[\mathrm{sgn}(w_{j,l})\frac{(|w_{j,l}| - \lambda)}{2}e^{(1/2-|w_{j,l}|/\lambda)}\right] = 0$$

Thus, $\lim_{w_{j,l}\to\lambda^+} \hat{w}_{j,l} = \lim_{w_{j,l}\to\lambda^-} \hat{w}_{j,l} = 0$, proving continuity at $|w_{j,l}| = \lambda$.

2) **Asymptotic Behavior:** When $|w_{j,l}| \to +\infty$, the deviation between $\hat{w}_{j,l}$ and $w_{j,l}$ gradually decreases, overcoming the fixed bias issue in soft threshold functions. The function has $y = x$ as its asymptote.

3) **Bias Reduction:** The improved function reduces the constant deviation between original and estimated coefficients compared to traditional soft thresholding.
4) **Higher-Order Differentiability:** The improved threshold function satisfies higher-order differentiability when $|w_{j,l}| \neq \lambda$, facilitating various mathematical transformations and processing.
5) **Impact of Adjustable Parameters:** When $a = 0$ and $b = 0$, the function becomes a soft threshold; when $a \to \infty$ and $b \to \infty$, it approaches a hard threshold function. Thus, the improved function can flexibly adjust between soft and hard thresholds based on practical requirements. Empirical testing shows optimal denoising performance when $a = 1$, $b = 1$, $m = 2$, and $t = 1$.

In summary, the improved threshold function overcomes the discontinuity of hard threshold functions and the constant bias issue in soft threshold functions.

Denoising Algorithm Steps: a) Select the db5 wavelet basis and perform multi-level orthogonal wavelet transform on the noisy image to obtain decomposition coefficients $w_{j,l}$; b) Estimate noise variance δ and compute the fixed threshold λ using Equation (3). Process coefficients using Equation (9) to obtain estimated coefficients $\hat{w}_{j,l}$; c) Reconstruct the denoised image $\hat{f}_{j,l}$ using the estimated coefficients and low-frequency components.

The algorithm flowchart is shown in Figure 3 [Figure 3: see original paper].

3 Experimental Results and Performance Analysis

To validate the proposed threshold function, tests were conducted using the db5 wavelet basis for various decomposition levels. Test images (Lena and Peppers) of size 512×512 were contaminated with Gaussian noise of different variances ($\delta = 0.01, 0.03, 0.05$). Six methods were compared: hard threshold, soft threshold, hard-soft compromise, reference [**?**], reference [**?**], and the proposed method.

When $\delta = 0.01$ and $n = 3$, the denoised Lena and Peppers images are shown in Figures 4 [Figure 4: see original paper] and 5 [Figure 5: see original paper]. Visually, the images produced by the proposed method appear clearer with more effective noise removal.

Objective Metrics: Performance was evaluated using: - **MSE:** $\text{MSE} = \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}[f(j,l) - \hat{f}(j,l)]^2$ (lower is better) - **PSNR:** $\text{PSNR} = 10\log_{10}\left(\frac{255^2}{\text{MSE}}\right)$ (higher is better) - **SNR:** $\text{SNR} = 10\log_{10}\left\{\frac{\sum_{i=1}^{M}\sum_{j=1}^{N}[f(j,l)]^2}{MN\times\text{MSE}}\right\}$

Lena Image Results (n=3): Table 1 shows that compared to traditional hard/soft thresholds, recent improved functions increase PSNR by approximately 8 dB and SNR by about 7.5 dB while significantly reducing MSE. The proposed method achieves a maximum PSNR of 30.7349 dB—an improvement of approximately 11 dB over traditional soft thresholding.

Peppers Image Results (n=3): Table 2 presents similar improvements for the Peppers image.

Parameter Sensitivity Analysis: Figure 6 [Figure 6: see original paper] shows PSNR versus noise variance for Peppers image ($n = 3$). As noise variance increases, PSNR decreases for all methods, but the proposed method maintains the highest PSNR even at $\delta = 0.11$.

Figure 7 [Figure 7: see original paper] shows SNR versus noise variance, demonstrating that the proposed method consistently achieves higher SNR across all variance levels.

Figure 8 [Figure 8: see original paper] illustrates PSNR versus total decomposition levels (n) for $\delta = 0.01$. PSNR decreases as decomposition levels increase, but the proposed method outperforms all others at each level.

Figure 9 [Figure 9: see original paper] shows SNR versus decomposition levels, confirming the proposed method' s superior performance across different decomposition depths.

In conclusion, varying noise variance and decomposition levels produce different denoising effects. The proposed threshold function consistently improves PSNR and SNR while reducing MSE compared to five alternative methods.

4 Conclusion

This paper proposes an improved threshold image denoising function that incorporates adjustable parameters to optimize wavelet coefficient estimation. Comparative analysis across different noise variances and decomposition levels demonstrates that the new threshold selection rule provides flexibility by allowing level-dependent thresholds. The adjustable parameters enable optimal coefficient estimation. Experimental results confirm that the proposed method effectively removes noise while preserving useful signal information, yielding superior denoising performance both visually and quantitatively compared to existing approaches.

References

[1] Zhou Xianchun, Wu Ting, Shi Lanfang, et al. A kind of wavelet transform image denoising method based on curvature variation regularization[J]. *Acta Electronica Sinica*, 2018, 46(3): 621-628.

[2] Tang Hui, Liu Zengli, Chen Lin, et al. Wavelet image denoising based on the new threshold function[J]. *Applied Mechanics and Materials*, 2013, 347-350: 2231-2235.

[3] Luo Yuan, Tan Qin, Zhang Yi. Improved threshold function denoising algorithm based on Teager energy operator[J/OL]. *Application Research of Computers*, 2019, 36(5). [2018-09-30]. http://kns.cnki.net/kcms/detail/51.1196.TP.20180314.1730.066.html.

[4] Zhang Jianhua, Zhu Qiang, Zhang Jinrong, et al. A novel algorithm for threshold image denoising based on wavelet construction[J]. *Cluster Computing*, 2018: 1-8.

[5] Yi Qingming, Chen Mingmin, Shi Min. Improved wavelet de-noising method for infrared image application[J]. *Computer Engineering and Applications*, 2016, 52(1): 173-177.

[6] Zhu Fang, Liu Wei. A new quaternion wavelet image denoising method based on NIG model[J]. *Infrared Technology*, 2017, 39(10): 928-935.

[7] Zhang Lian, Qin Huafeng, Yu Chengbo. Research of denoising method based on wavelet threshold[J]. *Computer Engineering and Applications*, 2008, 44(9): 172-173+199.

[8] Wang Bei, Zhang Genyao, Li Zhi, et al. Wavelet threshold denoising algorithm based on new threshold function[J]. *Journal of Computer Applications*, 2014, 34(5): 1499-1502.

[9] Qin Dongdong, Chen Zhijun, Yan Xueqin. Wavelet image denoising with multilevel thresholding function[J]. *Computer Engineering*, 2017, 43(6): 202-206.

[10] Wang Rui, Zhang Youchun. New threshold function in wavelet threshold de-noising[J]. *Computer Engineering and Applications*, 2013, 49(15): 215-218.

[11] Cui Jinge, Chen Bingquan, Xu Qing, et al. A wavelet threshold image denoising algorithm based on a new kind of sign function[J]. *Telecommunication Science*, 2017, 33(1): 45-52.

[12] Huang Yuchang, Hou Dewen. Fingerprint de-noising based on improved wavelet threshold function[J]. *Computer Engineering and Applications*, 2014, 50(6): 179-181, 209.

[13] Hu Ran, Guo Chengcheng, Yang Jianfeng. Video denoising algorithm based on wavelet threshold and PCA[J]. *Computer Science*, 2016, 43(4): 290-293.

[14] Yang Huixian, Wang Xusi, Xie Penghe, et al. Infrared image denoising based on improved threshold and inter-scale correlation of wavelet transform[J]. *Acta Automatica Sinica*, 2011, 37(10): 1167-1174.

[15] Donoho D L, Johnstone I M. Adapting to unknown smoothness via wave shrinkage[J]. *Journal of the American Statistical Association*, 1995, 90(432): 1200-1224.

[16] Song Yajun, Yang Chen, Yang Jinbao. Image denoising algorithm using regional threshold by wavelet-based contourlet transform[C]//Proc of Interna-

tional Conference on Identification, Information and Knowledge in the Internet of Things. Washington DC: IEEE Computer Society, 2018: 33.

[17] Zhou Xifeng, Zhu Wenwen, Guo Qiangang. The denoising of ultrasonic signal based on asymptotic semi-soft thresholding function[J]. *Journal of Detection & Control*, 2011, 33(2): 35-39.

[18] Chen Zhuan, Hu Zhifeng. Remote sensing image denoising based on improved wavelet threshold algorithm[J]. *Bulletin of Surveying and Mapping*, 2018(4): 28-31.

[19] Zhang Chengzhi, Feng Huajun, Xu Zhihai, et al. Piece-wise noise variance estimation of images based on wavelet transform[J]. *Journal of Zhejiang University: Engineering Science*, 2018, 52(9): 1804-1810.

[20] Zhou Fengbo, Li Changgeng, Zhu Hongqiu. Research on threshold improved denoising algorithm based on lifting wavelet transform in UV-Vis spectrum[J]. *Spectroscopy and Spectral Analysis*, 2018, 38(2): 506-510.

Note: Figure translations are in progress. See original paper for figures.