Heteroscedastic GNSS Positioning Error Model Testing Postprint

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Abstract

In the Global Navigation Satellite System (GNSS), positioning accuracy is an important performance indicator; an accurate positioning error model facilitates understanding of the system' s operating mode and also provides guidance for further improving system performance. When the ranging errors of satellites satisfy the conditions of mutual independence and uniform variance, the positioning error equals the ranging error standard deviation multiplied by the geometric dilution of precision factor. However, in practical applications, due to the different propagation paths of signals from different satellites, the statistical characteristics of ranging errors are difficult to be identical. Especially when satellite signals are affected by multipath interference and ionospheric scintillation, the ranging errors among different satellites exhibit significant differences. This paper presents a theoretical positioning error model for non-uniform ranging errors established using the singular value decomposition method; using measured data from Beijing and Hong Kong stations, the positioning errors affected by multipath effects and ionospheric scintillation are analyzed. The results demonstrate that their statistical characteristics are consistent with those given by the positioning error model with non-uniform variance, validating the correctness and effectiveness of the model.

Full Text

A Test of GNSS Positioning Error Model with Non-Uniform Variance

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Abstract

In Global Navigation Satellite Systems (GNSS), positioning accuracy is a critical performance metric, and an accurate positioning error model is essential for understanding system behavior and providing guidance for further performance improvements. When satellite ranging errors satisfy the conditions of mutual independence and uniform variance, the positioning error equals the ranging error standard deviation multiplied by the Geometric Dilution of Precision (GDOP). However, in practical applications, the statistical characteristics of ranging errors across different satellites are unlikely to be identical due to varying signal propagation paths. This is particularly true when satellite signals are affected by multipath interference and ionospheric scintillation, which can cause significant differences in ranging errors among satellites. This paper introduces a theoretical positioning error model for non-uniform ranging errors based on Singular Value Decomposition (SVD). Using measured data from Beijing and Hong Kong stations, we analyze positioning errors affected by multipath effects and ionospheric scintillation. The results demonstrate that the statistical characteristics of positioning errors are consistent with those predicted by the non-uniform variance positioning error model, thereby validating its correctness and effectiveness.

Keywords: positioning error; geometric dilution of precision; ranging error; multipath interference; ionospheric scintillation

1 Introduction

In GNSS satellite positioning error analysis, the primary model used assumes uniform ranging error variance, where positioning error equals the ranging error standard deviation multiplied by GDOP when satellite ranging errors are independent and identically distributed with uniform variance. GDOP depends solely on satellite geometry and is inversely proportional to the volume of the tetrahedron or polyhedron formed by the user receiver and visible satellites. Numerous studies have focused on satellite selection algorithms to minimize GDOP values. Ranging error represents the residual error after correcting satellite pseudorange measurements, including uncompensated tropospheric delay, ionospheric delay, multipath interference, and receiver noise. In actual positioning solutions, the statistical characteristics of ranging errors across satellites are difficult to maintain as identical due to inter-satellite differences and varying propagation paths, violating the uniform variance assumption. This is especially pronounced when certain satellite signals suffer from multipath effects or ionospheric scintillation, causing their ranging errors to increase significantly.

While many studies have investigated the relationships among multipath interference, ionospheric scintillation, and positioning errors, providing statistical analyses and experimental results demonstrating these effects, no unified conclusion has been established describing these relationships. Some researchers have proposed eliminating satellites with large ranging errors, but this approach alters the satellite constellation geometry and may degrade GDOP. Although this measure helps maintain positioning accuracy to some extent, it is impractical for general user receivers. Therefore, in practical positioning solutions with non-uniform ranging errors, particularly when satellite signals are affected by multipath interference or ionospheric scintillation, ensuring navigation positioning accuracy and implementing practical solutions in general user receivers requires examining how non-uniform ranging errors affect positioning precision. Previous research has investigated the impact of non-uniform pseudorange errors on GPS positioning and proposed a positioning error model.

This paper aims to validate the positioning error model for non-uniform satellite ranging errors. First, we decompose the geometry matrix using SVD to derive the positioning error covariance expression and establish the positioning error model. Next, using measured data from Beijing and Hong Kong stations, we perform positioning solutions on satellite signals affected by multipath effects and ionospheric scintillation, comparing the results with the theoretical model under non-uniform ranging errors. Finally, conclusions are presented.

2 Positioning Error Model Based on SVD Method

In linear algebra, SVD is a fundamental matrix decomposition method that is theoretically equivalent to least squares solutions. The advantage of SVD lies in its ability to provide the matrix pseudoinverse, yielding concise, self-constrained expressions for the solution. The singular values and orthogonal vectors obtained from SVD offer geometric and theoretical insights into the problem. Moreover, SVD solutions exhibit numerical stability, addressing rounding issues that affect conventional least squares solutions, particularly when equations are inherently singular. The SVD solution represents the optimal approximation in the least-squares sense.

In GNSS satellite positioning, the positioning error equation can be expressed as:

$$\Delta\rho = G\Delta x + \varepsilon$$

where $\Delta\rho$ represents the difference between measured and estimated satellite pseudorange values, ε denotes ranging error, Δx represents the unknown corrections to user receiver position and clock bias, and G is the geometry matrix. Using SVD to decompose the geometry matrix G, we can derive the positioning error covariance expression and establish the positioning error model.

The SVD decomposition of geometry matrix G is given by:

$$G = U\Sigma V^T$$

The pseudoinverse of matrix G is:

$$G^+ = V\Sigma^+U^T$$

where U and V are orthogonal matrices whose column vectors are the eigenvectors of GG^T and G^TG, respectively. Σ and Σ^+ are diagonal matrices containing the singular values σ_i and σ_i^+ $(i = 1, 2, 3, 4)$ of G and G^+, with $\sigma_i^+ = 1/\sigma_i$ for $\sigma_i > 0$ and $\sigma_i^+ = 0$ otherwise.

Let Δx denote positioning error. The positioning error covariance can be expressed as:

$$\text{Cov}(\Delta x) = E[\Delta x \Delta x^T] = G^+E[\varepsilon\varepsilon^T](G^+)^T$$

2.1 Uniform Variance Positioning Error Model Based on SVD Method

When satellite ranging errors follow independent, zero-mean normal distributions with identical variance σ_ε^2, the ranging error model characteristics are:

$$E[\varepsilon\varepsilon^T] = \sigma_\varepsilon^2 I$$

From equation (7), the positioning error covariance matrix becomes:

$$\text{Cov}(\Delta x) = \sigma_\varepsilon^2 G^+(G^+)^T = \sigma_\varepsilon^2(G^TG)^{-1} = \sigma_\varepsilon^2 H$$

where $H = (G^TG)^{-1}$ is the GDOP matrix. This real symmetric matrix can be expressed as:

$$H = V\Lambda V^T$$

The four column vectors v_i $(i = 1, 2, 3, 4)$ of V are the eigenvectors of matrix H, and:

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

contains the four eigenvalues of H. The positioning error distribution forms a four-dimensional ellipsoid where the vectors v_i define the directions of the four principal axes, and $\sqrt{\lambda_i}$ represent the semi-axis lengths of the ellipsoid.

2.2 Non-Uniform Variance Positioning Error Model Based on SVD Method

When satellite ranging errors cannot satisfy the condition of identical variance, let $\sigma_{\varepsilon_n}^2$ denote the ranging error variance of the n-th satellite. The ranging error model characteristics are:

$$E[\varepsilon\varepsilon^T] = \mathrm{diag}(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, ..., \sigma^2_{\varepsilon_N})$$

From equation (7), the positioning error covariance matrix becomes:

$$\mathrm{Cov}(\Delta x) = G^+ \mathrm{diag}(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, ..., \sigma^2_{\varepsilon_N})(G^+)^T$$

This shows that the positioning error model becomes more complex under non-uniform ranging errors.

Assuming $\sigma^2_{\varepsilon_{\min}}$ is the minimum ranging error variance among N satellites, the total positioning error covariance matrix from equation (12) can be expanded as:

$$\mathrm{Cov}(\Delta x) = \sigma^2_{\varepsilon_{\min}}(G^T G)^{-1} + \sum_{n=1}^{N}(\sigma^2_{\varepsilon_n} - \sigma^2_{\varepsilon_{\min}}) g_n^T g_n$$

where g_n is the n-th row vector of matrix G^+. The positioning error covariance consists of two parts: the uniform variance positioning error component and the sum of positioning error increments caused by ranging error variances exceeding $\sigma^2_{\varepsilon_{\min}}$. The increment term depends not only on $\sigma^2_{\varepsilon_{\min}}$ and G but also on the variance increments $(\sigma^2_{\varepsilon_n} - \sigma^2_{\varepsilon_{\min}})$ and the n-th row vector of U matrix.

Equation (13) shows that $\mathrm{Cov}(\Delta x)$ is a real symmetric matrix. Let v'_i denote the four column vectors representing the eigenvectors of $\mathrm{Cov}(\Delta x)$, and λ'_i $(i = 1, 2, 3, 4)$ represent its eigenvalues. According to the eigenvalue interlacing theorem, the relationship between λ'_i and λ_i is:

$$\lambda'_i = \lambda_i + \Delta\lambda_i$$

where:

$$\Delta\lambda_i = \sum_{n=1}^{N}(\sigma^2_{\varepsilon_n} - \sigma^2_{\varepsilon_{\min}})\|u_n^T v'_i\|^2$$

The positioning error covariance matrix can be expressed using eigenvalues and eigenvectors as:

$$\mathrm{Cov}(\Delta x) = \sum_{i=1}^{4} \lambda'_i v'_i (v'_i)^T$$

Thus, when satellite ranging errors are non-uniform, the positioning error distribution forms a four-dimensional ellipsoid where v'_i defines the four principal axes, with v_1' as the major axis and $\sqrt{\lambda_i'}$ as the semi-axis lengths. Compared with the uniform variance assumption, this ellipsoid' s principal axes shift toward a specific direction related to both constellation geometry and the positions of satellites with increased ranging error variance. The ellipsoid' s axis lengths increase due to ranging error variance increments, with the major axis experiencing the most significant elongation. When $\sigma_{\varepsilon_n}^2 \gg \sigma_{\varepsilon_{\min}}^2$, the variance increments substantially stretch the positioning error ellipsoid, primarily along the major axis direction.

3 Positioning Results Analysis Based on SVD Method

This study validates the SVD-based positioning error model using measured data from Beijing and Hong Kong stations. Data periods when satellite signals were affected by multipath interference and ionospheric scintillation were selected for positioning solutions, as these conditions cause significant increases in satellite ranging errors. According to the positioning error model, the major axis of the positioning error ellipsoid should exhibit noticeable deflection and elongation under such conditions.

The research first obtains satellite ranging errors by fitting and detrending satellite pseudorange measurements. Next, positioning solutions are performed on satellites affected by multipath interference and ionospheric scintillation using the SVD method to obtain actual positioning results. Using equation (9), the eigenvectors and eigenvalues of the positioning error ellipsoid principal axes are calculated under the uniform ranging error assumption. Using equations (12) and (16), the eigenvectors and eigenvalues are obtained for the actual positioning error ellipsoid under non-uniform ranging errors. The model' s validity is verified by comparing the error ellipsoid characteristics with actual positioning results.

3.1 Positioning Solution Parameters

Location Coordinates: Two IGS stations were selected to examine the effects of multipath and ionospheric scintillation: Beijing station (BJFS) at 39.6084°N, 115.8925°E and Hong Kong station (HKSL) at 22.3659°N, 114.1787°E. Hong Kong' s low-latitude location experiences frequent ionospheric scintillation events.

Satellite Positions and Pseudorange Measurements: GPS satellite measurements from April 25-27, 2014, March 20-22, 2014, June 20-22, 2014, September 22-24, 2014, December 21-23, 2014, February 27, 2014, and March 10, 2014 were used for positioning solutions. Pseudorange measurements and satellite positions were obtained from 30-second sampling rate observation files and navigation messages provided by the IGS website.

3.2 Ranging Error Extraction Method

Pseudorange measurements contain various errors that can be categorized by source into satellite-related errors, signal propagation-related errors, and receiver-related errors. Based on their variation characteristics, these errors can be classified as biases or noise. Biases change slowly over time and can be directly measured or predicted using mathematical models. In contrast, noise varies rapidly, exhibits statistical characteristics, and is difficult to predict precisely.

Observation files record satellite pseudorange values, which show certain trends containing geometric range and pseudorange measurement biases. Detrending the satellite pseudorange versus time curve yields residuals that represent the ranging errors ε. This study uses polynomial fitting to detrend satellite pseudorange variation curves, determining polynomial coefficients through root-mean-square values and coefficient of determination R^2 to achieve the most accurate ranging error extraction.

3.2.1 Multipath Effects on Ranging Error Multipath effects occur when receiver antennas capture GPS signals reflected by surrounding objects, with each reflected signal potentially undergoing multiple reflections before reaching the antenna. Multipath degrades carrier phase and pseudorange measurement accuracy and can cause code phase loss-of-lock and signal disappearance. Generally, low-elevation satellite signals are more susceptible to multipath effects—the lower the satellite elevation angle, the higher the probability of multipath occurrence, the lower the pseudorange measurement accuracy, and the larger the ranging error.

[Figure 1: see original paper] shows the standard deviation of ranging error σ_ε versus elevation angle for six satellites (PRN14, 15, 18, 21, 22, 24) measured at Beijing station on April 26, 2014, over 24 hours with an elevation mask angle of 10°. The figure demonstrates that as elevation angle decreases, multipath effects cause σ_ε to increase from 0.25 m to 2 m, allowing elevation angle to serve as an indicator for multipath-affected ranging errors.

3.2.2 Ionospheric Scintillation Effects on Ranging Error Ionospheric scintillation causes rapid fluctuations in radio signals propagating through the ionosphere and can severely interrupt signals, representing a significant factor affecting space radio system performance. The low-latitude equatorial anomaly region is one of the most frequently affected and severely impacted areas globally. Ionospheric scintillation can increase user receiver measurement errors.

ROTI (Rate of TEC Index) indicates ionospheric scintillation severity, with larger ROTI values representing more intense scintillation. [Figure 2: see original paper] shows the relationship between ranging error standard deviation σ_ε and ROTI for Hong Kong station satellites affected by ionospheric scintillation in March 2014. The σ_ε values vary between 0.2 m and 3 m, showing an increasing trend with ROTI, enabling ROTI-based identification of ionospheric scintillation effects on ranging errors.

3.3.1 Positioning Error Analysis Under Multipath Effects

Since satellite constellation geometry varies over time, data periods with minimal constellation changes were selected for model validation. During April 25-27, 2014, from 00:20 to 00:35 UTC (epochs 40-70), the azimuth and elevation variations of visible satellites at Beijing station were less than 5°, maintaining essentially identical constellation geometry. With an elevation mask angle of 10°, satellites with elevations between 10° and 30° were considered affected by multipath. During this 15-minute interval, six satellites (PRN14, 15, 18, 21, 22, 24) were visible, with only PRN14 experiencing multipath effects.

[Figure 3: see original paper] shows the σ_ε variation over time for each satellite on April 26, 2014, demonstrating that actual satellite ranging errors are non-uniform, with significant increases when satellites experience multipath interference.

Positioning solutions were performed in the local East-North-Up (ENU) coordinate system, where the E-N plane represents the horizontal direction. [Figure 4: see original paper] shows the horizontal projection of positioning results at Beijing station on April 25, 26, and 27, 2014. The 30 horizontal positioning results from 00:20-00:35 are marked with red dots, while all 2,880 daily results are shown as blue dots. Satellite elevation angles of 10°, 20°, and 30° correspond to circles with radii of 2 m, 4 m, and 6 m, respectively, with satellite azimuths corresponding to positioning error directions. The mean azimuth and elevation of multipath-affected PRN14 during the 15-minute interval are marked with red stars, while other satellites are marked with yellow stars.

For each of the 30 positioning results, the geometry matrix G was decomposed via SVD to obtain eigenvectors v_i and eigenvalues λ_i of the positioning error ellipsoid principal axes under the uniform ranging error assumption. The projection of mean eigenvector $\bar{v}$ onto the horizontal plane is shown as yellow line segments. Using extracted σ_ε values with equations (12) and (16), the actual positioning error ellipsoid eigenvectors v_i' and eigenvalues λ_i' were obtained. The projection of mean vector $\bar{v}'$ onto the horizontal plane is shown as red line segments, with triangles marking three times the mean eigenvalue projection $3\sqrt{\bar{\lambda}'}$ along the red line.

[Figure 5: see original paper] presents horizontal positioning distributions at Beijing station during the spring equinox, summer solstice, autumn equinox, and winter solstice periods. The figures show results from: March 20-22, 2014 (04:10-04:25 UTC), June 20-22, 2014 (02:30-02:45 UTC), September 22-24, 2014 (01:35-01:50 UTC), and December 21-23, 2014 (04:45-05:00 UTC). Satellite elevations of 10°, 20°, and 30° correspond to circles with radii of 8/3 m, 16/3 m, and 8 m, respectively.

presents the mean GDOP values, mean ranging error variance increments $\sum \Delta\sigma_{\varepsilon}^2$ (where $\Delta\sigma_{\varepsilon}^2 = \sigma_{\varepsilon_n}^2 - \sigma_{\varepsilon_{\min}}^2$), mean projections of positioning error ellipsoid semi-axis lengths under uniform variance assumption ($3\sqrt{\lambda}$), and mean projections of actual positioning error ellipsoid semi-axis lengths ($3\sqrt{\lambda'}$) for 30 epochs during specified time intervals.

The results demonstrate that actual positioning results align well with the non-uniform variance positioning error model. In terms of axis direction, the actual positioning error ellipsoid major axis rotates significantly compared to the uniform variance assumption, with the rotation angle related to constellation geometry and the position of satellites with increased σ_ε. The positioning points during selected intervals distribute around the actual error ellipsoid major axis direction. Regarding axis length, the actual error ellipsoid axes are significantly elongated compared to the uniform variance case, with elongation magnitude related to constellation geometry and ranging error variance increments. The actual positioning results fall within the $3\sqrt{\lambda'}$ range, consistent with the central limit theorem.

3.3.2 Positioning Error Analysis Under Ionospheric Scintillation

ROTI values identify satellites affected by ionospheric scintillation. Analysis reveals that PRN7 and PRN9 on February 27, 2014 (16:00-17:00 UTC) and PRN7 on March 10, 2014 (18:00-19:00 UTC) experienced strong ionospheric scintillation events.

During February 27, 2014, from 16:25 to 16:40 UTC, constellation geometry changed minimally. Seven satellites (PRN1, 7, 9, 11, 19, 28, 32) were visible, with PRN19 and PRN32 affected by multipath and PRN7 and PRN9 affected by ionospheric scintillation.

[Figure 6: see original paper] shows the σ_ε variation over time for these seven satellites, demonstrating significant increases when signals experience multipath interference or ionospheric scintillation.

[Figure 7: see original paper] displays the horizontal positioning results at Hong Kong station on February 27, 2014. Daily results are marked with blue dots, while results from 16:25-16:40 UTC are marked with red dots. Multipath-affected satellites PRN19 and PRN32 are marked with red stars, ionospheric scintillation-affected satellites PRN7 and PRN9 with green stars, and other satellites with yellow stars. Calculations yield $\sqrt{\lambda} = 2.3149$ m and $\sqrt{\lambda'} = 3.7268$ m. Yellow and red line segments represent projections of error ellipsoid major axes under uniform variance and actual conditions, respectively, with triangles marking $3\sqrt{\lambda'}$.

[Figure 8: see original paper] shows horizontal positioning results at Hong Kong station on March 10, 2014. From 17:15-17:30 UTC, ten satellites (PRN1, 4, 7, 8, 9, 11, 17, 20, 28, 32) were visible, with PRN7, 11, and 32 affected by multipath.

PRN7 experienced both multipath and ionospheric scintillation (marked with black stars). Calculations yield $\sqrt{\lambda} = 2.1574$ m and $\sqrt{\lambda'} = 3.5646$ m.

The positioning distribution figures and calculations confirm that actual positioning results match the non-uniform variance positioning error model in both axis direction and length. When multiple satellites exhibit significantly increased ranging error standard deviations, the error ellipsoid major axis elongation becomes more pronounced.

4 Conclusion

This study investigates positioning error characteristics under non-uniform GNSS satellite ranging errors, presenting a non-uniform variance positioning error model derived through SVD decomposition. The model indicates that with non-uniform ranging errors, positioning errors distribute as a four-dimensional ellipsoid in space, with axis directions and lengths obtained via SVD decomposition of the positioning error covariance matrix. Compared with the uniform variance assumption, the non-uniform variance positioning error ellipsoid becomes deflected and elongated due to ranging error variance increments. When certain satellites exhibit significantly increased ranging error variance, the major axis deflects noticeably and the ellipsoid elongates primarily along this axis.

The model was validated using measured data from Beijing and Hong Kong stations. Satellite pseudorange measurements were detrended to extract ranging errors, followed by positioning solution analysis of satellites affected by multipath interference and ionospheric scintillation. Results show that when satellite signals experience multipath interference or ionospheric scintillation, ranging error standard deviations increase significantly. The actual positioning error ellipsoid exhibits major axis rotation and elongation compared to the uniform variance assumption. Actual positioning results align with the non-uniform variance error ellipsoid major axis direction, with distribution ranges matching the ellipsoid semi-axis lengths. The non-uniform variance positioning error model accurately describes the relationship among GDOP, ranging errors, and positioning errors, providing a foundation for further positioning accuracy research.

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