Mathematical principle of m×n resistor networks

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Abstract

Unified processing and research of multiple network models have been implemented, achieving a new theoretical breakthrough through the establishment of two new theorems for evaluating the exact electrical characteristics (potential and resistance) of complex m×n resistor networks via the Recursion-Transform method with potential parameters (RT-V), which apply to a variety of different types of lattice structures with arbitrary boundaries, such as nonregular m×n rectangular networks and nonregular m×n cylindrical networks. Our research provides analytical solutions for the electrical characteristics of complex networks (finite, semi-infinite, and infinite), which were previously unsolved problems. As applications of these theorems, a series of analytical solutions for potential and resistance of complex resistor networks have been discovered. In particular, three novel mathematical propositions were discovered when comparing resistances in two resistor networks, and many interesting trigonometric identities were also discovered.

Full Text

Preamble

Mathematical Principle of m×n Resistor Networks

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Abstract

This paper implements unified processing and research of multiple network models, achieving a new theoretical breakthrough by establishing two novel theorems for evaluating the exact electrical characteristics (potential and resistance) of complex m×n resistor networks using the Recursion-Transform method with potential parameters (RT-V). The approach applies to various lattice structures with arbitrary boundaries, including non-regular m×n rectangular networks and non-regular m×n cylindrical networks. Our research yields analytical solutions for the electrical characteristics of complex networks (finite, semi-infinite, and infinite) that have not been previously solved. As applications of these theorems, a series of analytical solutions for potential and resistance in complex resistor networks are discovered. In particular, three novel mathematical propositions emerge when comparing resistances in two resistor networks, and many interesting trigonometric identities are also discovered.

Key words: complex network, RT-V method, electrical properties, boundary conditions, trigonometric identity, Laplace equation
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I. Introduction

Many complex scientific problems can be simulated using resistor network models, encompassing numerous electrical and non-electrical problems in physics, engineering, and mathematics. The progress of circuit theory not only promotes the development of integrated circuits and electrification science but also fosters interdisciplinary development in natural science. Resistor network models are so important that issues from various disciplines can be studied through network simulation, such as conduction in anisotropic disordered systems [1], percolation and conduction [2], anisotropy in electrical conductivity [3], nonlinear localized modes in two-dimensional electrical lattices [4], electric circuit networks equivalent to chaotic quantum billiards [5], photonic crystal circuits [6], manifesting the evolution of eigenstates from quantum billiards [7], dynamical signatures of fractionalization [8], processing of hexagonally sampled two-dimensional signals [9], topological insulators and superconductors [10], topological properties of linear circuit lattices [11], topological spin excitations [12], three-dimensional printed meshes [13], topological insulator surfaces [14], fractional-order circuit networks [15], mean field theory [16, 17], finite-size corrections of the dimer model [18], lattice Green' s functions [19-22], resistance distance [23], and so on. In particular, the two important equations of Poisson and Laplace [24, 25] can be simulated by resistor network models [26]. Additionally, a real planar network of graphene exists in nature.

Calculating the equivalent resistance between two arbitrary lattice sites in a resistor network is always an important yet difficult problem, requiring not only circuit theory but also innovative algebra. For example, when the boundary of a resistor network is arbitrary, it is usually very difficult to obtain the exact potential and resistance of complex networks with arbitrary boundaries. In fact, the boundary acts like a wall or trap, affecting the solution of the problem. Therefore, practical needs require us to create new theories to accurately calculate the electrical characteristics (voltage and resistance) of complex circuit networks.

Let us review the research history of resistor networks. In 1845, Kirchhoff established the basic circuit theory (the node current law and the circuit voltage law). After 150 years, Cserti [27] calculated the two-point resistance of infinite networks using Green' s function technique, focusing mainly on infinite lattices, with some applications appearing in later literature [28, 29]. In 2004, Wu [30] formulated a different approach (called the Laplacian matrix method) and derived explicit resistance formulas for arbitrary finite and infinite lattices with normative boundaries (such as free, periodic boundary, etc.) in terms of the eigenvalues and eigenvectors of the Laplacian matrix, which relies on two matrices along two perpendicular directions. Later, Laplacian matrix analysis was also applied to impedance networks [31], and after some improvements, several new resistor network problems were resolved [32-34]. However, the Laplacian approach cannot apply to networks with arbitrary boundaries since it is impossible to give explicit eigenvalues for arbitrary matrix elements (associated with arbitrary boundaries). Yet boundary conditions are important as they represent real cases occurring in real life.

In 2011, Tan pioneered a new technique for studying complex resistor networks [35], now called Recursion-Transform (RT) theory of resistor networks [26]. The RT method depends on one matrix containing one direction, which is obviously different from the Laplacian method that depends on two matrices along two directions. With the development of RT technique, many problems of non-regular networks with zero resistor edges have been resolved [36-45]. Additionally, the advantage of the RT method is that all resistance results are in a single summation, differing from the Laplacian approach which gave resistance results in the form of a double summation.

Recently, the RT method has been subdivided into two forms: one form is the matrix equation expressed by current parameters [38-44], simply called the RT-I method; another form is the matrix equation expressed by potential parameters [26, 45], simply called the RT-V method. Summarizing previous applications of RT (including RT-I and RT-V), it is not hard to see that previous research relied on the zero resistor boundary, such as the globe network [36, 44] belonging to cylindrical networks with two zero resistor boundaries, the cobweb network [26, 40] belonging to cylindrical networks with one zero resistor boundary, the fan network [37, 45] belonging to rectangular networks with one zero resistor boundary, and the hammock network [34, 43] belonging to rectangular networks with two zero resistor boundaries. Obviously, how to study complex networks without zero resistor boundaries using the RT method remains a question.

[Figure 1: see original paper]

This paper develops a new technique and improves RT theory to allow us to study arbitrary resistor networks without relying on zero resistor boundaries, which can derive the electrical properties (potential and resistance) of arbitrary m×n complex networks with complex boundaries. Here we build two new theorems that lead to large problems being resolved. Our study shows the universal RT method is very interesting and useful for solving complex networks. We focus on researching the electrical properties (potential and resistance) of Fig.1 and Fig.2 for two complex m×n resistor networks with two arbitrary boundaries using the advanced RT-V method, which have not been resolved before. It is worth emphasizing that non-regular complex networks with two arbitrary boundaries are multi-purpose network models because they can produce various geometrical structures as shown in Figs.5-10. Thus a large number of resistor network problems will be resolved by this paper.

[Figure 2: see original paper]

From the above analysis, Professor Wu [30] was the first to give several accurate equivalent resistance formulas for regular resistor networks using the Laplacian matrix method. For comparative study, we introduce two main results for resistor networks from Ref.[30].

Case-1. Consider Fig.1 with a regular m×n rectangular network, where n and m are the maximum coordinate values of (n, m), resistors r and r0 are bonded respectively in the horizontal and vertical directions. The resistance formula for Fig.1 is:

120rrr 0121212(,)11mnrrRddxxyymn 122,1,21111011[cos()cos()]222(1)(1)(1cos)(1cos)mnxjixjiijjjiCyCymn

where arbitrary two nodes in the network.

Case-2. Consider Fig.2 with a cylindrical m×n resistor network, where n and m are the maximum coordinate values of (n, m), resistors r and r0 are bonded respectively in the horizontal and vertical directions. The resistance formula for Fig.2 is:

120rrr 0121212(,)11mnrrRddxxyymn 122,1,21111011[cos()cos()]222(1)(1)(1cos)(1cos)mnxjixjiijjjiCyCymn

where

The above results were found for the first time by Wu. Later Refs.[32-34] improved the Laplacian matrix method to make it applicable to regular cobweb and hammock networks. However, the improved Wu method still cannot resolve resistor networks with arbitrary boundaries, such as the networks with two arbitrary boundaries shown in Fig.1 and Fig.2. In addition, the equivalent resistance in Eqs.(1) and (2) are in double summation form, not single sum.

[Figure 3: see original paper]

II. RT-V Theory and Poisson Equation

Consider two kinds of complex m×n resistor networks shown in Fig.1 and Fig.2, where n and m are the maximum coordinate values of (n, m). Assume the origin of the rectangular coordinate system, and denote nodes of the network by coordinates {x, y}. Assume the electric current J goes from the input to the output. Denote the nodal potential of the sub-network as shown in Fig.3, and express the nodal potential at. We will study the complex resistor networks in four steps.

The first step involves setting up a discrete Poisson equation based on the sub-network of Fig.3. By Kirchhoff' s law to establish the nodal potential equations along the vertical direction, we achieve a discrete static field equation (or Poisson equation) for any network:

2012121212()(,)111mnryyrRddyyxxnmm 121222„„21111102cos2()1(1)(1)(1cos2)(1cos)mnxjxjxjxjiijij

where , and contains the input and output conditions of the current, denote second order discrete equation, and , Eq.(3) reduces to the discrete Laplace equation. For the arbitrary network together with the upper and lower boundary conditions, by Eq.(3) we are led to:

, (4)

where are respectively column matrixes, and reads is the matrix built along the vertical direction. For Fig.1 and Fig.2, the:

, (5)

, (6)

, (7)

where , and is the resistor between in Fig.2, when belongs to Fig.1; when), the belongs to Fig.2. The purpose of introducing is to express two different resistor networks uniformly.

The second step considers the boundary conditions of the left and right edges in the networks of Fig.1 and Fig.2. Applying Kirchhoff' s law to each of the left and right boundaries, we obtain two matrix equations of boundary conditions:

, (8)

, (9)

where Eq.(7). is the identity matrix, matrix is given by

Equations (4)-(9) are all the equations we need to compute the node potential. However, it is impossible for us to get the solution of the above equations directly. Thanks to the RT theory of Tan that gave the matrix transform method [26, 38-40], we create the new technique here. In the following we give the transformation technology based on RT-V theory.

The third step creates matrix transform. Firstly, we work out the eigenvalue of matrix 22()(),()kyyxyxxxxhVrI 0hrr 12()„()yxyyyyIJ 2()()()()112yyyyxxxxxVVVV 2()(1)()(1)2yyyy
()()()()012TyyyyknIIII I 1m A1m A12002(1)00002(1)002mhbhhbhhhhhhhbhhhbh
, which is given by solving determinantal equation of (just), yields:

) (10)

where , and for Fig.1, for Fig.2.

Next, we transform Eqs.(4)-(9) by the following approaches:

, (11)

, (12)

where . Assuming is the row vectors of matrix Pm+1, such as

Thus, we multiply Eq.(4) from the left-hand side by Pm+1, we get:

, (14)

. (13)

where Eqs.(11) and (12) are used.

Similarly, applying to Eqs.(8) and (9), we are led to:

, (15)

. (16)

The above Eqs.(10)-(16) are all essential equations for evaluating the node potential.

The fourth step solves the matrix equations (13)-(16). Selecting the reference potential, by Eqs.(14)-(16), we obtain after some algebra and reduction the solution:

, (17)

where for Fig.1, for Fig.2, and is defined in Eq.(26) below, and have:

) (18)

where are, respectively, defined in Eqs.(19)-(25) below.

The RT-V Theory. The above method of establishing recursive matrix equations with voltage parameters, implementing matrix transform and obtaining the solutions of matrix equations is called RT-V theory. The detailed content of the RT-V theory (Recursion-Transform theory with potential parameters) can be found in the above four steps in Eqs.(3)-(18). Here we give a summary of the RT-V theory consisting of four steps: The first step creates a main matrix equation of potential distributions along the Y axis (such as building Eqs.(3)-(7)); The second step builds the constraint equations (including boundary conditions) of nodal potentials (such as setting up Eqs.(8) and (9)); The third step diagonalizes the matrix relation to reduce the equations from two dimensions to one dimension (such as converting Eq.(4) to Eq.(14) et al.); The fourth step figures out the analytic solution of the equations (such as the results of Eqs.(17) and

(18)), then makes the inverse matrix transform by Eq.(12) to derive the analytical solutions of nodal potential and gives the resistance formula by Ohm' s law.

III. Two Theorems of Resistor Networks

A. Several Definitions

In order to facilitate and simplify the expression of the solutions of matrix equations, we define several variables for later use:

, (19)

. (20)

And define variables for later use by:

, (22)

. (23)

, (24)

. (25)

The above definitions are applicable throughout the entire article. All of these definitions are meant to illustrate the following two fundamental theorems, and we always assume that the electric current J goes from the input to the output in our entire paper.

B. Two Fundamental Theorems

Theorem-1. Consider the arbitrary resistor networks of Fig.1 and Fig.2 whose maximum coordinate value is (n, m). Then the potential of node in the resistor ,kiC,ii ,1cos()2kyikiCy cos()kyykiCyy 21cos(1cos)1iiihhhh 21cos(1cos)1iiihhhh (1)(1)ibim 0fo Fig.11for Fig.2bb ()ikF(),isx (),iks ()ikG()()()ikkkiiiiF ()()()1iiikkkFFF ()()(),1(1)iiisxxsxFhF 0s if sssssiiixxxnxsixxiiixxxnxsxxxxx ()()()()112211(2)(1)(1)iiiinnnnGFhhFhhF 111(,) network can be written as:

, (26)

where , and is defined in Eq.(13), is the conjugate complex of , and is the solution of the matrix equation (14) together with the boundary condition equations. Formula (26) is a general formula suitable for any resistor network model.

In particular, when selecting as the reference potential, the potential of in the resistor networks can be written as:

where is the conjugate complex of (which is just a real number), and is a piecewise function:

, (27)

, (28)

is given by (18) which is the solution of equations (13)-(16).

Theorem-2. Consider the arbitrary resistor networks of Fig.1 and Fig.2 whose maximum coordinate value is (n, m). Then the resistance between any two nodes in the network is given by:

. (29)

where is the solution of the matrix equation (14) together with the boundary condition equations. Formula (29) is a general formula suitable for any resistor network model.

In particular, for the networks of Fig.1 and Fig.2, the resistance between two nodes can be written as:

. (30)

where is the case of Fig.1, and is the case of Fig.2, and is given by (18) which is the solution of equations (13)-(16).

The above two new theorems contain a wide variety of geometric structures of network models, which can produce many new results of potential and resistance, and can create new mathematical identities (see section 6). In the following, we prove the correctness of the two theorems.

C. Proof of Theorems

Consider the m×n resistor network with two arbitrary boundaries shown in Fig.1 and Fig.2. In the introduction, we have built the key Eqs.(4)-(9) by the RT-V theory, and converted the equations to Eqs.(14)-(16), and derived Eqs.(17) and (18). Now we will work out the exact eigenvalues of matrix in Eq.(7). Eq.(10) can be derived by solving equation , and then we need to consider two cases below.

One is for Fig.1. Substituting Eq.(10) into (11) with , we get the eigenvectors:

, (31)

where obtained , and . By careful calculation, the inverse matrix can be easily:

, (32)

where [ ]T denotes matrix transpose.

Thus, the term appeared in Eq.(13) can be specifically rewritten as:

) (34)

Another is for Fig.2. Substituting Eq.(10) into (11) with , the eigenvector is obtained after some algebra and derivation:

, (35)

where . According to strict calculations, the inverse matrix reads:

. (36)

Thus, the term appeared in Eq.(13) can be specifically rewritten as:

1m Adet0mt AE0b 1m A01111101121212coscoscoscoscoscosmmmmmmvvvvvv P12kvk (1)iim

We find that Eq.(32) and Eq.(36) can be rewritten as a unified form below:

, (39)

where , and is the conjugate complex of Using Eq.(12), we have , expanding this matrix equation, then we get:

, (40)

Equation (40) agrees with formula (26) that we need to verify.

Further, we get by comparing equation (39) with equations (32) and (36).

And when selecting , we have Eq.(17). Substituting Eqs.(17), (33) and (37) into Eq.(40), then Eq.(27) can be verified immediately.

Next, we verify Eqs.(29) and (30). By Ohm’ s law, we have:

Substituting Eq.(26) with Substituting Eq.(27) with Eq.(30). Thus, two theorems are verified.

. (41)

into Eq.(41), we therefore obtain Eq.(29). into Eq.(41), we immediately obtain In subsequent sections we consider applications of theorems to arbitrary lattices. In all applications, we stipulate all parameters in Eqs.(18)-(39) apply to all resistor networks, and denote the resistors along the two principal directions by r and r0 except for resistors on the left-right boundaries, and the input and output nodes of current are respectively at.

IV. Electrical Properties of Complex Rectangular Network

A. Nodal Potential of Complex Rectangular Network

Consider the non-regular resistor network shown in Fig.1, where the maximum coordinate is (n, m). Selecting as the reference potential, the potential of any in the finite and semi-infinite networks can be written as:

, (42)

, (43)

where , and are, respectively, defined in Eqs.(19)-(25). For Eq.(43), there be with finite. In particular, when (meaning the input and output nodes of currents are at the same vertical axis), formulae (42) and (43) reduce to:

, (44)

. (45)

Proof of Eq.(42). For Fig.1, substituting Eq.(34) with into (18), we achieve:

) (46)

Substituting Eq.(46) and (34) into (27) with , we therefore achieve Eq.(42).

For proving Eq.(43), when with finite , it can be got a limit by using Eqs.(20)-(25):

. (47)

So, substituting Eq.(47) into (42) with , we therefore verified Eq.(43).

Formula (42) is a meaningful result because the network of Fig.1 is very complex and has not been resolved before, which contains a lot of different network models since the different boundary resistors can produce different geometric structures. Here several special applications of formula (42) are given below.

Application 1. When , Fig.1 degrades into a regular m×n rectangular network. The potential of a node in the network is:

, (48)

(001)21()2miixJVxr (,)dxy1122()(),,0(),1(,)11(1cos)iimxxyixxyimniyiiniCCUxyxxrrCJmmG 1212,,,
where reduces to In particular, when , potential formula (48) reduces further to:

. (49)

Application 2. When), Fig.1 degrades into a fan network as shown in Fig.4(a), where r and are the respective resistors along longitude (radius) and latitude (arc) directions, and the resistor element on the outer arc is (an arbitrary boundary resistor). The potential of a in the fan network can be written as:

, (50)

where we redefine Please note that a non-regular fan network (the outer arc resistor is arbitrary) is a scientific conundrum, which has not been solved before. Ref.[26] has researched just the regular fan network (the outer arc resistor is), but our formula (50) with is different from the result in Ref.[26] because two results depend on different matrices along the orthogonal direction.

[Figure 4: see original paper]

Application 3. When , Fig.1 degrades into a hammock network as shown in Fig.4(b). The potential of a node in the hammock network can be written as:

, (51)

where we redefine In particular, when are respectively at the left and right poles, the potential of Eq.(51) reduces to:

, (52)

Application 4. Consider the input current J is at on the left edge, and the output current J is at on the right edge. The potential of a node in the non-regular resistor network of Fig.1 is:

, (53)

where is defined in Eq.(23).

In particular, when , the potential (53) reduces to:

, (54)

Application 5. Consider a regular rectangular network of Fig.1 with , when is on the left edge, and is on the right edge, the potential of a node the rectangular network is:

In particular, when , the potential equation (55) reduces to:

, (55)

, (56)

Application 6. Consider is at the bottom edge, and is on the top edge. The potential of a node in the non-regular resistor network of Fig.1 is:

, (57)

where is defined in Eq.(24), and In particular, when , the potential equation (57) reduces to:

, (58)

, the potential equation (57) reduces to:

Application 7. Consider , and are on two diagonal nodes, the potential of:

, (59)

(,)22(1)UxynxrJm 11(0,)dy22(,)dny(,)dxy mn12()()12„21„,()1(,)212(1)1(1cos)iimnxyixyiyiiiinrCrCUxynxrCJ a node in the non-regular resistor network is:

where is defined in Eq.(23).

In particular, when , the potential of Eq.(60) reduces to:

, (60)

, (61)

Application 8. Assume Fig.1 is a semi-infinite network, and but n, x and y are finite. Consider is on the left edge, and is on the right edge. When , the potential of a node in the semi-infinite rectangular network is:

, (62)

where . Please note that these definitions apply to all such issues as appear below. Eq.(62) can be derived by taking the limit of Eq.(55).

Application 9. Assume Fig.1 is a semi-infinite network, where but n, x and y are finite. When , taking the limit in Eq.(54), we achieve the potential in a semi-infinite network:

, (63)

Application 10. Assume Fig.1 is an infinite network, but finite. Taking the limit in Eq.(43), we have the potential in the infinite rectangular network:

, (64)

Notice that Eqs.(62) and (63) belong to the case of a semi-infinite network, while Eq.(64) belongs to the case of an infinite network.

B. Resistance of Complex m×n Rectangular Network

Consider an m×n rectangular network with two arbitrary boundaries shown in Fig.1, where the maximum coordinate is (n, m). Defining , the resistance between two nodes in the finite and semi-infinite networks are respectively:

, (65)

, (66)

where , and are, respectively, defined in Eqs.(19)-(25). For Eq.(66), there be with finite . Eq.(66) can be derived by taking the limit in Eq.(65).

Proof of Eq.(65). For Fig.1, substituting Eq.(42) with into (41), then Eq.(65) is verified.

Eq.(65) is an exact expression which still contains a variety of resistance results with all kinds of boundary conditions because the left and right boundaries are arbitrary resistors. For clearly understanding formula (65), we set , m or n as special values, and give several special cases to understand its application and meaning.

Case 1. When , the network of Fig.1 degrades into a rectangular network with an arbitrary right boundary, then formula (65) reduces to:

, (67)

where reduces to

Case 2. When , the network of Fig.1 degrades into a normal rectangular network, then formula (65) reduces to:

, (68)

where reduces to . This problem has been researched in Ref.[30], and gave Eq.(1) with a double sum. Clearly, our result (68) is different from Eq.(1). Two different results will be compared in the last section. This also shows that the equivalent resistance can be expressed in different forms.

Case 3. When , the network of Fig.1 degrades into a non-regular fan network with an arbitrary boundary as shown in Fig.4(a). By Eq.(65), we obtain the resistance of a fan network:

, (69)

where is re-defined as

In particular, when , the network of Fig.1 degrades into a normal network, then formula (69) reduces to:

, (70)

where is re-defined as . This case has been researched in [37], but the result is different from Eq.(70); however, they are equivalent to each other. The reason is that they chose different matrices along different directions. This also shows that the equivalent resistance can be expressed in different forms.

Case 4. When , the network degrades into a fan network with double resistor edge, then formula (65) reduces to:

, (71)

where is re-defined as

Case 5. When , the network of Fig.1 degrades into a hammock network, so formula (65) reduces to:

, (72)

where is re-defined as . This problem has been researched in Ref.[34], but the result is different from Eq.(72)], the reason is that they chose different matrices along different directions.

In particular, when are at the left and right poles, Eq.(72) reduces to:

. (73)

Case 6. When two nodes are on the same vertical axis, from (65) we have the resistance between two nodes:

Where are defined in (24) and (25).

In particular, when , formula (74) reduces to:

. (74)

. (75)

Case 7. When both are at the left edge, formula (65) reduces to:

. (76)

In particular, when , formula (76) reduces to:

. (77)

It can be found that Eq.(77) agrees with the result in Ref.[41]; clearly both of these results are verified indirectly by each other.

Case 8. When both are at the same horizontal axis, from (65), we have the resistance:

, (78)

where are defined in (24) and (25).

Especially, when both are at the bottom edge, Eq.(78) reduces to:

. (79)

are two corner points at the bottom edge, Eq.(79) reduces to a neat result, namely:

. (80)

Case 9. When is on the left edge and is on the right edge, formula (65) reduces to:

where

In particular, when , Eq.(81) reduces to:

, (81)

. (82)

Case 10. When is at the bottom edge and is on the top edge, then formula (65) reduces to:

, (83)

where are defined in (24) and (25).

Case 11. When is at the coordinate origin, and is an arbitrary point, by Eq.(65), the equivalent resistance is:

where are defined in Eqs.(24) and (25).

In particular, when , Eq.(84) reduces to:

, (84)

, (85)

Case 12. When are two diagonal nodes, by (65) we have the resistance between two maximally separated nodes:

where

In particular, when , Eq.(86) reduces to:

, (86)

. (87)

Please note that Eq.(87) is a desired equivalent resistance between two maximum separated nodes in an arbitrary m×n resistor network. This is an interesting result because it is simple and easy to research the asymptotic expansion for the maximum resistance. Refs.[46, 47] studied the asymptotic expansion by making use of the result (1). Obviously, the concise Eq.(87) is more conducive to the study of the asymptotic expression of the maximum resistance. In addition, there are similarities between equations (80) and (87), but only minor differences, where Eq.(80) is the resistance between two corner points at the bottom edge, and Eq.(87) is the resistance between two maximum separated nodes on the diagonal line.

Case 13. When , with , and are finite. Taking the limit of Eq.(76), we have the resistance in the semi-infinite network:

. (88)

Obviously, case 13 is a semi-infinite network problem. The reason why the equivalent resistance is independent of the right boundary is that the network is infinite on the right.

Case 14. When , and are finite. It means that the network is finite at the bottom and left but infinite at the top and right. Taking the limit of Eq.(88), we have:

where . This definition applies to all such issues as appear below.

, (89)

In particular, when , and , but is finite, taking the limit of Eq.(88), we have the equivalent resistance on the left edge:

. (90)

Eqs.(89) and (90) belong to the problem of semi-infinite network, which has not been solved before.

Case 15. When two arbitrary nodes , but are finite, we have the resistance between:

, (91)

Formula (91) can be proved by taking the limit in Eq.(66). Eq.(91) shows that the equivalent resistance of an infinite network is independent of the boundary conditions.

Notice that Eqs.(88)-(90) belong to the case of a semi-infinite network, while Eq.(91) belongs to the case of an infinite network.

Since Eq.(65) is a profound and versatile formula that contains multiple outcomes, for readers to better understand the meaning of Eqs.(65) and (90), we illustrate the meaning and usefulness through four simple examples below.

[Figure 5: see original paper]

Case 16. When , the resistor network of Fig.1 degrades into a resistor network as shown in Fig.5. The equivalent resistance between any two nodes in the resistor network can be written as:

, (92)

, (93)

where are, respectively, defined in Eq.(24) and (25), and:

. (94)

[Figure 6: see original paper]

Case 17. When , the resistor network of Fig.1 degrades into an arbitrary resistor network as shown in Fig.6, where there be four arbitrary resistor elements ( ), the equivalent resistance between any two nodes can be written as:

. (95)

. (96)

. (97)

where (k=1,2), and are, respectively, defined in Eqs.(24) and (25). And , with:

. (98)

From the above derivation, we find that formula (65) is a generalized result, which is applicable to many network problems and summarized a variety of complex network models since it contains six arbitrary elements (.

Case 18. Consider a regular m×n rectangular network, when (the network is finite at the left but infinite at the bottom, top and right.), by Eq.(90), the resistance on the left edge is:

In particular, when , Eq.(99) reduces to:

. (99)

. (100)

Case 19. Consider a regular m×n rectangular network, when (the network is finite at the left but infinite at the bottom, top and right.), by Eq.(90) we have the resistance on the left edge:

In particular, when , from Eq.(101) we get:

. (101)

. (102)

Eqs.(99)-(102) are two novel results which are given for the first time.

V. Electrical Properties of Complex Cylindrical Network

A. Nodal Potential of Complex Cylindrical Network

Consider the non-regular cylindrical network shown in Fig.2, where the maximum coordinate value is (n, m). Selecting as the reference potential, defining , the potential of any node in the finite and semi-infinite networks can be written as:

, (103)

, (104)

where are, respectively, defined in Eqs.(22)-(25). For Eq.(96), there be with finite m. Eq.(104) can be derived by taking the limit in Eq.(103).

In particular, when , formula (103) and (104) reduce to:

, (105)

, (106)

Proof of Eq.(103). For Fig.2, substituting Eq.(38) into Eq.(18), we achieve:

. (107)

The substitution of (107) into Eq.(27) yields:

, (108)

Because the elements in the network are real numbers, the potential must be real numbers. Thus, extracting the real part of Eq.(108) produces Eq.(103).

Formula (103) is a meaningful result because the network of Fig.2 is very complex and has not been resolved before, contains a lot of resistor network models, where each of the different boundary resistor represents a different network structure. So Formula (103) can create many interesting results. In the following applications we always assume that the current J goes from except for special instructions.

Application 1. Consider an arbitrary cylindrical network of Fig.2 with (103). We have the nodal potential:

where reduces to

In particular, when Eq.(109) reduces to:

, (109)

, (110)

[Figure 7: see original paper]

Application 2. Consider a cylindrical network of Fig.2. When , Fig.2 degrades into a cobweb network as shown in Fig.7(a). By (103) we have the nodal potential:

, (111)

where is redefined as

In particular, when is at left edge, and is at right edge. Eq.(111) reduces:

, (112)

Please note that the cobweb network with an arbitrary boundary has not been resolved before; previous work only studied the normal cobweb network (the boundary resistor is) [26]. Eq.(112) is an original result.

Application 3. Consider an arbitrary globe network shown in Fig.7(b). That is to say that Fig.2 degrades into a globe network when , from (103) we have the nodal potential:

, (113)

where we redefine

In particular, when is at left pole, and is at right pole, Eq.(113) reduces to:

. (114)

Formula (114) is very simple and very interesting because the potential distribution is only related to the x and has nothing to do with y, which shows the nodal potential is equal in the same latitude.

Application 4. Consider a non-regular cylindrical network of Fig.2. Assume is on the left edge, and is on the right edge. By (103) we have the nodal potential:

, (115)

where is defined in Eq.(23), and

In particular, when , Eq.(115) reduces to:

, Eq.(115) reduces to:

, Eq.(115) reduces to:

, (116)

, (117)

. (118)

Application 5. Consider a non-regular cylindrical network of Fig.2. When the node are located on the left edge, Eq.(103) reduces to:

, (119)

where , and is defined in Eq.(23).

Application 6. Consider a non-regular cylindrical network of Fig.2. When the node are located on the right edge, Eq.(103) reduces to:

, (120)

One knows the potential function has important application value for solving the Laplace equation. In this paper, the analytical solutions of node potential functions under various conditions are given, which provides a new theory for practical application. In particular, these simple equations of Eqs.(119) and (120) are very interesting and meaningful for applications.

B. Resistance of Complex m×n Cylindrical Network

Consider a complex m×n cylindrical network with two arbitrary boundaries shown in Fig.2, where the maximum coordinate value is (n, m), and defining , The equivalent resistance between two nodes in the finite and semi-infinite cylindrical networks are respectively:

, (121)

, (122)

where are, respectively, defined in Eqs.(24)-(25).

For Eq.(122), there be with finite m. Eq.(122) can be derived by taking the limit in Eq.(121).

Proof of Eq.(121). For Fig.2, substituting Eq.(103) with into Eq.(41), we therefore achieve (121).

Formula (121) is an exact and exciting result because the network of Fig.2 is very complex and has not been resolved before, and contains a lot of resistor network models, where each of the different boundary resistor represents a different network structure. In particular, when taking some specific value for , Eq.(121) gives rise to a series of special cases below.

Case 1. Consider a non-regular cylindrical network of Fig.2. When , the resistance of Eq.(121) reduces to:

, (123)

where reduces to

Case 2. Consider a normal cylindrical network of Fig.2 with , the resistance of Eq.(121) reduces to:

, (124)

where reduces to

Case 3. Consider a non-regular cylindrical network of Fig.2. When , the left boundary collapses to a pole, the network of Fig.2 degrades into a cobweb network with an arbitrary boundary resistor r2 as shown in Fig.7(a). We have the equivalent resistance:

, (125)

where is re-defined as

In particular, when , the network of Fig.7(a) degrades into a regular cobweb network, the resistance of Eq.(125) reduces to:

, (126)

where is redefined as

Please note that case 3 has been researched in Ref.[37], but the result is different from Eq.(126); however, they are equivalent to each other. The reason is that they chose different matrices along different directions, where Ref.[37] set up matrix along the longitude, but this paper sets up matrix along the latitude.

Case 4. When , the left and right boundary collapse respectively to two poles, the network of Fig.2 degrades into a globe network as shown in Fig.7(b). We have:

, (127)

where is re-defined as

Please note that case 4 has been researched in Ref.[36], but the result is different from Eq.(127); however, they are equivalent to each other. The reason is that they chose different matrices along different axes. This also shows that the equivalent resistance can be expressed in different forms.

Case 5. Consider a non-regular cylindrical network of Fig.2. When are on the same latitude, the resistance of Eq.(121) reduces to:

, (128)

where , and is defined in Eq.(23).

In particular, when , the network of Fig.2 degrades into a regular cylindrical network, the resistance of Eq.(128) reduces to:

, (129)

Especially, when are on the left edge, Eq.(129) reduces to:

Case 6. Consider a non-regular cylindrical network of Fig.2. When both are on the same horizontal axis, we have:

, (130)

Case 7. When is at the coordinate origin, and is an arbitrary point, formula (121) reduces to:

. (131)

where are defined in Eqs.(24) and (25).

In particular, when , Eq.(132) reduces to:

, (132)

, (133)

Case 8. Consider a non-regular cylindrical network of Fig.2. When is on the left edge and is on the right edge, the resistance between two edges is:

In particular, when , Eq.(134) reduces to:

, (134)

. (135)

, Eq. (134) reduces to:

, (136)

Case 9. Consider a non-regular cylindrical network of Fig.2. When is at the bottom edge, is an arbitrary node, and , but are finite, by (128) we have:

, (137)

where

In particular, when , Eq.(137) reduces to:

, but is finite, Eq.(137) reduces to:

. (138)

, (139)

Notice that Eqs.(137) and (138) belong to the case of a semi-infinite network, while Eq.(139) belongs to the case of an infinite network.

From the above results we know formula (121) and (122) are two general results which contain many results in a variety of lattice structures, can produce many new resistance formulae, and can create new identities (see Section 6).

It is essential to take into account formula (121) again in order to help the reader further understand its meaning. Here two simple examples are given below.

[Figure 8: see original paper]

Case 10. When , Fig.2 degrades into a 3D $\times$ resistor network as shown in Fig.8.

By Eq.(21) we have), substituting to Eq.(20) yields:

. (140)

So we have , and . By Eq.(121), we have the equivalent resistance between any two nodes:

, (141)

where represents the nodes of , and are, respectively, defined in Eqs.(24) and (25).

Eq.(141) is a general formula, which can produce two specific results below.

Consider the resistance between two nodes , there be , Eq.(141) reduces to:

, (142)

Consider the resistance between two nodes), there be , Eq.(141) reduces to:

, (143)

Fig.8 is a simple and common network model, but getting the equivalent resistance has always been a difficult problem because of the complexity of the boundary conditions. Eq.(141) is given for the first time, which provides a new theoretical basis for practical application.

[Figure 9: see original paper]

Case 11. When , Fig.2 degrades into a 3D × resistor network as shown in Fig.9.

We can get the equivalent resistance between any two nodes by Eq.(121). By Eq.(21) we have , substituting it to Eq.(20) yields:

. (144)

So we have , and . By Eq.(121), we have:

, (145)

where represents the nodes of , and are, respectively, defined in Eqs.(24) and (25).

Eq.(145) is a general formula, which can produce three specific results below.

Consider the resistance between two nodes , there be , Eq.(145) reduces to:

, (146)

Consider the resistance between two nodes , there be , Eq.(145) reduces to:

Consider the resistance between two nodes , there be , Eq.(145) reduces to:

, (147)

. (148)

Case 11 tells us again that the general formula (121) is a meaningful and multi-purpose result since just a 3D ×n resistor network has rich contents and many functions such as Eqs.(145)-(148).

VI. Comparison and Trigonometric Identities

A. Proposition-1. A General Trigonometric Identity-1

Defining , and . When natural numbers, and , we have the trigonometric identity:

where , (149)

. (150)

Please note that identity (149) is found for the first time by this paper. Identity (149) reduces a double sum to a single sum, which provides a new proposition and research method for mathematicians.

Proof of Proposition-1. Consider a regular m×n rectangular network shown in Fig.1 with , Ref.[30] gave a resistance formula (1) by the Laplacian matrix method, which is in the form of double sum. However, this paper gives Eq.(68) by the RT-V method, where the condition and network structure agree with Ref.[30]. Obviously, the two results with different form in two different articles are necessarily equivalent because they are from the same network with the same coordinates. Comparing formula (68) with formula (1), we immediately obtain identity (149).

We find Eq.(149) is an interesting identity for simplifying the double sum to be a single sum.

In particular, when taking particular values of , we have the following simple trigonometric identities.

Deduction 1. When , Eq.(149) reduces to:

In particular, when , Eq.(151) reduces to:

. (151)

. (152)

, there be , Eq.(153) reduces to:

. (153)

, (154)

where

Deduction 2. When , Eq.(149) reduces to:

. (155)

, Eq.(155) reduces to:

(155)

. (156)

where

Deduction 3. When , we have , Eq.(149) reduces to:

. (157)

, (158)

where

In particular, when , Eq.(158) reduces to:

. (159)

The above discoveries are interesting because they are found not in mathematics but in physics. Obviously, according to identity (149) we can derive a series of special trigonometric equalities when different coordinates (xi, yi) are made.

B. Proposition-2. A General Trigonometric Identity-2

Defining , and . When natural numbers, and , we have the trigonometric identity:

where , (160)

. (161)

Proof of Proposition-2. Consider a regular m×n cylindrical network shown in Fig.2 with , we obtain a resistance formula (116) by the RT-V method. However Ref.[30] gave another resistance formula (2) by means of the Laplacian matrix method. Comparing formula (124) with Eq.(2), we immediately obtain identity (160).

In particular, when setting special number values of , we have the following identities.

Deduction 1. When , from (160), we have:

In particular, when , we have , then Eq.(162) reduces to:

. (162)

where , with

Deduction 2. When , Eq. (160) reduces to:

, (163)

. (164)

. (165)

Deduction 3. When , we have , from (160), we have:

, (166)

where are defined in Eq.(164).

Please note that Eq.(166) is different from Eq.(158) because their are different from each other.

Deduction 4. When , by (165), we have:

. (167)

C. Proposition-3. A General Trigonometric Identity-3

Defining , and and defining , and:

, (168)

, (169)

Assume natural numbers satisfy , when natural number satisfies and the arbitrary real number , we have:

. (170)

Proof of Proposition-3. Consider a normal m ×n cylindrical network, where the maximum coordinate value is (n, m-1). are on the same longitude, by Eq.(131), we have:

However, Ref.[48] gave another resistance formula (the parameters in Ref.[48] have been converted to be exactly the same as those in Fig.2):

. (171)

, (172)

Thus formula (171) is equal to Eq.(172) since they have the same parameters in the same network. By Eq.(171) equals (172) to yield Eq.(170). That means, proposition-3 holds.

Deduction 1. When , and is given by (169), Eq.(170) reduces to:

, (173)

where , and is given by Eq.(168).

Deduction 2. When is given by (169), Eq.(170) reduces to:

where , and is given by Eq.(164).

Deduction 3. Since , substituting to (174) together with (169), which yields:

. (174)

where , and is given by Eq.(164).

Deduction 4. When , Eq.(170) reduces to:

where the following identity is used:

. (175)

, (176)

. (177)

We find that identity (170) is interesting because the left-hand side of the identity is the sum over n but the right-hand side is the sum over m, which provides a new mathematical identity for the application of mathematics.

VII. Conclusion and Comment

This paper set up a universal RT-V theory (Recursion-Transform theory with potential parameters) and reveals the basic principle of electrical characteristics of complex resistor networks for the first time. Two theorems (Theorem-1 and Theorem-2) are proposed, and explicit electrical characteristics (potential and resistance) formulae of complex networks are given, which contain results for finite and infinite networks.

It must be emphasized that previous theories (mainly Green's function technique and Laplacian matrix method) cannot solve resistor networks with complex boundaries, because Green' s function technique is usually used to solve infinite network problems, and the Laplacian matrix method depends on the solution of two eigenvalues which relies on two matrices along two orthogonal directions. Using the RT-V method to study resistor networks just relies on one matrix along one vertical direction, which avoids the confusion of another matrix with arbitrary elements that cannot be solved explicitly, and also gives concise results in a single summation, such as all equations given by this paper.

As applications of the two theorems, the analytical solutions of the electrical characteristics (including potential function and equivalent resistance) in complex m×n resistor networks with arbitrary boundaries are given, and many interesting results of various types of resistor networks are produced. Please note that a non-regular m×n rectangular network (see Fig.1) contains arbitrary fan (see Fig.4(a)) and hammock (see Fig.4(b)) networks; a non-regular m×n cylindrical network (see Fig.2) contains arbitrary cobweb (see Fig.7(a)) and globe (see Fig.7(b)) networks. Thus the analytical solutions of the electrical characteristics given by this paper have general significance, which applies to a wide variety of lattice structures, and means the Laplace equation with complex boundary conditions is resolved by modeling resistor networks. The trigonometric identities given show that our research work established new research ideas and approaches for the study of mathematical identities.

In addition, resistance formulae (65), (66), (121) and (122) et al. can be extended to impedance networks since the grid elements can be either resistors or impedances in Fig.1 and Fig.2. For example, assume:

, (178)

that we can therefore study the arbitrary m×n RLC network if we do a plural analysis [31, 41] to the resistance results obtained in this paper.

Our resistance formulae can also apply to fractional circuits. For example, the frequency domain impedance of fractional capacitance and inductance are respectively [15]:

, (179)

. (180)

When we replace r and r0 with , and use the plural analysis [31, 41], the equivalent impedances of the fractional RLC network with arbitrary boundaries can be obtained.

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Note: Figure translations are in progress. See original paper for figures.