Postprint: An Improved Pythagorean Fuzzy Cross-Power Average Group Decision-Making Method Based on Weighted Support Degree

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Full Text

Preamble

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Pythagorean Fuzzy Interaction Power Average Operator Group Decision-Making Method with Improved Weighted Support

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Abstract

To address multi-attribute group decision-making problems in Pythagorean fuzzy environments, this paper first combines Pythagorean fuzzy numbers with power average operators to creatively develop a novel improved weighted support measure. Based on this, we propose a Pythagorean fuzzy interaction power average operator with improved weighted support and discuss its properties, thereby establishing a decision-making method that can reflect interactions among decision attributes under Pythagorean fuzzy backgrounds. Finally, we apply the method to smart city evaluation. The case analysis demonstrates that the proposed method can solve practical multi-attribute group decision-making problems and can be further applied to intelligent logistics, pattern recognition, artificial intelligence, and other fields.

Keywords: Pythagorean fuzzy number; power average operator; weighted support; multi-attribute group decision-making

0 Introduction

Due to the uncertainty of the objective world, fuzzy set theory proposed by Zadeh [**?**] in 1965 has been extensively studied and widely applied in decision analysis, pattern recognition, optimal control, artificial intelligence, and many other fields. With social development and deepening research, scholars have paid increasing attention to more advanced fuzzy sets that can profoundly characterize the nature of uncertainty, making it a research hotspot that has yielded fruitful results, such as interval-valued fuzzy sets, type-2 fuzzy sets, interval type-2 fuzzy sets, and fuzzy multisets proposed on the basis of classical fuzzy sets. Given the widespread hesitation uncertainty in real life, Bulgarian scholar Atanassov [**?**] proposed intuitionistic fuzzy sets in 1986 based on fuzzy sets, which consist of membership and non-membership degrees to accurately depict the hesitation between affirmation and negation in human judgment. Since then, domestic and international scholars have conducted extensive and in-depth explorations of intuitionistic fuzzy sets, achieving numerous excellent results and gradually perfecting intuitionistic fuzzy set theory.

However, with social progress and development, decision-making problems in real life have become increasingly complex. In intuitionistic fuzzy environments, decision-making requires that the sum of membership and non-membership degrees of expert evaluations be less than 1, but this condition is often not fully satisfied in practice. To break through this limitation, Yager et al. [**?**, **?**] extended intuitionistic fuzzy sets and proposed Pythagorean fuzzy sets, which satisfy the condition that the sum of membership and non-membership degrees may exceed 1 but their squared sum does not exceed 1. Compared with intuitionistic fuzzy sets, Pythagorean fuzzy sets are more flexible and flexible, enabling more delicate and comprehensive characterization of uncertainty, and have become a hot topic in fuzzy set theory research both domestically and internationally \cite{5~15}. In multi-attribute decision-making problems, aggregation operators form the foundation of many decision-making methods, making research on aggregation operators particularly important in Pythagorean fuzzy environments. Garg [?] proposed probabilistic Pythagorean fuzzy weighted average and probabilistic Pythagorean fuzzy geometric average operators and applied them to optimal product strategy selection. Based on TOPSIS methodology, Zhang et al. [?] proposed a new decision-making model in Pythagorean fuzzy environments and introduced score functions, operational rules, and distance measures. Peng et al. [?] proposed comparison methods for Pythagorean fuzzy numbers and established a new group decision-making model using Pythagorean fuzzy aggregation operators. Wei et al. [?] studied Hamacher aggregation operators and multi-attribute decision-making problems with dual hesitant Pythagorean fuzzy information, proposing dual hesitant Pythagorean fuzzy aggregation operators based on Hamacher operators for integrating dual hesitant Pythagorean fuzzy information, and applied them to supplier selection examples to verify their practicality and effectiveness.

The above studies, whether on aggregation operators or decision-making methods, mostly assume that variables or attributes are independent of each other. However, this is often not the case in reality. Therefore, Zeng et al. [?] proposed a Pythagorean fuzzy ordered weighted averaging weighted average distance operator that considers the importance of each attribute in aggregated information, and based on this operator, introduced a hybrid TOPSIS method called PFOWAWAD-TOPSIS for Pythagorean fuzzy MCDM problems. Zhang et al. [?] considered interrelationships among parameters and proposed generalized Pythagorean fuzzy Bonferroni mean and generalized Pythagorean fuzzy Bonferroni geometric mean operators.

Based on the above literature review, existing research has not yet considered the situation where information among attributes or experts is interrelated (mutually supportive), which is extremely common in real life and urgently needs theoretical support. In the field of information fusion and computational science, Yager addressed similar problems by defining a support concept that can objectively characterize correlations among variables through similarity between individual variables and others, avoiding the subjectivity and arbitrariness of manually depicting variable correlations, and subsequently proposed a power average aggregation operator that can consider interrelationships among data information [?]. This technique has attracted considerable attention from scholars. Liu et al. [?] proposed a series of aggregation operators including neutrosophic number weighted power average, neutrosophic number weighted geometric power average, and generalized neutrosophic number weighted power average in neutrosophic information contexts, and applied them to multi-attribute group decision-making problems based on derivations of the operators'properties. Song et al. [?] combined interval numbers with power average operators and proposed a new interval number power average aggregation operator for decision analysis. In hesitant fuzzy environments, Liu et al. [?] proposed a generalized power hesitant fuzzy ordered weighted average operator based on power average operators and generalized means to handle multi-attribute decision-making problems under hesitant fuzzy environments. Gao et al. [?] proposed new cloud algorithms such as cloud possibility and cloud support degree, which can be used to compare clouds and determine weights, and combined cloud support degree with power average aggregation operators to propose a new linguistic aggregation operator based on cloud models called cloud generalized power ordered weighted average operator. He Xia et al. [?] defined Pythagorean fuzzy power average operators, Pythagorean fuzzy power ordered weighted average operators, Pythagorean fuzzy power geometric operators, and Pythagorean fuzzy power ordered weighted geometric operators, and studied their properties respectively.

The above studies fully demonstrate the powerful capability of power average operators in characterizing variable correlations, but they also have certain limitations: the importance weights reflecting variables themselves are not embodied in power average operators. To reflect the importance of variables themselves, the concept of support degree can be introduced to objectively reflect the credibility of each evaluation data through support relationships among data. However, existing support degrees have certain defects. Moreover, the above Pythagorean fuzzy number operations do not consider that there are also certain associations and mutual influences between membership and non-membership degrees of different Pythagorean fuzzy numbers. Therefore, this paper adopts Pythagorean cross-operation rules [?].

Based on the above literature analysis and summary, inspired by the idea of power average operators and under the condition that existing support degrees have defects, this paper proposes a new improved weighted support degree. In Pythagorean fuzzy environments, considering the interrelationships among Pythagorean fuzzy numbers, we propose a Pythagorean fuzzy cross power average operator with improved weighted support, establish its decision-making model, and finally demonstrate the effectiveness and feasibility of the operator and model through case analysis.

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1 Preliminary Knowledge

This chapter briefly summarizes the basic concepts of intuitionistic fuzzy sets, Pythagorean fuzzy sets, and power average operators.

Definition 1 [?] Let X be a universe of discourse. An intuitionistic fuzzy set on X is defined as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ represent the membership and non-membership degrees of element x belonging to A, respectively, and satisfy $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Definition 2 [**?**, **?**] Let X be a universe of discourse. A Pythagorean fuzzy set on X is defined as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ represent the membership and non-membership degrees of element x belonging to A, respectively, and satisfy $\mu_A^2(x)+\nu_A^2(x) \leq 1$ for all $x \in X$. The hesitation degree of x belonging to A is $\pi_A(x) = \sqrt{1 - \mu_A^2(x) - \nu_A^2(x)}$. A Pythagorean fuzzy number (PFN) is denoted as $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$, and the set of all PFNs is called Pythagorean fuzzy sets (PFS).

Considering that there exist certain associations and mutual influences between membership and non-membership degrees of different PFNs, Liu Weifeng et al. [**?**] defined cross-impact addition, scalar multiplication, multiplication, and power operations for PFNs, and proposed the Pythagorean fuzzy interaction weighted averaging operator.

Definition 3 [**?**] Let $\alpha_1 = \langle \mu_{\alpha_1}, \nu_{\alpha_1} \rangle$ and $\alpha_2 = \langle \mu_{\alpha_2}, \nu_{\alpha_2} \rangle$ be two PFNs. Then:

$$\alpha_1 \oplus \alpha_2 = \left\langle \sqrt{\mu_{\alpha_1}^2 + \mu_{\alpha_2}^2 - \mu_{\alpha_1}^2\mu_{\alpha_2}^2 - \nu_{\alpha_1}^2\nu_{\alpha_2}^2}, \sqrt{\nu_{\alpha_1}^2 + \nu_{\alpha_2}^2 - \mu_{\alpha_1}^2\nu_{\alpha_2}^2 - \nu_{\alpha_1}^2\mu_{\alpha_2}^2} \right\rangle$$

$$\alpha_1 \otimes \alpha_2 = \left\langle \sqrt{\mu_{\alpha_1}^2\mu_{\alpha_2}^2 + \mu_{\alpha_1}^2\nu_{\alpha_2}^2 - \mu_{\alpha_1}^2\mu_{\alpha_2}^2\nu_{\alpha_2}^2 - \mu_{\alpha_1}^2\mu_{\alpha_2}^2\nu_{\alpha_1}^2}, \sqrt{\nu_{\alpha_1}^2\nu_{\alpha_2}^2 + \mu_{\alpha_1}^2\nu_{\alpha_2}^2 - \mu_{\alpha_1}^2\nu_{\alpha_1}^2\nu_{\alpha_2}^2 - \mu_{\alpha_2}^2\nu_{\alpha_1}^2\nu_{\alpha_2}^2} \right\rangle$$

$$\lambda\alpha = \left\langle \sqrt{1 - (1 - \mu_\alpha^2)^\lambda}, \sqrt{1 - (1 - \nu_\alpha^2)^\lambda} \right\rangle, \lambda > 0$$

$$\alpha^\lambda = \left\langle \sqrt{1 - (1 - \mu_\alpha^2)^\lambda}, \sqrt{1 - (1 - \nu_\alpha^2)^\lambda} \right\rangle, \lambda > 0$$

Definition 4 [**?**] Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ $(i = 1, 2, \ldots, n)$ be a set of PFNs, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be a weight vector satisfying $\omega_i \geq 0$ and $\sum_{i=1}^n \omega_i = 1$. Then the Pythagorean fuzzy interaction weighted averaging (PFIWA) operator is defined as:

$$\text{PFIWA}_\omega(\alpha_1, \alpha_2, \ldots, \alpha_n) = \bigoplus_{i=1}^n \omega_i\alpha_i = \left\langle \sqrt{1 - \prod_{i=1}^n (1 - \mu_{\alpha_i}^2)^{\omega_i}}, \sqrt{1 - \prod_{i=1}^n (1 - \nu_{\alpha_i}^2)^{\omega_i}} \right\rangle$$

To effectively compare two PFNs, literature [**?**] provides score and accuracy functions for Pythagorean fuzzy numbers.

Definition 5 [**?**] Let $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$ be a PFN. Its score function is defined as:

$$S(\alpha) = \mu_\alpha^2 - \nu_\alpha^2, \quad S(\alpha) \in [-1, 1]$$

Definition 6 [?] Let $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$ be a PFN. Its accuracy function is defined as:

$$h(\alpha) = \mu_\alpha^2 + \nu_\alpha^2, \quad h(\alpha) \in [0,1]$$

Based on the score and accuracy functions, multiple PFNs can be compared.

Definition 7 [?] Let $\alpha_1 = \langle \mu_{\alpha_1}, \nu_{\alpha_1} \rangle$ and $\alpha_2 = \langle \mu_{\alpha_2}, \nu_{\alpha_2} \rangle$ be two PFNs. Then: - If $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 > \alpha_2$ - If $S(\alpha_1) = S(\alpha_2)$ and $h(\alpha_1) > h(\alpha_2)$, then $\alpha_1 > \alpha_2$ - If $S(\alpha_1) = S(\alpha_2)$ and $h(\alpha_1) = h(\alpha_2)$, then $\alpha_1 \sim \alpha_2$

Definition 8 [?] Let $\alpha_1 = \langle \mu_{\alpha_1}, \nu_{\alpha_1} \rangle$ and $\alpha_2 = \langle \mu_{\alpha_2}, \nu_{\alpha_2} \rangle$ be two PFNs. The distance between them is defined as:

$$d(\alpha_1, \alpha_2) = \frac{1}{2} \left(|\mu_{\alpha_1}^2 - \mu_{\alpha_2}^2| + |\nu_{\alpha_1}^2 - \nu_{\alpha_2}^2| + |\pi_{\alpha_1}^2 - \pi_{\alpha_2}^2| \right)$$

To objectively and effectively characterize correlations among multiple variables, Yager [?] proposed a power average (PA) aggregation operator based on similarity among variables through a support degree concept.

Definition 9 [?] The power average operator is a mapping $PA : \mathbb{R}^n \to \mathbb{R}$ defined as:

$$PA(a_1, a_2, \dots, a_n) = \frac{\sum_{i=1}^n (1 + T(a_i)) a_i}{\sum_{i=1}^n (1 + T(a_i))}$$

where $T(a_i) = \sum_{j=1, j \neq i}^n \text{Sup}(a_i, a_j)$, and $\text{Sup}(a_i, a_j)$ represents the support degree between a_i and a_j, satisfying the following properties: - $\text{Sup}(a_i, a_j) \in [0,1]$ - $\text{Sup}(a_i, a_j) = \text{Sup}(a_j, a_i)$ - If $|a_i - a_j| < |a_p - a_q|$, then $\text{Sup}(a_i, a_j) \geq \text{Sup}(a_p, a_q)$

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2 An Improved Weighted Support

The objective weighting method based on support degree considers correlations among variables (attributes), but the importance of variables (attributes) themselves cannot be ignored. Yager addressed this issue by providing a weighted form of relative support degree [?]:

$$V_i = \frac{\omega_i (1 + T(a_i))}{\sum_{i=1}^n \omega_i (1 + T(a_i))}$$

where ω_i is the weight of a_i.

Although Equation (8) appears directly applicable to this research, careful examination reveals certain defects in this weighted form. Below we analyze three special cases:

a) When all $\mathrm{Sup}(a_i, a_j) = c$ $(0 \le c \le 1)$, the original expression can be simplified to:

$$V_i = \frac{\omega_i(1+(n-1)c)}{\sum_{i=1}^{n} \omega_i(1+(n-1)c)} = \frac{\omega_i}{\sum_{i=1}^{n} \omega_i} = \omega_i$$

This result shows that when $\mathrm{Sup}(a_i, a_j) = c$, the mutual support (relationship) is identical, or correlation differences are not considered, and the weight plays a dual role, which seems contrary to the original importance weight.

b) When $\omega_i = 1/n$, the original expression can be simplified to:

$$V_i = \frac{\frac{1}{n}(1+\sum_{j\neq i} \mathrm{Sup}(a_i, a_j))}{\sum_{i=1}^{n} \frac{1}{n}(1+\sum_{j\neq i} \mathrm{Sup}(a_i, a_j))} = \frac{1+\sum_{j\neq i} \mathrm{Sup}(a_i, a_j)}{\sum_{i=1}^{n}(1+\sum_{j\neq i} \mathrm{Sup}(a_i, a_j))}$$

This result shows that when $\omega_i = 1/n$, it can be treated as each component having equal weight, or weight differences being ignored, but this expression seems inconsistent with the original support degree.

c) The original $\mathrm{Sup}(a_i, a_j)$ appears complex but is essentially a similarity index. If $\mathrm{Sup}(a_i, a_j) = c$ $(0 \le c \le 1)$, then $T(a_i) = (n-1)c$, and the relative support degree becomes $V_i = \frac{\omega_i(1+(n-1)c)}{\sum_{i=1}^{n} \omega_i(1+(n-1)c)}$, which satisfies $\sum_{i=1}^{n} V_i = 1$. Equation (6) can be simplified to:

$$PA(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} V_i a_i$$

Based on the above three analyses, this paper proposes a natural and reasonable improved weighted support degree as:

$$V_i = \frac{\omega_i + T(a_i)}{\sum_{i=1}^{n}(\omega_i + T(a_i))}$$

Considering the above three special cases, in contrast to Yager' s proposed weighted support degree, the improved weighted support degree proposed in this paper is more consistent and reasonable. Below, based on the improved weighted support degree and combined with cross-operation rules, we provide the definition and properties of the Pythagorean fuzzy cross power average operator with improved weighted support.

3 Pythagorean Fuzzy Cross Power Average Operator Based on Improved Weighted Support

Definition 10 Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ $(i = 1, 2, \dots, n)$ be a set of PFNs, and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be a weight vector satisfying $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$. The Pythagorean fuzzy cross power average operator with improved weighted support (W-PFIPA) is defined as:

$$\text{W-PFIPA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \bigoplus_{i=1}^{n} \frac{\omega_i + T(\alpha_i)}{\sum_{j=1}^{n}(\omega_j + T(\alpha_j))} \alpha_i$$

where $T(\alpha_i) = \sum_{j=1, j\neq i}^{n} \text{Sup}(\alpha_i, \alpha_j)$, and $\text{Sup}(\alpha_i, \alpha_j)$ is the support degree between α_i and α_j, satisfying: - $\text{Sup}(\alpha_i, \alpha_j) \in [0, 1]$ - $\text{Sup}(\alpha_i, \alpha_j) = \text{Sup}(\alpha_j, \alpha_i)$ - If $d(\alpha_i, \alpha_j) < d(\alpha_p, \alpha_q)$, then $\text{Sup}(\alpha_i, \alpha_j) \geq \text{Sup}(\alpha_p, \alpha_q)$

where $d(\alpha_i, \alpha_j)$ is the distance between α_i and α_j.

Theorem 1 Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ $(i = 1, 2, \dots, n)$ be a set of PFNs. Then the W-PFIPA operator can be expressed as:

$$\text{W-PFIPA}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left\langle \sqrt{1 - \prod_{i=1}^{n} (1 - \mu_{\alpha_i}^2)^{\frac{\omega_i + T(\alpha_i)}{\sum_{j=1}^{n}(\omega_j + T(\alpha_j))}}}, \sqrt{1 - \prod_{i=1}^{n} (1 - \nu_{\alpha_i}^2)^{\frac{\omega_i + T(\alpha_i)}{\sum_{j=1}^{n}(\omega_j + T(\alpha_j))}}} \right\rangle$$

Proof. First, we prove by mathematical induction:

When $n = 1$, the result obviously holds.

When $n = 2$, the equation holds.

Assume the equation holds when $n = k$, then when $n = k + 1$, the equation also holds. Therefore, for any n, the equation must hold.

Second, using Definition 3, we have:

$$\bigoplus_{i=1}^{n} \frac{\omega_i + T(\alpha_i)}{\sum_{j=1}^{n}(\omega_j + T(\alpha_j))} \alpha_i = \left\langle \sqrt{1 - \prod_{i=1}^{n} (1 - \mu_{\alpha_i}^2)^{\frac{\omega_i + T(\alpha_i)}{\sum_{j=1}^{n}(\omega_j + T(\alpha_j))}}}, \sqrt{1 - \prod_{i=1}^{n} (1 - \nu_{\alpha_i}^2)^{\frac{\omega_i + T(\alpha_i)}{\sum_{j=1}^{n}(\omega_j + T(\alpha_j))}}} \right\rangle$$

Thus, Theorem 1 is proved.

Next, we discuss the properties of the W-PFIPA operator and provide proofs.

Theorem 2 Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ $(i = 1, 2, \dots, n)$ be a set of PFNs. The W-PFIPA operator has the following properties:

(a) **Idempotency.** If $\alpha_i = \alpha$ for all $i = 1, 2, \dots, n$, then:

$$\text{W-PFIPA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha$$

(b) **Boundedness.** Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ $(i = 1, 2, \ldots, n)$ be a set of PFNs. Then:

$$\alpha_{\min} \leq \text{W-PFIPA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha_{\max}$$

where $\alpha_{\min} = \langle \min_i \mu_{\alpha_i}, \max_i \nu_{\alpha_i} \rangle$ and $\alpha_{\max} = \langle \max_i \mu_{\alpha_i}, \min_i \nu_{\alpha_i} \rangle$.

(c) **Commutativity.** Let $\alpha_i = \langle \mu_{\alpha_i}, \nu_{\alpha_i} \rangle$ $(i = 1, 2, \ldots, n)$ be a set of PFNs, and $(\alpha'_1, \alpha'_2, \ldots, \alpha'_n)$ be any permutation of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then:

$$\text{W-PFIPA}(\alpha'_1, \alpha'_2, \ldots, \alpha'_n) = \text{W-PFIPA}(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

Proof. (Detailed proof steps would follow here, showing the mathematical derivations for each property.)

Below, we discuss two special cases of the W-PFIPA operator:

(a) When all $\text{Sup}(\alpha_i, \alpha_j) = c$ $(0 \leq c \leq 1)$, Equation (10) can be equivalently simplified to:

$$\text{W-PFIPA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \bigoplus_{i=1}^{n} \frac{\omega_i + c\sum_{j\neq i} 1}{\sum_{j=1}^{n}(\omega_j + c\sum_{i\neq j} 1)} \alpha_i = \bigoplus_{i=1}^{n} \frac{\omega_i + (n-1)c}{\sum_{j=1}^{n} \omega_j + n(n-1)c} \alpha_i$$

When $c = 1$, the W-PFIPA operator degenerates into the Pythagorean fuzzy cross power average operator, i.e.:

$$\text{W-PFIPA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \bigoplus_{i=1}^{n} \frac{1}{n} \alpha_i = \text{PFIPA}(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

When $c = 0$, the W-PFIPA operator becomes:

$$\text{W-PFIPA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \bigoplus_{i=1}^{n} \omega_i \alpha_i$$

4 A Correlation-Based Decision Model

In this chapter, we establish a decision-making model using the W-PFIPA operator. Let the set of alternatives be $A = \{A_1, A_2, \ldots, A_m\}$, the set of decision attributes be $C = \{C_1, C_2, \ldots, C_n\}$, the expert group be $D = \{d_1, d_2, \ldots, d_k\}$, the attribute weight set be $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$, and the expert weight set be $r = (r_1, r_2, \dots, r_k)^T$ with $r_s \in [0,1]$ and $\sum_{s=1}^{k} r_s = 1$. The evaluation value given by decision-maker d_s for alternative A_i regarding attribute C_j is a PFN denoted as $\alpha_{ij}^{(s)} = \langle \mu_{ij}^{(s)}, \nu_{ij}^{(s)} \rangle$, thus obtaining the Pythagorean fuzzy decision matrix $Q^{(s)} = (\alpha_{ij}^{(s)})_{m\times n}$. Based on this information, we can rank or select the m alternatives.

The specific steps of the decision-making method based on the W-PFIPA operator are as follows:

a) Calculate the support degree matrix of the expert group. The distance between two PFNs reflects the difference between them:

$$d(\alpha_{ij}^{(s)}, \alpha_{ij}^{(t)}) = \frac{1}{2}\left(|\mu_{ij}^{(s)2} - \mu_{ij}^{(t)2}| + |\nu_{ij}^{(s)2} - \nu_{ij}^{(t)2}| + |\pi_{ij}^{(s)2} - \pi_{ij}^{(t)2}|\right)$$

b) Calculate the associated weights. The support degree of expert d_s for alternative A_i regarding attribute C_j is:

$$\text{Sup}(\alpha_{ij}^{(s)}) = \sum_{t=1, t\neq s}^{k} \text{Sup}(\alpha_{ij}^{(s)}, \alpha_{ij}^{(t)})$$

The associated weight can weaken the weight of expert evaluation information that differs significantly from group opinions and deviates far from the group, without requiring subjective weight assignment.

c) Use the W-PFIPA operator to aggregate the group decision matrix and obtain the comprehensive group decision matrix $Q = (\alpha_{ij})_{m\times n}$:

$$\alpha_{ij} = \text{W-PFIPA}(\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \dots, \alpha_{ij}^{(k)}) = \bigoplus_{s=1}^{k} \frac{r_s + \text{Sup}(\alpha_{ij}^{(s)})}{\sum_{t=1}^{k}(r_t + \text{Sup}(\alpha_{ij}^{(t)}))} \alpha_{ij}^{(s)}$$

d) Use the PFIWA operator [**?**] to calculate comprehensive evaluation values:

$$\alpha_i = \text{PFIWA}_{\omega}(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in}) = \bigoplus_{j=1}^{n} \omega_j \alpha_{ij}$$

e) Use the score function in Definition 5 to calculate the comprehensive evaluation values of each alternative.

f) Use Definition 7 to rank the comprehensive attribute values of alternatives from largest to smallest to obtain the optimal alternative.

g) End.

5 Case Study

In recent years, as China' s urbanization process has entered a period of rapid development, a series of "urban diseases" such as environmental pollution, resource shortage, and public safety have emerged and become increasingly serious, hindering sustainable urban development. Smart cities are an excellent remedy for "urban diseases," and the concept of smart cities has become a focus and hotspot of research in various fields \cite{18~21}. Smart cities use information and communication technologies to sense, analyze, and integrate key information from core urban operation systems, thereby making intelligent responses to various needs including people' s livelihood, environmental protection, public safety, urban services, and industrial and commercial activities. Their essence is to use advanced information technology to achieve intelligent urban management and operation, thereby creating a better life for urban residents and promoting harmonious and sustainable urban growth. In the information age, IoT, big data, cloud computing, and other information technologies are widely applied in all aspects of urban construction. The application of advanced information technology promotes information sharing in fields such as urban livelihood, public safety, and industrial and commercial activities. Research on smart city evaluation is beneficial for urban development and improving residents' quality of life. On the other hand, a comprehensive and systematic smart city evaluation also provides new path choices for urban development and enhances the country' s overall economic strength.

In this study, we assume that after preliminary screening by an expert group, four cities A_1, A_2, A_3, A_4 remain for evaluation and analysis. Due to space limitations, detailed information about the four cities is not described here. Based on existing literature reviews and after several rounds of discussion, five decision attributes are selected for this evaluation: C_1: infrastructure construction, C_2: human capital investment, C_3: natural and socio-economic environment, C_4: information and communication technology, and C_5: government service level. The attribute weights are $\omega = (0.30, 0.10, 0.25, 0.20, 0.15)^T$.

To ensure the scientificity and objectivity of this smart city evaluation, three experts $D = \{d_1, d_2, d_3\}$ from research institutions, government departments, and non-profit organizations were invited to participate. The expert group weight is $r = (0.35, 0.45, 0.20)^T$. At the beginning of the evaluation, due to time pressure, limited reference materials, and limited personal experience with relevant issues, the expert group expressed willingness to use PFNs to represent judgments on relevant issues, which helps reflect their uncertainty about evaluation problems. The evaluation data from each expert are shown in Tables 1 to 4 .

Table 1 Pythagorean fuzzy matrix given by expert 1
Table 2 Pythagorean fuzzy matrix given by expert 2
Table 3 Pythagorean fuzzy matrix given by expert 3

The detailed calculation steps are as follows:

a) Calculate the weighted support degree matrix of the expert group.

b) Calculate the associated weights.

c) Use the W-PFIPA operator to aggregate the group decision matrix and obtain the expert group decision matrix shown in Table 4 .

d) Use the PFIWA operator to calculate comprehensive evaluation values:

e) Use Definition 5 to calculate the score function for each alternative:

f) Based on the score function values, the ranking of alternatives is:

Table 5 Score value, sorting result, and optimal plan

6.1 Comparative Analysis of Experimental Results

To verify the effectiveness of the proposed operator and decision-making method, we selected methods from literature [?], [?], [?], [?], and [?] to calculate comprehensive evaluation values and ranking results. The score values, ranking results, and optimal alternatives are shown in Table 5.

Through comparative analysis, the optimal alternative in this paper is the same as those obtained by the PFWA operator in literature [?], the TOPSIS method in literature [?], the PFPA operator in literature [?], the PFIWA operator in literature [?], and the PFPWA operator in literature [?], all being A_1. However, the score values and ranking results obtained by each method are slightly different. The main reasons for these differences may be:

a) The above models adopt different aggregation methods, and the focus of their aggregation operators varies. Literature [?] and literature [?] mainly study cases where attributes or experts are independent, without considering the existence of mutual support among attributes or experts, hence the different ranking results.

b) The ranking results for alternatives 2, 3, and 4 in literature [?], literature [?], and literature [?] slightly deviate from this paper' s results because the PFWA operator used in literature [?] is a subjective weighting method that fails to reflect differences between data and experts. Literature [?] obtains ranking results by calculating the closeness of each alternative to positive and negative ideal solutions. Although literature [?] uses power average operators and support degree to obtain weights, the applied support degree has flaws as analyzed above. Moreover, this method does not use cross-operation rules, thus failing to reflect differences in data information, resulting in different ranking outcomes.

c) The different ranking results in literature [?] are due to defects in the weighted support degree used (which have been improved in this paper), leading to relatively large deviations in score values.

The multi-attribute decision-making method based on the proposed Pythagorean fuzzy cross power average operator with improved weighted support fully considers the mutual support degree among information and can objectively assign expert weights. Moreover, the operator aggregation calculation is simple and easy to operate. Based on the above analysis, the conclusions drawn in this paper are reasonable and effective.

In the above case analysis, the proposed Pythagorean fuzzy power average integration operator with improved weighted support has been well applied. As an extension of intuitionistic fuzzy sets, Pythagorean fuzzy sets can more fully characterize real situations. The combination of improved weighted support and power average operators well considers the mutual support degree among attribute information, making the decision-making environment more stable while enriching Pythagorean fuzzy aggregation operator theory.

6.2 Model Comparative Analysis

To further verify the rationality of the proposed model, we selected models from literature [?], literature [?], literature [?], and literature [?] for comparative analysis. The results are shown in Table 6 .

Table 6 Model comparison

Model	Considers Support Degree	Applies Cross-Operation
This paper: W-PFIPA	Yes	Yes
Literature [?]	No	No
Literature [?]	Yes	No
Literature [?]: PFIWA	No	Yes
Literature [?]: PFPWA	Yes	No

As shown in Table 6, this paper considers the support degree among data in the information aggregation process, meaning it can weaken the weight of expert evaluation information that differs significantly from and deviates far from group opinions. Although literature [?] and literature [?] also consider support degree, their support degrees have defects that lead to result deviations. Moreover, this paper uses more reasonable cross-operation rules, making the results more appropriate.

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7 Conclusion

In multi-attribute group decision-making processes, due to the complexity of real-world problems, attribute values are suitable for representation by

Pythagorean fuzzy numbers. Meanwhile, to reflect the importance of variables themselves, we improved the concept of original weighted support degree and combined it with power average operators to propose a Pythagorean fuzzy power average operator with improved weighted support. We then established a decision-making method that can reflect interactions among decision attributes under Pythagorean fuzzy backgrounds, fully considering the information support degree among experts and effectively avoiding biases caused by subjective weighting.

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Note: Figure translations are in progress. See original paper for figures.