# Reconstruction of the Faraday Dispersion Function Using Compressed Sensing: A Postprint

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## Abstract

The observed polarization quantities and the Faraday dispersion function constitute a Fourier transform pair, with the Faraday dispersion function reflecting the magnetic field structure of both the emission region and the propagation path of radiation. Accurately reconstructing the Faraday dispersion function through this relationship is of great importance for studying magnetic fields in the Milky Way and extragalactic galaxies. A reconstruction method for the Faraday dispersion function based on compressed sensing has been proposed, whose simulation results outperform traditional methods; however, its practical utility remains uncertain. This work primarily investigates whether this method remains feasible when applied to actual observational frequency ranges, and conducts large-sample statistical experiments. The results indicate that the reconstruction outcomes are influenced by various factors and exhibit considerable randomness. After performing a secondary least-squares fitting on the reconstruction results near their peaks, the reconstructed Faraday depth becomes closer to the true value.

## Full Text

## Reconstruction of Faraday Dispersion Function Based on Compressive Sensing Method

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### Abstract

The observed polarization quantities and the Faraday dispersion function form a Fourier transform pair, and the Faraday dispersion function reflects the magnetic field structure of the emission region and the propagation path. Accurately reconstructing the Faraday dispersion function through this relationship is crucial for studying the magnetic fields of the Milky Way and extragalactic galaxies. A compressive sensing-based method for reconstructing the Faraday dispersion function has been proposed, with simulation results showing superiority over traditional methods, but its practical utility remains unknown. This paper investigates whether the method remains feasible when applied to actual observational frequency ranges and conducts large-sample statistical experiments. The results show that the reconstruction is affected by various factors and exhibits significant randomness. After performing a least-squares fit near the peaks of the reconstructed results, the recovered Faraday depth becomes closer to the true value.

**Keywords:** Compressive sensing; Polarization; Faraday dispersion function; Faraday depth

Faraday rotation is a physical phenomenon where the polarization angle of linearly polarized electromagnetic waves rotates as they propagate through magnetized interstellar medium. The rotation of the polarization position angle can be expressed as $\phi(r)\lambda^2$, where $\lambda$ is the observed wavelength and $\phi(r)$ is the Faraday depth at the emission location $r$, given by:

$$\phi(r) = \int n_e B_\parallel \, dl \tag{1}$$

where $n_e$ is the thermal electron density, $B_\parallel$ is the line-of-sight magnetic field component, and $dl$ is the line-of-sight path element. The observed polarization quantity $P$ can then be written as [**?**]:

$$P(\lambda^2) = Q(\lambda^2) + iU(\lambda^2) = \int F(\phi) \exp(2i\phi\lambda^2) \, d\phi \tag{2}$$

where $F(\phi)$ is the Faraday dispersion function (FDF), a function of Faraday depth, and $Q$ and $U$ are Stokes parameters that depend on the observed wavelength (or frequency).

From equation (3), we see that the polarization quantity $P$ and the Faraday dispersion function $F(\phi)$ form a Fourier transform pair. $F(\phi)$ represents the polarization intensity and polarization position angle at Faraday depth $\phi$, reflecting the magnetic field structure in the emission region and along the propagation path. To study the magnetic fields in interstellar medium, we need to reconstruct the Faraday dispersion function.

Several methods have been proposed to reconstruct the Faraday dispersion function. Brentjens & de Bruyn introduced Faraday rotation measure synthesis (RMS), which reconstructs the FDF through the Fourier relationship between observed polarization quantities and the Faraday dispersion function. Due to limited observational bandwidths, the FDF obtained by RMS is a convolution of the true dispersion function with a window function, requiring deconvolution for complex sources. Wavelet methods similar to RMS have also been developed [?, ?]. Additionally, some methods assume specific emission models to reconstruct the FDF, such as the Q-U fitting method [?].

This paper focuses on the compressive sensing method (also called compressed sampling, see principles in [?, ?]). The advantage of this method is that it can potentially reconstruct the complete original signal using extremely sparse sampling data. Compressive sensing theory is based on the sparsity of signals and achieves accurate reconstruction through coded measurements and reconstruction algorithms. Several compressive sensing-based algorithms for reconstructing the Faraday dispersion function have been proposed. For example, Li et al. established algorithms using the minimum $\ell_1$ norm method for Faraday-thin sources ($\lambda^2\Delta\phi \ll 1$, where $\Delta\phi$ is the extent of the source in Faraday depth domain, i.e., the width of $F(\phi)$), Faraday-thick sources ($\lambda^2\Delta\phi \gg 1$), and sources containing both Faraday-thin and Faraday-thick components. Their simulations showed that this method can reconstruct the Faraday dispersion function and, when compared with RMS, the compressive sensing method produced better results in terms of both numerical values and errors [?]. Andrecut et al. employed a matching pursuit algorithm to develop a reconstruction method for sources containing both Faraday-thin and Faraday-thick components [?]. However, these methods remain in the theoretical simulation stage, and their practical utility has not been thoroughly investigated. For instance, Li et al. used an excessively large frequency (wavelength) range, did not add noise to their experimental data, and did not examine or discuss the impact of various parameter settings in the algorithm on reconstruction results, nor did they use a sufficiently large sample size. In this paper, we build upon Li et al.’ s method [?] and conduct numerical simulations from a practical observational perspective to investigate these issues more deeply and explore the method’ s practicality. We assume the Faraday dispersion function is sparse (i.e., the source contains one or multiple Faraday-thin components) and have written a program in Python to reconstruct the Faraday dispersion function. We adopt a frequency range of 1.1–3.1 GHz, corresponding to the L-band observation range of the Australia Telescope Compact Array (ATCA) [?].

## 1 Reconstruction Method

Our problem involves reconstructing an unknown signal $\mathbf{x}_0$ from its linear projection $\mathbf{y} \in \mathbb{R}^M$ through a measurement matrix $\mathbf{A} \in \mathbb{R}^{M\times N}$ ($M \ll N$), i.e., $\mathbf{y} = \mathbf{A}\mathbf{x}_0$. In compressive sensing theory, assuming the unknown signal $\mathbf{x}_0$ is sparse, it can be accurately reconstructed from measurements $\mathbf{y}$ by solving the minimum $\ell_1$ norm problem:

$$\mathbf{x} = \min \|\mathbf{x}_0\|_{\ell_1}$$

If the unknown signal $\mathbf{x}_0$ is not sparse, we can obtain a sparse representation through some transform (such as wavelet transform), i.e., $\mathbf{x}_0 = \omega\mathbf{X}$, where $\mathbf{X}$ is the sparse representation of the signal in the $\omega$ transform domain. Then we have:

$$\text{s.t. } \mathbf{A}\mathbf{x}_0 = \mathbf{A}\omega\mathbf{X} = \mathbf{y} \tag{7}$$

$$\mathbf{x} = \min \|\mathbf{X}\|_{\ell_1}$$

We rewrite equation (2) in matrix form:

$$\mathbf{Y}\mathbf{f} = \mathbf{P} \tag{8}$$

Here $\mathbf{f}$ is the Faraday dispersion function, an $N$-dimensional vector; $\mathbf{Y}$ is the discrete representation of the Fourier transform between $\mathbf{P}$ and $\mathbf{f}$, an $M \times N$ matrix; and $\mathbf{P}$ is the observed polarization quantity, an $M$-dimensional vector, where $M$ is the number of observed frequency channels, each receiving polarization in different wavebands.

Since the Faraday dispersion function $\mathbf{f}$ is a complex function, we can separate its real and imaginary parts as $\mathbf{f}.real$ and $\mathbf{f}.imag$. Equation (8) then becomes:

$$\begin{bmatrix} \mathbf{C} & -\mathbf{S} \\ \mathbf{S} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{f}.real \\ \mathbf{f}.imag \end{bmatrix} = \begin{bmatrix} \mathbf{P}.real \\ \mathbf{P}.imag \end{bmatrix}$$

where $\mathbf{C}$ and $\mathbf{S}$ are both $M \times N$ matrices with elements $C_{ij} = \cos(2\phi_j\lambda_i^2)$ and $S_{ij} = \sin(2\phi_j\lambda_i^2)$, respectively, for $i = 1, 2, \dots, M$ and $j = 1, 2, \dots, N$. The vector $\begin{bmatrix} \mathbf{f}.real \\ \mathbf{f}.imag \end{bmatrix}$ is a $2N$-dimensional vector, and $\mathbf{P} = \begin{bmatrix} \mathbf{P}.real \\ \mathbf{P}.imag \end{bmatrix}$ is a $2M$-dimensional vector.

We set the number of observation channels $M$ to 100, with a wavelength range from 0.096 m to 0.273 m (corresponding to frequencies of 1.1-3.1 GHz). To better describe our experimental setup, we adopt two definitions from Brentjens & de Bruyn: the maximum Faraday depth $|\phi_{max}|$ and the full width at half maximum (FWHM) of the rotation measure spread function (RMSF), $\Delta\phi_{FWHM}$. These two quantities are determined by the maximum wavelength squared $\lambda_{max}^2$, the minimum wavelength squared $\lambda_{min}^2$, and the interval between observation channels (i.e., the interval between squares of adjacent wavelengths) $\delta\lambda^2$:

$$|\phi_{max}| = \frac{\pi}{\delta\lambda^2} \tag{10}$$

$$\Delta\phi_{FWHM} = \frac{2\sqrt{3}}{\lambda_{max}^2 - \lambda_{min}^2} \tag{11}$$

In our experiments, we add Gaussian noise with mean zero and standard deviation $\sigma = 1$ (intensity units are arbitrary, hereafter the same) to $Q$ and $U$. We set the resolution in Faraday depth space $\delta\phi$ to be 1/4 of $\Delta\phi_{FWHM}$. The total number of grid points $N$ (the number of coordinate points $\phi_i$ in $\phi$ space, $i = 1, 2, \ldots, N$) is then:

$$N = \mathrm{int}\left(\frac{2|\phi_{max}|}{\delta\phi}\right) \tag{12}$$

where int() denotes the integer part. Table 1 lists the main experimental parameters.

**Table 1 Experimental Parameters**

- $\Delta\phi_{FWHM}$: $9.2 \times 10^{-3}$ rad m $^2$
- $\lambda^2_{max}$: $7.5 \times 10^{-2}$ m$^2$
- $\lambda^2_{min}$: $6.6 \times 10^{-4}$ m$^2$
- $|\phi_{max}|$: 2625 rad m $^2$
- $N$: 53
- $\delta\phi$: 13.7 rad m $^2$

## 2 Experimental Results

### 2.1 General Case

In our simulation experiments, we assume a Faraday dispersion function containing five Faraday-thin sources: $F(-164.1) = 3 + 6i$, $F(-82.0) = -4 + 3i$, $F(0.0) = 2 + 3i$, $F(68.4) = -8 - 5i$, and $F(136.7) = 5 + 3i$, with corresponding signal-to-noise ratios (SNRs) of 6.7, 5.0, 3.6, 9.4, and 5.8. We attempt to reconstruct this Faraday dispersion function, and the results are shown in Figure 1a [Figure 1: see original paper]. We use the weighted mean difference in $\phi$ ($d_{wm}$) to compare the reconstructed function $f$ with the original Faraday dispersion function $f$ [**?**]:

$$d_{wm} = \frac{\sum_i \phi_i p_i}{\sum_i p_i} - \frac{\sum_i \phi_i' p_i'}{\sum_i p_i'}$$

where $p$ represents the polarization intensity. The figure shows that without noise, the model accurately reconstructs the original image, while with noise, the reconstruction exhibits deviations. The $d_{wm}$ values are $5.80 \times 10^{-5}$ and 11.78 for noiseless and noisy cases, respectively.

We find that the Faraday dispersion function reconstructed through compressive sensing varies with the polarization intensity magnitude and phase of the Faraday-thin sources, as well as the separation $\Delta\phi$ between two sources in Faraday depth, sometimes failing to accurately reconstruct the original image. We test two scenarios: (1) Keeping the separation $\Delta\phi$ between the five

Faraday-thin sources unchanged while varying their magnitudes and phases, e.g., $F(-164.1) = 3 - 6i$, $F(-82.0) = -4 + 3i$, $F(0.0) = 5 + 3i$, $F(68.4) = 8 - 5i$, $F(136.7) = -5 + 4i$, with SNRs of 6.7, 5.0, 5.8, 9.4, and 6.4. The reconstruction results are shown in Figure 1b. (2) Keeping magnitudes and phases unchanged while varying the Faraday depth separation $\Delta\phi$, e.g., $F(-150.4) = 3 + 6i$, $F(-68.4) = -4 + 3i$, $F(0.0) = 2 + 3i$, $F(109.4) = -8 - 5i$, $F(164.1) = 5 + 3i$, with SNRs of 6.7, 5.0, 3.6, 9.4, and 5.8. The reconstruction results are shown in Figure 1c.

When $\Delta\phi$ is fixed, the first three sources are not well reconstructed, with errors in both phase and magnitude; the $d_{wm}$ values are 6.31 and 17.03 for noiseless and noisy cases, respectively. When magnitude and phase are fixed, the last four sources show significant deviations from the original; the $d_{wm}$ values are 8.18 and 17.11, respectively. We also find that without noise, when the separation $\Delta\phi$ between these Faraday-thin sources is sufficiently large ($\Delta\phi > 90$ rad m $^{2}$), the model can accurately reconstruct the original image regardless of changes in magnitude, phase, or $\Delta\phi$, and even with noise, the errors remain small. Of course, in all these tests, the Faraday depths of the sources were set on the grid points $N$ of this method.

In practical applications, we typically do not know the Faraday depths of the Faraday dispersion function components or the separation $\Delta\phi$ between sources beforehand. The Faraday depths of components we aim to reconstruct are random and generally not on the grid points $N$ set by this method, and $\Delta\phi$ is also random. Therefore, we cannot be certain whether the model can reconstruct the Faraday dispersion function well. Here we randomly select a dispersion function containing five Faraday-thin sources (with Faraday depths not on grid points $N$): $F(-114) = -6 - i$, $F(-60) = 7 - 4i$, $F(48) = 3 - 5i$, $F(102) = -3.5 + 3i$, $F(210) = 9 + 3i$, with corresponding SNRs of 6.1, 8.1, 5.8, 4.6, and 9.5. The test results are shown in Figure 1d, with $d_{wm}$ values of 0.40 and 10.20.

Figure 1 [Figure 1: see original paper] shows: top row (a, b), bottom row (c, d). In each panel, from top to bottom: original data, noiseless reconstruction, and noisy reconstruction. Thick solid lines show amplitude, thin solid lines show real parts, and dashed lines show imaginary parts.

In each simulation, we generate random Gaussian noise, so the reconstruction results differ each time. In our experiments, the reconstruction accuracy of the Faraday dispersion function varies with the number of sources, their magnitudes, phases, and noise level, and also depends on parameters set in the method, such as the number of observation channels $M$ and the resolution $\delta\phi$ in $\phi$ space.

## 2.2 Large Sample Experiment

To more intuitively observe the reconstruction performance of this method, we conducted large-sample statistical experiments to examine the effects of three factors on reconstruction results: noise, the separation $\Delta\phi$ between two sources in Faraday depth, and the resolution $\delta\phi$ in $\phi$ space (different $\delta\phi$ leads to different total grid points $N$). We use the number of reconstructed sources and the weighted mean difference in $\phi$ to characterize reconstruction quality [**?**].

We first examine the impact of signal-to-noise ratio (SNR) on reconstruction results. Here we assume the Faraday dispersion function contains only one Faraday-thin source, fixing its Faraday depth at $\phi = 20$ rad m ². We randomly simulate 1000 sources with SNRs ranging from 3 to 60 (here noise is still set to 1, so SNR corresponds to the polarization intensity of the Faraday dispersion function). The resolution in $\phi$ space, $\delta\phi$, remains at 1/4 of $\Delta\phi_{FWHM}$. Since only one Faraday-thin source is present, we directly use the difference in $\phi$ ($\phi_{recovered} - \phi_{original}$) to describe reconstruction performance, excluding signals with SNR less than 2. The results are shown in Figure 2a [Figure 2: see original paper].

The figure shows that for the single-source case, the number of sources can be accurately reconstructed, while the difference in $\phi$ always takes two values, meaning the reconstructed $\phi_{recovered}$ appears on adjacent grid points (here the set $\phi_{original}$ is not on a grid point). To reduce reconstruction error, we perform a least-squares fit near the peak of the reconstruction results (all subsequent experiments adopt this fitting), with results shown in Figure 2b. After fitting, the difference in $\phi$ is reduced by a factor of 2 to 4 for most sources.

Since the reconstructed Faraday depth $\phi$ always falls on adjacent grid points, we might ask whether we can reduce the reconstruction error in $\phi$ by decreasing the resolution of $\phi$ space.

Figure 2 [Figure 2: see original paper] shows: (a) without fitting, (b) with fitting. In each panel, from top to bottom: number of reconstructions, difference in $\phi$.

We next examine the effects of separation $\Delta\phi$ between two sources and resolution $\delta\phi$ in $\phi$ space on reconstruction results. Here we assume a Faraday dispersion function containing two Faraday-thin sources, fixing one source’ s polarization intensity $p = 13.235$, polarization position angle $\chi = 2.749$, and Faraday depth $\phi = 10.256$ rad m ². The second source is randomly simulated with 1000 instances having polarization intensity (equivalent to SNR) ranging from 3 to 30, polarization position angle ranging from 0 to $2\pi$, and separation $\Delta\phi$ ranging from 5 to 200. We repeat this experiment for different resolution values $\delta\phi$ (i.e., $\delta\phi$ taking 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, and 1/9 of $\Delta\phi_{FWHM}$), excluding signals with SNR less than 1. The results are shown in Figure 3 [Figure 3: see original paper].

Figure 3 [Figure 3: see original paper] shows: first row (a) $\delta\phi = \Delta\phi_{FWHM}/2$, (b) $\delta\phi = \Delta\phi_{FWHM}/3$; second row (c) $\delta\phi = \Delta\phi_{FWHM}/4$, (d) $\delta\phi = \Delta\phi_{FWHM}/5$; third row (e) $\delta\phi = \Delta\phi_{FWHM}/6$, (f) $\delta\phi = \Delta\phi_{FWHM}/7$; fourth row (g) $\delta\phi = \Delta\phi_{FWHM}/8$, (h) $\delta\phi = \Delta\phi_{FWHM}/9$. In each panel, from top to bottom: number of reconstructions, weighted mean difference $d_{wm}$.

To compare reconstruction performance under different resolutions, we examine the accuracy of correctly reconstructed source numbers for various resolutions, as shown in Figure 4 [Figure 4: see original paper]. The figure shows that when $\delta\phi$ is 1/3 or 1/4 of $\Delta\phi_{FWHM}$, the accuracy is highest, reaching approximately 83%. In the earlier experiment on noise effects, we asked whether decreasing resolution could reduce reconstruction error. This experiment shows that resolution $\delta\phi$ should not be made arbitrarily small.

Figure 4 [Figure 4: see original paper] shows: x-axis represents the ratio $\delta\phi/\Delta\phi_{FWHM}$, y-axis represents reconstruction accuracy.

Using the experiment with $\delta\phi = \Delta\phi_{FWHM}/4$ as an example, we randomly selected several reconstructed images, shown in Figure 5 [Figure 5: see original paper]. The figure demonstrates that after least-squares fitting of the reconstructed images, the Faraday depth $\phi_{fitting}$ is closer to the input model' s Faraday depth $\phi_{original}$, but the polarization intensity of fitted (or reconstructed) sources $p_{fitting}$ is consistently lower than that of the input model $p_{original}$. In Figure 4c (and 4a, 4b), when the separation between two sources $\Delta\phi > 50$, $d_{wm}$ exhibits periodic oscillations as $\Delta\phi$ increases. We believe these oscillations may be related to the position of the second source' s Faraday depth relative to the grid. To investigate this, we marked $d_{wm}$ values when the second source is near grid point $\phi_i$, $\phi_i + \frac{1}{2}\delta\phi$, and $\phi_i + \frac{1}{4}\delta\phi$ on the plot of $d_{wm}$ versus $\Delta\phi$, as shown in Figure 6 [Figure 6: see original paper]. We find these oscillations have a period of $\delta\phi$.

Figure 5 [Figure 5: see original paper] shows examples of reconstructed images when $\delta\phi = \Delta\phi_{FWHM}/4$.

Figure 6 [Figure 6: see original paper] shows $d_{wm}$ varying with separation $\Delta\phi$.

## Conclusion

Measuring the polarization of diffuse radio emission at multiple frequencies is an effective method for studying Galactic magnetic fields. Current Faraday rotation measure synthesis utilizes the Fourier transform pair relationship between polarization intensity and the Faraday dispersion function, obtaining the FDF through inverse Fourier transform of observed polarization intensity [**?**]. Due to limited observational bandwidths, the resulting solution requires deconvolution. Therefore, a more robust method is needed to address this problem.

In this paper, we apply the compressive sensing-based Faraday dispersion function reconstruction method to actual observational frequency ranges and conduct experiments on factors that may affect reconstruction performance. Experimental results show that reconstruction accuracy varies significantly with the number of Faraday-thin sources, their polarization intensities, phases, and the separation $\Delta\phi$ between two sources in Faraday depth, exhibiting considerable randomness. In large-sample statistical experiments, we find that reconstruction accuracy is highest when the resolution $\delta\phi$ in $\phi$ space is 1/3 or 1/4 of $\Delta\phi_{FWHM}$. After performing least-squares fitting near each peak of the reconstruction results, the fitted Faraday depth $\phi$ values are closer to the original values. Through refitting after compressive sensing reconstruction, we can obtain more accurate Faraday depth $\phi$ values. Therefore, this method can serve as a complement to existing methods. However, the polarization intensities reconstructed by this method are generally low, and the reconstruction performance exhibits significant randomness. Therefore, we need continuous improvements to enhance its stability and accuracy for better application to actual observational data.

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*Note: Figure translations are in progress. See original paper for figures.*