Logic and Some Problems in Its Application to the Foundations of Mathematics

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Date: 2018-09-28T00:00:00+00:00

Abstract

To fundamentally eliminate the various paradoxes existing in the foundations of mathematics and establish mathematics on a highly reliable basis, it is discovered that formal logic can only be applied within a domain of discourse (called the feasible domain) where the three fundamental laws—the law of identity, the law of contradiction, and the law of excluded middle—all hold; otherwise, various errors, including paradoxes, will arise. Within the applicable scope of formal logic, i.e., the feasible domain, paradoxes do not exist provided that the premises are reliable and the reasoning is rigorous. Based on this conclusion, we analyze the causes of formation of some historically well-known paradoxes such as the liar paradox and the barber paradox, simultaneously point out some logical errors in the application of Peano axioms in the foundations of mathematics and in the proofs of Cantor' s theorem, the nested interval method, and the diagonal argument, and propose unified suggestions for defining natural numbers, rational numbers, and irrational numbers that can avoid these errors.

Full Text

Problems of Logic and Its Application in Mathematical Foundations

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Abstract

In order to fundamentally eliminate various paradoxes existing in mathematical foundations and establish mathematics on a highly reliable basis, this paper finds that formal logic can only be applied within a discussion domain where the three fundamental laws—the law of identity, the law of non-contradiction, and the law of excluded middle—all hold true. This domain is termed the "feasible domain." When discussions stray outside this feasible domain, various errors, including paradoxes, inevitably arise. Within the feasible domain, however, paradoxes cannot exist as long as the premises are reliable and the derivations are rigorous. Based on this conclusion, the paper analyzes the formation causes of several historically famous paradoxes, including the liar paradox and the barber paradox. Simultaneously, it identifies logical errors in the application of Peano axioms and in the proofs of Cantor' s theorem, the interval nesting method, and the diagonal argument. The paper proposes a unified approach to defining natural numbers, rational numbers, and irrational numbers that can avoid these errors.

Keywords: logic; paradox; feasible domain; mathematical foundation; Peano axioms; Cantor' s theorem; interval nesting method; diagonal argument

Chinese Library Classification Number: O143

At the end of the 19th century, confronted with what seemed a perfectly complete theoretical edifice in physics, people firmly believed that only cosmetic work remained and that little groundbreaking progress lay ahead. Problems that classical physics struggled to explain, such as the wave theory of light and black-body radiation, were regarded merely as "two small clouds in a clear sky" that would eventually be blown away. Yet these very clouds ultimately shook the foundations of physics, precipitating revolutionary change.

Similarly, some clouds appear to drift above the clear sky of logic. For instance, if logic is universally applicable, why do numerous paradoxes exist that logic cannot explain? Conversely, if logic is not universally applicable, how do we demarcate its boundaries? Any mathematician of considerable accomplishment, upon entering the domain of mathematical foundations, typically feels dizzy: here exist not only paradoxes unseen in other rigorous branches of mathematics, but also numerous counterintuitive notions. Some so-called proofs, while bold and free, lack the rigor to withstand careful scrutiny.

Clearly, a fundamental clarification of these clouds hanging over logic and mathematical foundations is necessary. This paper represents a preliminary attempt at such work.

2. The Origin of Logical Rules

From a problem-simplification perspective, treating logical rules as innate or a priori, as certain transcendental philosophers do, appears effective. However, these philosophers cannot answer how unicellular organisms eventually evolved "innately correct" logical rules in the human brain. In reality, we cannot observe any phenomena in newborns that could be considered applications of logical rules. Therefore, we have no reason to arbitrarily assume that thinking rules, including logical rules, are innate. Humans possess learning abilities. When individuals intentionally, unintentionally, or even accidentally succeed in practice using certain information-processing patterns, these patterns are preserved in memory, while those leading to failure are forgotten. Thus, positive and negative feedback from practice gradually shapes information-processing patterns that help individuals succeed. In other words, individual information-processing patterns are acquired through "trial and error."

These patterns encompass both simple conditioned and unconditioned reflexes and information processing involving thought. For humans, thinking plays a crucial role in information processing. These thinking patterns can also be called individual thinking habits. Consequently, individual thinking habits accumulate gradually through trial and error.

Humans, however, possess language and communication abilities. Since evolution cannot create logical rules, when people exchange their successful thinking habits through language, those habits that achieve consensus and become widely disseminated can only be the source of logical rules. Thus, fundamentally speaking, logical rules originate from practice.

3. The Limitations of Logic and the Causes of Paradoxes

The benefit of logic lies in its ability to help organize sensory material. Through defining concepts, we can conveniently identify related matters. Through logical rules, we can further organize the relationships among these concepts. For instance, if we call knowledge obtained through near-complete induction "axioms," we can use deduction to form a vast deductive system encompassing both known and unknown knowledge. Euclid' s *Elements* and Newton' s *Principia Mathematica* exemplify this approach. Theorems proven from reliable axioms using logical laws with broad practical foundations are often more reliable than empirical laws derived directly from limited practice, thus forming the main research method of rigorous sciences like mathematics: logical proof. However, obtaining reliable logical conclusions clearly requires two conditions: reliable axioms and reliable logical rules—neither of which necessarily holds.

First, if axioms are inductions from practical knowledge, since the scope of practice is always limited, we cannot guarantee that axioms necessarily hold. For example, Newton' s three laws of mechanics derive from summarizing practical knowledge. While sufficiently reliable at low speeds and macroscopic scales, they no longer hold at microscopic and high-speed scales.

The birth of non-Euclidean geometry made people realize that axioms need not originate from practice but can be constructed through free human thought. When machine reasoning became reality, freely constructing some axioms and then using machines to obtain vast deductive systems became effortless. If all these could become academic knowledge, humans would be unable to master such massive amounts of knowledge. Selecting truly useful knowledge from this sea of "knowledge" would require practical utility as the measuring standard.

For instance, if non-Euclidean geometry had not found application in relativity theory, people would probably have long forgotten it. Therefore, even knowledge derived from free thinking cannot ultimately escape the constraints of practice.

Besides the uncertain reliability of axioms, the reliability of logic itself does not always hold. The reason is simple: since logical rules originate from thinking habits that seem to universally succeed in practice, and since the scope of practice is always limited, we cannot guarantee the universality of logic. The existence of certain paradoxes serves as proof.

The most famous paradox is the so-called liar paradox: "Everything he says is a lie." If this statement is true, it means he has at least told one truth, implying that not everything he says is a lie, making the statement false—thus creating a contradiction. Similarly, if the statement is false, it means everything he says is true, but the statement itself is false, again creating a contradiction. Whether the statement is true or false, the result is contradictory.

The liar paradox has perplexed many wise minds throughout history and remains imperfectly resolved. If we believe formal logic is always correct, this paradox is forever unsolvable. However, if we do not blindly worship formal logic but view it as having a scope of application, the problem is not difficult to solve. In fact, formal logic is based on the three laws: identity, non-contradiction, and excluded middle. Therefore, only within a discussion domain where these three laws all hold—hereinafter called the feasible domain of formal logic, or simply the feasible domain—is formal logic reliable. The cause of the liar paradox lies simply in the fact that within the discussed scope, the law of non-contradiction no longer holds, or the discussion has left the feasible domain. Actually, there are two statements here: this particular statement is true, but his other statements are false. Clearly, truth and falsehood contradict each other; therefore, we cannot combine this true statement with other false statements (into "all statements") for discussion, otherwise we enter a contradictory non-feasible domain where paradoxes naturally arise. The correct formulation should be: except for this statement, everything else he says is a lie.

The famous barber paradox similarly results from mixing two classes of people with contradictory behaviors (those who shave themselves and those who do not) into one discussion, thereby deviating from the feasible domain. This paradox can be resolved by excluding the barber himself from the advertisement. Thus, whether the barber shaves himself or not, it does not contradict the advertisement. The reasoning is not complicated: since the set of people who shave themselves includes the barber but may also include some villagers who are thrifty or enjoy shaving themselves, the advertisement' s scope does not include such people but only those who do not shave themselves. In other words, if we use set S to represent those who do not shave themselves, S' s definition excludes not only the barber but also those villagers who shave themselves. If we then treat the barber as an element of set S to study his behavior, this directly contradicts S' s definition, violates the law of non-contradiction, deviates from the feasible domain, and makes paradoxes unsurprising.

Some paradoxes do not necessarily result from discussions leaving the feasible domain. For example, as is well known, pursuit problems should be studied using velocity. If we surreptitiously replace the velocity concept with the space concept while ignoring time issues, we obtain the so-called Zeno' s paradox. In Zeno's "Achilles" argument, the time required for pursuit is actually a convergent infinite series. Although the series has infinitely many terms, its sum is finite. Therefore, Zeno' s paradox merely stems from a mistaken assumption: that the sum of infinitely many numbers must be infinite. With the concept of convergent infinite series, Zeno' s paradox has actually been thoroughly resolved.

The chicken-and-egg paradox does not actually exist. This is because chickens evolved gradually from other species, tracing back to unicellular organisms that reproduce through cell division, where "egg" and "chicken" are indistinguishable. Therefore, this paradox arises only from the false premise that organisms cannot evolve. Since the subsequent derivation has no problems, it can actually serve as evidence that "organisms are evolving." Recent literature has provided similar explanations from an evolutionary perspective, though its answer seems to transform the question "which came first, the chicken or the egg?" into "which came first, the bird or the bird' s egg," which is not entirely satisfactory.

Thus, so-called paradoxes either result from discussions deviating from the feasible domain or from discussions that are not rigorous or introduce false assumptions or imaginings.

4. Logical Problems in Mathematical Foundations

Among all sciences, mathematics is the most rigorous. In most mature areas of mathematics, proofs are typically very rigorous, and paradoxes are rarely seen. However, in mathematical foundations, paradoxes exist alongside many proofs that appear clever but cannot withstand careful scrutiny.

For instance, logicism in mathematical foundations claims that all mathematical knowledge can be constructed using logic. However, as previously argued, logic is merely a summary of successful thinking habits; therefore, logic itself does not contain specific knowledge of any particular discipline. Attempting to construct specific knowledge of any discipline using only logical knowledge is doomed to fail.

For example, some works claim to define addition using Peano axioms and prove that 1+1=2. However, the vague concept of successor in these axioms already implicitly contains the practical knowledge that 1+1=2; otherwise, we cannot explain why the successor of 1 is 2 rather than 3. In fact, if we stipulate that the successor of 1 is 3—for instance, defining natural numbers as 0, 1, 3, 4, 5, 6⋯or 0, 1, 3, 5, 7, 9⋯—we do not violate any of the five Peano axioms, but this would inevitably lead to the erroneous conclusion that 1+1=3. Using a successor concept that implicitly contains 1+1=2 to then define addition and prove 1+1=2 is clearly a meaningless exercise full of logical circularity. This example concretely demonstrates that specific scientific knowledge cannot be derived using only logical knowledge. Gödel' s incompleteness theorems are essentially just a proof of this fact.

The proof of Cantor' s theorem (card(P(A)) > card(A)) is also erroneous. The proof' s key point is: let f be any function from A to the power set P(A). To prove that f cannot be surjective, it suffices to exhibit a subset of A that is not in the image of f. This is B = {a | a A, a f(a)}. Cantor proved that there exists no A such that f( ) = B, meaning B is not in the image of f.

The theorem' s proof appears concise and clever but cannot withstand rigorous scrutiny. Let us first examine a concrete example: 1111111,1,1, ···,1···,···2!2!3!2!3!!n . Obviously, a one-to-one correspondence h can be established with the set of natural numbers N: 111111:11,21,31,···,1···,··· 2!2!3!2!3!!hnn . Since one-to-one correspondence includes surjection, set (1) and the set of natural numbers are mutually surjective. The last element of infinite set (1) is exactly the base of natural logarithms e, so set (1) can also be expressed as 1111111,1,1,···,1···,···e2!2!3!2!3!!n . However, since every natural number is finite, there exists no n N such that 1111···e2!3!!n . Can we therefore conclude that h is not surjective?

The cause of this contradiction lies in: if an infinite set has a last element, that element can only be the result of reaching infinity, while natural numbers cannot reach infinity. This does not prevent establishing a one-to-one correspondence between all other elements that have not reached infinity in the two infinite sets.

Historically, the debate between potential infinity and actual infinity has never been resolved. However, since logic is merely a summary of human thinking experience and cannot create any specific knowledge out of nothing, this mathematical problem, which cannot be logically proven by more universal knowledge, can only be addressed directly from practice.

From a practical perspective, we can naturally induce two types of infinite sets: the first type has a last element (as in example 2); the second type has no last element—for instance, any natural number has a successor, so the set of natural numbers has no last element. Obviously, the first type of infinite set can be described using either potential infinity or actual infinity: when described using potential infinity, although the last element is not written out, its existence is not denied. The second type of infinite set, however, can only be described using potential infinity to avoid conflict with practice.

This classification is not only natural but also consistent with practice and objectively exists. Thus, for the first type of infinite set, the existence of an element that is not the image of a mapping does not indicate that the mapping is necessarily not surjective (as in the example above), yet this is precisely the basis of Cantor' s theorem proof!

In fact, in Cantor' s theorem proof, the constructed B can be exactly the last element of P(A), which is a first-type infinite set. Let A = {1, 2, 3 ···} and P(A) = { {}, {1}, {2}, {1,2}, {3}, {1,3}, {2,3}, {1,2,3} ···{n}···{1,2,3,···}···}. Since a last element exists, P(A) belongs to the first type of infinite set. Let f: 1→{}, 2 → {1}, 3→ {2}, 4 →{1,2}······

Some might argue that we cannot reach infinity and therefore e does not exist. This is logically untenable. For example, we cannot reach the sun, yet the sun still exists. Because 1 {}, 2 {1}, 3 {2}, 4 {1,2} ······we can obtain B = {1, 2, 3, 4, ···}. 可见，这时 B 恰好是第一类无限集 P(A) 的最后一个元素！而如前所述：最后一个元素不是映射的像并不表明映射一定不是满射的，康托尔的证明不成立！

Cantor' s error lies in extending the definition of surjection based on finite sets to infinite sets where it may not apply, or in confusing the distinction between finite and infinite sets.

In the closely related proof of the uncountability of real numbers, errors also result from lack of rigor. When using the interval nesting method to prove the uncountability of real numbers, one continuously divides intervals to "avoid" the listed real numbers and finally obtains a "new" real number at the limit point where interval length equals zero, thereby contradicting the assumption that all real numbers were listed. Obviously, before the limit point, intervals remain divisible and can continuously "avoid" the listed real numbers. However, the limit point itself is not an interval and cannot continue using interval division to "avoid" the listed real numbers—that is, we cannot exclude the possibility that the limit point is among the listed real numbers. The proof by contradiction fails!

When using the famous diagonal method to prove the uncountability of real numbers, one first assumes real numbers are countable and lists them all, then cleverly uses the diagonal to construct a "new" real number different from all listed real numbers, thereby contradicting the assumption. However, this method requires the diagonal to traverse all listed real numbers so that it contains some decimal digit of each listed real number, thereby enabling construction of a "new" real number different from any listed one and making the proof by contradiction valid. This is impossible: taking decimal expansions as an example, n decimal places correspond to 10^n possible decimals, while the diagonal can only traverse n of them. Whether n is finite or infinite, this fact cannot change. Therefore, the proof by contradiction also fails.

Some might argue that if we uniformly adopt the approach of potential infinity and exclude the last element from infinite sets, the definition of surjection can be extended to any infinite set. However, if we do this, B would not be an element within the infinite set, which not only contradicts the definition of P(A) but also means that even if constructed, it cannot prove that f is not surjective. Cantor' s theorem remains unproven.

Thus, regardless of whether real numbers are countable or not, at least so far, no rigorous proof has demonstrated that real numbers are uncountable.

These proofs all employ proof by contradiction. Although the proof processes appear concise and clever, the results contradict human intuition. Intuitionists in the history of mathematics considered proof by contradiction unreliable but could not provide convincing reasons. In fact, within the feasible domain, there is no reason to believe that proof by contradiction based on the law of excluded middle cannot hold. However, any mathematical proof should aim for absolute rigor; any slight lack of rigor may lead to absurd results, and proof by contradiction is no exception. In fact, errors caused by lack of rigor in proof by contradiction are often more deeply hidden and harder to detect, requiring even greater caution to obtain reliable results.

In mathematical foundations, errors and confusion caused by non-rigorous proofs have long existed and remain uncorrected to this day. This paper only points out some examples.

These errors and confusion should also relate to unreasonable definitions of numbers. From the perspective of mathematical history, humans discovered natural numbers first, then rational numbers, and finally irrational numbers. Therefore, using natural numbers to define rational numbers (as ratios of two natural numbers) and then using rational numbers to define irrational numbers (through Dedekind cuts) is natural and widely adopted to this day. However, this does not mean this definitional approach is unique or even sufficiently reasonable.

As is well known, from a logical perspective, the modern interpretation of definition is a precise and concise description of the essential characteristics of a thing or the intension and extension of a concept. However, what exactly are the essential characteristics of a thing? This obviously depends on people' s level of understanding; therefore, for the same thing or class of things, different levels of understanding may yield different definitions.

Using natural numbers to define rational numbers and then rational numbers to define irrational numbers conforms to the human cognitive process and seems natural. However, definitions that conform to the human cognitive process usually cannot capture the essential characteristics of things. This is because human understanding rarely touches upon the essence of things from the beginning. For example, definitions based on initially encountered natural numbers divide numbers into various types without providing their common points or unifying them, leaving many contradictions and unresolved debates such as countability versus uncountability.

Is there a definition that better captures essence? Literature [12] describes the method of expressing irrational numbers using infinite decimals. The author believes we can uniformly define natural numbers, rational numbers, and irrational numbers using infinite decimals: rational numbers are infinite decimals that become all zeros or repeating decimals from some digit onward (zero can also be considered a repeating block); integers, including natural numbers, are infinite decimals that are all zeros from the first decimal place onward.

This definition is not only concise but also identifies the common point of these three types of numbers: infinite decimals, thereby unifying them.

When the objects we seek to understand appear diverse and scattered, we can hardly claim to have found their essence; when they appear unified and harmonious, we can at least consider ourselves approaching their essence. Seeking commonalities through abstraction of concrete objects is at least one method for finding essential characteristics. Therefore, we can regard infinite decimals as the essential characteristic of these three types of numbers, thereby unifying real numbers. Consequently, all unnecessary debates arising from failure to grasp essential characteristics will cease to exist.

For example, consistent with human intuition, natural numbers, rational numbers, and irrational numbers can all be represented as points on the number line. Since their essence is infinite decimals and their status in measure theory is equal, they can easily be defined scientifically and clearly using the conventional logical method of genus (infinite decimals) plus specific difference.

5. Conclusion

Logic originates from practice. Within a domain of discourse (feasible domain) where the three laws of identity, non-contradiction, and excluded middle all hold, formal logic generates no paradoxes as long as premises are reliable and reasoning is rigorous, allowing mathematics to be built on a highly reliable foundation. This paper points out why logicism in mathematical foundations cannot succeed, identifies errors in the proofs of Cantor’ s theorem, interval nesting, and the diagonal argument, and provides a unified definition of natural numbers, rational numbers, and irrational numbers using the scientific logical method of genus plus specific difference that can avoid these errors.

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Note: Figure translations are in progress. See original paper for figures.