# Postprint: A Multi-granular Rough Decision Analysis Method Based on Possibility-degree Tolerance Relations

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## Abstract

In incomplete interval-valued decision information systems, a multi-granulation decision-theoretic rough model based on possibility tolerance relations is proposed by leveraging the respective advantages of possibility tolerance relations and multi-granulation decision-theoretic rough sets. First, the concept of possibility degree is introduced and a novel tolerance relation is defined. Then, optimistic and pessimistic multi-granulation decision-theoretic rough set models based on possibility tolerance relations are constructed, the lower and upper approximations of the models are presented, and related properties and theorems are proved. Finally, the effectiveness and applicability of the model are verified through an example. The results demonstrate that by adjusting the attribute similarity threshold , the model can possess certain fault tolerance capabilities and strong classification capabilities.

## Full Text

## Preamble

### Rough Analysis Method of Multi-Granularity Decision-Theoretic Rough Set Based on Possible Degree Tolerance Relation

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**Abstract:** In incomplete interval-valued decision information systems, this paper proposes a multi-granularity decision-theoretic rough model based on possible degree tolerance relation by leveraging the respective advantages of possible degree tolerance relation and multi-granularity decision-theoretic rough sets. First, we introduce the concept of possible degree and define a new tolerance relation. Then, we construct optimistic and pessimistic multi-granularity decision-theoretic rough set models based on possible degree tolerance relation, present the lower and upper approximations of these models, and prove their relevant properties and theorems. Finally, a case study demonstrates the effectiveness and applicability of the proposed model. The results show that by adjusting the attribute similarity threshold , the model exhibits certain fault tolerance capabilities and strong classification performance.

**Keywords:** incomplete interval-valued decision information system; tolerance relation; rough set

## 0 Introduction

Rough set theory, as a mathematical tool for analyzing and processing uncertainty, has been widely applied in numerous fields [1~4]. The classical rough set model [5,6] handles complete discrete information systems based on equivalence relations. However, in real-world scenarios, information systems often contain not only discrete data but also measurement values expressed as intervals within a certain range, and these systems are frequently incomplete. This imposes certain limitations on the practical application of traditional rough sets. To address this issue, many scholars have extended the classical rough set model [7~10]. Yang [11] studied incomplete interval-valued information systems and constructed a rough set model based on -dominance relation, computing lower and upper approximations through given judgment theorems and discernibility functions. Dai [12] et al. further defined maximum and minimum similarity degrees in incomplete interval-valued information systems and constructed a rough set model based on -weak similarity relation using attribute similarity to assess uncertainty in such systems.

In current research on incomplete interval-valued information systems, most scholars primarily use dominance relations [13~15] instead of equivalence relations to handle missing values. Some researchers have defined similarity tolerance relations from the perspectives of compatibility, difference, and opposition degrees of attribute sets, establishing rough set models accordingly [16-17]. Dai [18] et al. proposed possible degree tolerance relation in incomplete interval-valued information systems, effectively solving the missing value problem. From the perspective of granular computing, the aforementioned rough models for incomplete interval-valued information systems all classify the universe based on a single equivalence relation or a single dominance relation, unable to analyze and process problems from multi-granularity and multi-level perspectives. Qian et al. [19] proposed multi-granulation rough sets, which approximate targets using a family of indiscernibility relations, including optimistic and pessimistic multi-granulation rough sets. Building upon this foundation, scholars have proposed various extended models [20-23]. Sang et al. [24] utilized Bayesian decision theory to analyze probabilistic fusion relations in multi-granulation rough set models, constructing optimistic and pessimistic multi-granulation decision-theoretic rough set models.

For incomplete interval-valued decision information systems, this paper integrates possible degree tolerance relation into multi-granularity decision-theoretic rough models, constructing a novel multi-granularity decision-theoretic rough model. We verify the relevant properties of the model and prove its theorems. Additionally, we investigate how changes in the attribute similarity threshold affect classification. This model combines the advantages of possible degree tolerance relation and multi-granulation decision-theoretic rough sets, effectively expanding the model's applicability. Both theoretical analysis and experimental results demonstrate its effectiveness.

## 1.1 Incomplete Interval-Valued Decision Information System

**Definition 1 [16]** An information system is defined as a quadruple $S = (U, A, V, f)$, where $U = \{x_1, x_2, ..., x_n\}$ is a finite set of objects called the universe; $A = C \cup D$ is the attribute set, with $C$ being the condition attribute set and $D$ the decision attribute set; $V = \bigcup_{a\in A} V_a$ is the set of all attribute value domains; and $f : U \times A \to V$ is an information function such that $f(x, a) \in V_a$ for each $a \in A$ and $x \in U$.

**Definition 2 [19]** Let $S = (U, C \cup D, V, f)$ be an interval-valued decision information system. For any $a_k \in C$, let $V_{a_k}$ be the finite value domain of attribute $a_k$. If for any $x_i \in U$ and $a_k \in C$, $f(x_i, a_k) = [l_{ik}, r_{ik}]$ where $l_{ik} \le r_{ik}$, then $S$ is called an interval-valued decision information system. If there exists $x_i \in U$ and $a_k \in C$ such that $f(x_i, a_k) = *$ (where "*" denotes a missing value), indicating that the value of object $x_i$ on attribute $a_k$ is unknown, then $S$ is called an incomplete interval-valued decision information system.

## 1.2 Multi-Granularity Decision-Theoretic Rough Set

Let $S = (U, C \cup D, V, f)$ be a complete decision information system, where $R = \{R_1, R_2, ..., R_m\}$ is a family of $m$ attribute subsets on the condition attribute set $C$. Let $\Omega = \{\omega_1, \omega_2, ..., \omega_s\}$ be a finite set of $s$ states and $A = \{a_1, a_2, ..., a_n\}$ be a finite set of $n$ actions. Let $\lambda_{ij}^k$ denote the risk cost of taking action $a_i$ in state $\omega_j$ under the $k$-th granular structure, and $P(\omega_j|x)$ be the conditional probability of object $x$ being in state $\omega_j$. Assuming $P(\omega_j|x) = P(\omega_j|[x]_{R_k})$, the expected risk of taking action $a_i$ for a given object $x$ under $m$ granular structures can be expressed as:

$$R(a_i|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) = \sum_{j=1}^{s} \lambda_{ij} P(\omega_j|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) = \sum_{j=1}^{s}\sum_{k=1}^{m} \lambda_{ij}^k P(\omega_j|[x]_{R_k})$$

where $P(\omega_j|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m})$ is a unified probabilistic representation under different multi-granularity fusion strategies.

Given $m$ granular structures, where decision actions include positive region decision ($POS$), negative region decision ($NEG$), and boundary region decision ($BND$), with the decision set denoted as $\{a_1, a_2, a_3\}$, the corresponding expected losses are:

$$R(a_1|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) = \sum_{i=1}^{m} \lambda_{1i}^{c} P(X|[x]_{R_i})$$
$$R(a_2|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) = \sum_{i=1}^{m} \lambda_{2i}^{c} P(X|[x]_{R_i})$$
$$R(a_3|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) = \sum_{i=1}^{m} \lambda_{3i}^{c} P(X|[x]_{R_i})$$

where the decision cost function values satisfy $\lambda_{1i} \leq \lambda_{2i} < \lambda_{3i}$ for $i = 1, 2, 3$. The decision rules are as follows:

$$(P)\text{If } R(a_1|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \leq R(a_2|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \text{ and } R(a_1|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \leq R(a_3|[x]_{R_1},$$
$$(N)\text{If } R(a_2|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \leq R(a_1|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \text{ and } R(a_2|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \leq R(a_3|[x]_{R_1},$$
$$(B)\text{If } R(a_3|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \leq R(a_1|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \text{ and } R(a_3|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \leq R(a_2|[x]_{R_1},$$

Let $\alpha = \frac{\lambda_{23}-\lambda_{13}}{(\lambda_{23}-\lambda_{13})+(\lambda_{11}-\lambda_{21})}$, $\beta = \frac{\lambda_{22}-\lambda_{12}}{(\lambda_{22}-\lambda_{12})+(\lambda_{11}-\lambda_{21})}$, and $\gamma = \frac{\lambda_{21}-\lambda_{11}}{(\lambda_{21}-\lambda_{11})+(\lambda_{12}-\lambda_{22})}$. With $0 \leq \beta < \gamma < \alpha \leq 1$, the decision rules can be simplified as:

$$(P)\text{If } P(X|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \geq \alpha, \text{ decide } POS(X)$$
$$(N)\text{If } P(X|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) \leq \beta, \text{ decide } NEG(X)$$
$$(B)\text{If } \beta < P(X|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m}) < \alpha, \text{ decide } BND(X)$$

where $P(X|[x]_{R_1}, [x]_{R_2}, ..., [x]_{R_m})$ is the conditional probability.

**Definition 3 [19]** Let $S = (U, C \cup D, V, f)$ be a complete decision information system, $R = \{R_1, R_2, ..., R_m\}$ be $m$ attribute subsets on the condition attribute set $C$, and $X \subseteq U$ be any subset. Given $0 \leq \beta < \alpha \leq 1$, the lower approximation, upper approximation, and boundary region of the pessimistic multi-granulation decision-theoretic rough set are defined as:

$$\underline{R}_P(X)_\alpha = \{x \in U : \bigwedge_{i=1}^{m} (P(X|[x]_{R_i}) \geq \alpha)\} = \{x \in U : (P(X|[x]_{R_1}) \geq \alpha) \wedge (P(X|[x]_{R_2}) \geq \alpha) \wedge ... \wedge (P(X|[x$$
$$\overline{R}_P(X)_\beta = \{x \in U : \bigvee_{i=1}^{m} (P(X|[x]_{R_i}) > \beta)\} = \{x \in U : (P(X|[x]_{R_1}) > \beta) \vee (P(X|[x]_{R_2}) > \beta) \vee ... \vee (P(X|[x$$
$$BN_P(X)_{\beta,\alpha} = \overline{R}_P(X)_\beta - \underline{R}_P(X)_\alpha$$

## 2 Multi-Granularity Decision-Theoretic Rough Set Based on Possible Degree Tolerance Relation

**Definition 4 [18]** Let $S = (U, C \cup D, V, f)$ be an incomplete interval-valued decision information system, $a_k \in C$, and $x_i, x_j \in U$ be any two objects. When the attribute values of $x_i$ and $x_j$ on attribute $a_k$ are both not single-valued, the possible degree that $x_i$ and $x_j$ are similar with respect to attribute $a_k$ is defined as:

$$P_{a_k}(x_i \geq x_j) = \max\left\{1 - \max\left(\frac{l_{jk} - r_{ik}}{r_{ik} - l_{ik} + r_{jk} - l_{jk}}, 0\right), 0\right\}$$

where $f(x_i, a_k) = [l_{ik}, r_{ik}]$ and $f(x_j, a_k) = [l_{jk}, r_{jk}]$. When the attribute values of $x_i$ and $x_j$ on attribute $a_k$ are both single-valued, if they are equal, the possible degree of similarity between $x_i$ and $x_j$ with respect to attribute $a_k$ is 1; otherwise, it is 0. When one of the attribute values is missing (denoted by "*"), the possible degree of similarity is defined as 1.

**Definition 5** Let $S = (U, C \cup D, V, f)$ be an incomplete interval-valued decision information system, $a_k \in C$, and $x_i, x_j \in U$ be any two objects. When the attribute values of $x_i$ and $x_j$ on attribute $a_k$ are both not single-valued, the similarity degree between $x_i$ and $x_j$ with respect to attribute $a_k$ is defined as:

$$S_{a_k}(x_i, x_j) = 1 - |P_{a_k}(x_i \geq x_j) - P_{a_k}(x_j \geq x_i)|$$

where $P_{a_k}(x_i \geq x_j)$ is the possible degree that $x_i$ and $x_j$ are similar with respect to attribute $a_k$, and $|P|$ denotes the cardinality of set $P$.

**Definition 6** Given an incomplete interval-valued decision information system $S = (U, C \cup D, V, f)$, let $\omega$ $(0 \leq \omega \leq 1)$ be a given similarity threshold. The tolerance relation based on possible degree on the incomplete interval-valued information system is defined as:

$$\Gamma_\omega = \{(x_i, x_j) \in U \times U : S_{a_k}(x_i, x_j) \geq \omega, \forall a_k \in C\}$$

Clearly, $\Gamma_\omega$ satisfies reflexivity and symmetry but does not satisfy transitivity.

**Definition 7** Let $S = (U, C \cup D, V, f)$ be an incomplete interval-valued decision information system, $R = \{R_1, R_2, ..., R_m\}$ be $m$ attribute subsets on the condition attribute set $C$, and $X \subseteq U$ be any subset. Let $\omega$ $(0 \leq \omega \leq 1)$ be a given similarity threshold and $0 \leq \beta < \alpha \leq 1$. The lower approximation, upper approximation, and boundary region of the multi-granularity decision-theoretic rough set based on possible degree tolerance relation are defined as:

$$\underline{R}_O(X)^\omega_\alpha = \{x \in U : \bigvee_{i=1}^{m}(P(X|[x]^\omega_{R_i}) \geq \alpha)\} = \{x \in U : (P(X|[x]^\omega_{R_1}) \geq \alpha) \vee (P(X|[x]^\omega_{R_2}) \geq \alpha) \vee ... \vee (P(X|[x$$

$$\overline{R}_O(X)^\omega_\beta = \{x \in U : \bigwedge_{i=1}^{m}(P(X|[x]^\omega_{R_i}) > \beta)\} = \{x \in U : (P(X|[x]^\omega_{R_1}) > \beta) \wedge (P(X|[x]^\omega_{R_2}) > \beta) \wedge ... \wedge (P(X|[x$$

$$BN_O(X)^\omega_{\beta,\alpha} = \overline{R}_O(X)^\omega_\beta - \underline{R}_O(X)^\omega_\alpha$$

where $P(X|[x]^\omega_{R_i})$ is the conditional probability.

**Definition 8** Let $S = (U, C \cup D, V, f)$ be an incomplete interval-valued decision information system, $R = \{R_1, R_2, ..., R_m\}$ be $m$ attribute subsets on the condition attribute set $C$, and $X \subseteq U$ be any subset. Let $\omega$ $(0 \leq \omega \leq 1)$ be a given similarity threshold and $0 \leq \beta < \alpha \leq 1$. The lower approximation, upper approximation, and boundary region of the pessimistic multi-granularity decision-theoretic rough set based on possible degree tolerance relation are defined as:

$$\underline{R}_P(X)^\omega_\alpha = \{x \in U : \bigwedge_{i=1}^{m}(P(X|[x]^\omega_{R_i}) \geq \alpha)\} = \{x \in U : (P(X|[x]^\omega_{R_1}) \geq \alpha) \wedge (P(X|[x]^\omega_{R_2}) \geq \alpha) \wedge ... \wedge (P(X|[x$$

$$\overline{R}_P(X)^\omega_\beta = \{x \in U : \bigvee_{i=1}^{m}(P(X|[x]^\omega_{R_i}) > \beta)\} = \{x \in U : (P(X|[x]^\omega_{R_1}) > \beta) \vee (P(X|[x]^\omega_{R_2}) > \beta) \vee ... \vee (P(X|[x$$

$$BN_P(X)^\omega_{\beta,\alpha} = \overline{R}_P(X)^\omega_\beta - \underline{R}_P(X)^\omega_\alpha$$

**Theorem 1** Given an incomplete interval-valued decision information system $S = (U, C \cup D, V, f)$, let $R = \{R_1, R_2, ..., R_m\}$ be $m$ attribute subsets on the condition attribute set $C$, $X \subseteq U$ be any subset, and $\omega$ $(0 \leq \omega \leq 1)$ be a given similarity threshold. The following properties hold:

1. $\underline{R}_O(\emptyset)^\omega_\alpha = \underline{R}_O(\emptyset)^\omega_\beta = \emptyset$, $\underline{R}_P(\emptyset)^\omega_\alpha = \underline{R}_P(\emptyset)^\omega_\beta = \emptyset$
2. $\underline{R}_O(U)^\omega_\alpha = \underline{R}_O(U)^\omega_\beta = U$, $\underline{R}_P(U)^\omega_\alpha = \underline{R}_P(U)^\omega_\beta = U$
3. $\underline{R}_O(X)^\omega_\alpha \supseteq \underline{R}_O(X)^\omega_\beta$, $\overline{R}_O(X)^\omega_\alpha \subseteq \overline{R}_O(X)^\omega_\beta$
4. $\underline{R}_P(X)^\omega_\alpha \subseteq \underline{R}_P(X)^\omega_\beta$, $\overline{R}_P(X)^\omega_\alpha \supseteq \overline{R}_P(X)^\omega_\beta$

**Proof:** The properties can be easily proved from Definitions 7-9.

**Theorem 2** Given an incomplete interval-valued decision information system $S = (U, C \cup D, V, f)$, let $R = \{R_1, R_2, ..., R_m\}$ be $m$ attribute subsets on the condition attribute set $C$, $X \subseteq U$ be any subset, and $\omega$ $(0 \leq \omega \leq 1)$ be a given similarity threshold. The upper approximation of the optimistic multi-granularity decision-theoretic rough set based on possible degree tolerance relation is the union of the upper approximations under each granular classification rule.

**Proof:** For any $x \in \overline{R}_O(X)^\omega_\beta$, we have $\bigwedge_{i=1}^m (P(X|[x]^\omega_{R_i}) > \beta)$. This implies that for all $i \in \{1, 2, 3, ..., m\}$, $P(X|[x]^\omega_{R_i}) > \beta$. Therefore, $x \in \{x : P(X|[x]^\omega_{R_i}) > \beta\}$ for each $i$, which means $x \in \bigcup_{i=1}^m \overline{R}_i(X)^\omega_\beta$. Conversely, if $x \in \bigcup_{i=1}^m \overline{R}_i(X)^\omega_\beta$, then there exists at least one $i$ such that $P(X|[x]^\omega_{R_i}) > \beta$, which satisfies the condition for $\overline{R}_O(X)^\omega_\beta$. Thus, $\overline{R}_O(X)^\omega_\beta = \bigcup_{i=1}^m \overline{R}_i(X)^\omega_\beta$.

Similarly, we can prove that the upper approximation of the pessimistic multi-granularity decision-theoretic rough set based on possible degree tolerance relation is the intersection of the upper approximations under each granular classification rule.

**Definition 9** Let $S = (U, C \cup D, V, f)$ be an incomplete interval-valued decision information system, $R = \{R_1, R_2, ..., R_m\}$ be $m$ attribute subsets on the condition attribute set $C$, and $X \subseteq U$ be any subset. Let $\omega$ $(0 \le \omega \le 1)$ be a given similarity threshold and $0 \le \beta < \alpha \le 1$. The classification accuracy and classification quality under optimistic and pessimistic conditions are defined as:

$$\gamma^\omega_O(X)_{\beta,\alpha} = \frac{|\underline{R}_O(X)^\omega_\alpha|}{|\overline{R}_O(X)^\omega_\beta|}$$

$$\gamma^\omega_P(X)_{\beta,\alpha} = \frac{|\underline{R}_P(X)^\omega_\alpha|}{|\overline{R}_P(X)^\omega_\beta|}$$

**Theorem 3** Given an incomplete interval-valued decision information system $S = (U, C \cup D, V, f)$, let $R = \{R_1, R_2, ..., R_m\}$ be $m$ attribute subsets on the condition attribute set $C$, $X \subseteq U$ be any subset, and $\omega$ $(0 \le \omega \le 1)$ be a given similarity threshold. The multi-granularity decision-theoretic rough set model based on possible degree tolerance relation has the following properties:

1. When $\alpha = 1, \beta = 0$, the optimistic multi-granularity decision-theoretic rough set based on possible degree tolerance relation degenerates into the optimistic multi-granularity rough set under possible degree tolerance relation.
2. When $\alpha = 1, \beta = 0$, the pessimistic multi-granularity decision-theoretic rough set based on possible degree tolerance relation degenerates into the pessimistic multi-granularity rough set under possible degree tolerance relation.

**Proof:** This follows directly from Definitions 7-9.

**Theorem 4** Given an incomplete interval-valued decision information system $S = (U, C \cup D, V, f)$, let $R = \{R_1, R_2, ..., R_m\}$ be $m$ attribute subsets on the condition attribute set $C$, $X \subseteq U$ be any subset, and $\omega$ $(0 \le \omega \le 1)$ be a given similarity threshold. The following inclusion relations hold:

1. $\underline{R}_O(X)^\omega_\alpha \supseteq \underline{R}_P(X)^\omega_\alpha$
2. $\overline{R}_O(X)^\omega_\beta \subseteq \overline{R}_P(X)^\omega_\beta$

**Proof:** For any $x \in \underline{R}_P(X)^\omega_\alpha$, we have $\bigwedge_{i=1}^{m}(P(X|[x]^\omega_{R_i}) \geq \alpha)$. This implies that for all $i \in \{1, 2, 3, ..., m\}$, $P(X|[x]^\omega_{R_i}) \geq \alpha$. Therefore, there exists at least one $i$ such that $P(X|[x]^\omega_{R_i}) \geq \alpha$, which means $x \in \underline{R}_O(X)^\omega_\alpha$. Thus, $\underline{R}_P(X)^\omega_\alpha \subseteq \underline{R}_O(X)^\omega_\alpha$. The proof for the upper approximation is similar.

## 3 Case Study

Table 1 presents an incomplete interval-valued decision information system, where $U = \{x_1, x_2, ..., x_{20}\}$ is the universe, $C = \{a_1, a_2, ..., a_{10}\}$ is the condition attribute set, and $D = \{d\}$ is the decision attribute set. The family of condition attribute subsets is $R = \{R_1, R_2, R_3, R_4\}$, where $R_1 = \{a_1, a_2\}$, $R_2 = \{a_3, a_4, a_5\}$, $R_3 = \{a_6, a_7, a_8\}$, and $R_4 = \{a_9, a_{10}\}$. Let $\alpha = 0.75$, $\beta = 0.49$, and $\omega = 0.7$.

First, we partition the decision classes based on the decision attribute $d$ as follows: $U/IND(D) = \{\{x_1, x_2, x_4, x_6, x_7, x_9, x_{11}, x_{13}, x_{15}, x_{17}, x_{19}, x_{20}\}, \{x_3, x_5, x_8, x_{10}, x_{12}, x_{14}, x_{16}, x_{18}\}\}$. According to Definitions 4-9, we compute the lower and upper approximations of the multi-granularity decision-theoretic rough set for the incomplete interval-valued information system:

When $\omega = 0.7$, the optimistic multi-granularity decision-theoretic rough set based on possible degree tolerance relation yields: - $\underline{R}_O(D_1)^\omega_\alpha = \{x_1, x_2, x_6, x_{10}, x_{11}, x_{15}, x_{18}\}$ - $\overline{R}_O(D_1)^\omega_\beta = \{x_1, x_2, x_3, x_{19}, x_{20}\}$ - $\underline{R}_O(D_2)^\omega_\alpha = \{x_2, x_{10}, x_{18}\}$ - $\overline{R}_O(D_2)^\omega_\beta = \{x_1, x_2, x_3, x_6, x_7, x_8, x_{10}, x_{12}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}\}$

The pessimistic multi-granularity decision-theoretic rough set based on possible degree tolerance relation yields: - $\underline{R}_P(D_1)^\omega_\alpha = \{x_1, x_{21}\}$ - $\overline{R}_P(D_1)^\omega_\beta = \{x_1, x_2, x_4, x_5, x_9, x_{11}, x_{13}, x_{14}, x_{17}, x_{20}\}$ - $\underline{R}_P(D_2)^\omega_\alpha = \{x_2, x_{10}, x_{18}\}$ - $\overline{R}_P(D_2)^\omega_\beta = \{x_2, x_3, x_8, x_9, x_{10}, x_{12}, x_{16}, x_{17}, x_{18}\}$

When $\omega = 0.8$, the results are: - $\underline{R}_O(D_1)^\omega_\alpha = \{x_1, x_2, x_4, x_6, x_{11}, x_{13}, x_{15}\}$ - $\overline{R}_O(D_1)^\omega_\beta = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{17}, x_{18}, x_{19}, x_{20}\}$ - $\underline{R}_O(D_2)^\omega_\alpha = \{x_2, x_3, x_{10}, x_{12}, x_{18}\}$ - $\overline{R}_O(D_2)^\omega_\beta = \{x_2, x_3, x_8, x_9, x_{10}, x_{12}, x_{16}, x_{17}, x_{18}\}$

The pessimistic version yields: - $\underline{R}_P(D_1)^\omega_\alpha = \{x_1, x_{21}\}$ - $\overline{R}_P(D_1)^\omega_\beta = \{x_1, x_2, x_4, x_5, x_6, x_7, x_9, x_{11}, x_{13}, x_{14}, x_{15}, x_{17}, x_{19}, x_{20}\}$ - $\underline{R}_P(D_2)^\omega_\alpha = \{x_2, x_3, x_{10}, x_{12}, x_{18}\}$ - $\overline{R}_P(D_2)^\omega_\beta = \{x_2, x_3, x_8, x_{10}, x_{12}, x_{16}, x_{17}, x_{18}\}$

When $\omega = 0.85$, we obtain: - $\underline{R}_O(D_1)^\omega_\alpha = \{x_1, x_2, x_4, x_6, x_{11}, x_{13}, x_{15}, x_{20}\}$ - $\overline{R}_O(D_1)^\omega_\beta = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{17}, x_{18}, x_{19}, x_{20}\}$ - $\underline{R}_O(D_2)^\omega_\alpha = \{x_2, x_3, x_7, x_{10}, x_{12}, x_{18}, x_{19}\}$ - $\overline{R}_O(D_2)^\omega_\beta = \{x_2, x_3, x_8, x_9, x_{10}, x_{12}, x_{16}, x_{17}, x_{18}\}$

The pessimistic version yields: - $\underline{R}_P(D_1)^\omega_\alpha = \{x_1, x_2, x_4, x_{11}, x_{13}\}$ - $\overline{R}_P(D_1)^\omega_\beta = \{x_1, x_2, x_4, x_5, x_6, x_7, x_9, x_{11}, x_{13}, x_{14}, x_{15}, x_{17}, x_{19}, x_{20}\}$ - $\underline{R}_P(D_2)^\omega_\alpha = \{x_2, x_3, x_{10}, x_{12}, x_{18}\}$ - $\overline{R}_P(D_2)^\omega_\beta = \{x_2, x_3, x_8, x_{10}, x_{12}, x_{16}, x_{17}, x_{18}\}$

Using the optimistic multi-granularity decision-theoretic rough set based on possible degree tolerance relation as an example, we calculate the classification quality for three different attribute similarity threshold values. When $\omega = 0.7$, the classification qualities for the decision classes are 26.33%, 37.5%, and 55.75%. When $\omega = 0.8$, they are 40%, 50%, and 75%. When $\omega = 0.85$, they are 46%, 57.5%, and 85%.

Comparing these results with existing methods, the classification qualities obtained using the rough set method from reference [16] are 32.5%, 29.55%, and 36.5%; using reference [17] yields 42%, 46%, and 57.5%; and using reference [24] produces 32.5%, 36.5%, and 42%. The results demonstrate that the multi-granularity decision-theoretic rough model based on possible degree similarity tolerance relation proposed in this paper achieves higher classification quality when handling incomplete interval-valued decision information systems compared to existing models [16,17,24]. This improvement is attributed to the fact that the new model not only inherits the advantage of multi-granularity decision-theoretic rough sets in comprehensively considering different attribute subsets from multiple levels and perspectives, but also achieves fault tolerance and strong classification capability by adjusting the thresholds $\alpha$, $\beta$, and $\omega$.

Furthermore, to verify that the model can more effectively handle decision information systems containing incomplete interval values, we fix $\alpha = 0.75$ and $\beta = 0.49$ while varying $\omega$ to 0.7, 0.8, and 0.85. The results show that as the attribute similarity threshold $\omega$ increases, objects in the incomplete interval-valued decision information system can be classified more correctly. When $\omega$ becomes sufficiently large, all objects can be correctly classified. This indicates that adjusting the thresholds $\alpha$, $\beta$, and $\omega$ can reduce the impact of noise to a certain extent, endowing the model with fault tolerance and strong classification ability.

## 4 Conclusion

This paper integrates the respective advantages of possible degree tolerance relation and multi-granularity decision-theoretic rough sets, proposing a multi-granularity decision-theoretic rough model based on possible degree tolerance relation for incomplete interval-valued decision information systems. We provide complete definitions for both optimistic and pessimistic multi-granularity decision-theoretic rough set models and discuss their fundamental properties and measurement parameters in detail. In particular, we investigate how changes in the attribute similarity threshold affect classification quality for object partitioning. By using different values of $\omega$, we can obtain varying degrees of object classification, making the proposed model stable and flexible.

Future research will focus on decision rule acquisition and attribute reduction problems.

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*Note: Figure translations are in progress. See original paper for figures.*