Composition Rules of the Hodge Star Operator and the Exterior Differential Operator

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Date: 2018-10-08T00:00:00+00:00

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Full Text

Combination Rules of the Hodge Star Operator and the Exterior Derivative Operator

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Abstract

This paper systematically investigates the general combination rules for the Hodge star operator and exterior derivative operator acting on arbitrary differential form fields. First, we identify two combination operators that preserve the degree of differential forms and construct a new operator through their linear combination. Second, when combining arbitrary numbers of Hodge star operators and exterior derivative operators, we derive unified expressions for all formally distinct combination operators, which are composed of single Hodge star operators, single exterior derivative operators, and non-zero combinations of any two of them. On this basis, we analyze the interaction relationships among all operators and classify them according to how they alter the degree of differential forms. Finally, as an application, we discuss in detail how to construct Maxwell' s equations for electromagnetic fields from linear combinations of differential forms of the same degree.

Keywords: Hodge star operator; exterior derivative operator; Maxwell' s equations; general relativity; differential geometry
Classification: O 0412.1
Document code: A
Article ID: [DOI]

1. Introduction

The modern concept of differential forms, pioneered by the renowned mathematician Élie Cartan, finds extensive applications in mathematics (particularly geometry and topology) and numerous branches of physics. In mathematical physics and theoretical physics, differential forms have become an extremely effective tool for studying general relativity, gauge field theories (such as electromagnetic theory and Yang-Mills theory), supergravity, and string theory.

When working with differential forms, three fundamental operations are essential: the wedge product, the Hodge star operator (denoted by $*$), and the exterior differentiation operator (denoted by d). Since the wedge product is straightforward to understand, this paper focuses exclusively on the latter two operators. We provide their precise definitions and main properties below, with further details available in references [1-5].

The Hodge star operator, first introduced by the mathematician W. V. D. Hodge, is a linear map defined on the exterior algebra of a finite-dimensional oriented vector space. Acting on a p-form on an n-dimensional differentiable manifold, a single Hodge star operator transforms it into an $(n-p)$-form. However, the consecutive action of an even number of Hodge star operators leaves both the degree and the magnitude of the differential form' s components unchanged, potentially introducing only an overall sign change. On the other hand, the exterior derivative operator extends the ordinary differential operator from functions to differential form fields. When acting on a p-form, it treats each component as a function and performs partial differentiation, generating a $(p+1)$-form. The result of applying the exterior derivative twice consecutively to any differential form is identically zero.

Due to the intrinsic properties of the Hodge star and exterior derivative operators, when either acts on a differential form field twice or more consecutively, the former has virtually no effect (for an even number of applications), while the latter yields zero directly. Consequently, considering either operator in isolation ultimately provides limited transformation of the field. To expand the effects of these operators on differential forms, an effective approach is to consider all possible non-zero combinations, such as the codifferential operator [3]. The central question addressed in this paper is: what are the general combination rules governing these operators?

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2. Combination Operators Preserving the Degree of Differential Forms

In this section, we explore in detail how to construct new quantities whose components change in magnitude while preserving the degree of differential forms by applying the Hodge star operator and exterior derivative operator. We then present the general rules for combinations of two and three Hodge star operators and exterior derivative operators.

Consider an arbitrary p-form field A on an n-dimensional spacetime manifold. Following standard conventions, we denote the differential form as

$$A = \frac{1}{p!} A_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

where $A_{\mu_1 \cdots \mu_p}$ is a completely antisymmetric tensor of rank p. For convenience, throughout this paper we denote the Hodge star operator and exterior derivative operator as P_1 and P_2 respectively, referring to them collectively as fundamental operators. The subscripts "1" and "2" correspond to the operators "$*$" and "d" . Their actions on a p-form field A yield its dual differential form and exterior derivative:

$$P_1 A = *A = \frac{1}{p!(n-p)!} A_{\mu_1 \cdots \mu_p} \varepsilon^{\mu_1 \cdots \mu_p}{}_{\nu_{p+1} \cdots \nu_n} dx^{\nu_{p+1}} \wedge \cdots \wedge dx^{\nu_n}$$

$$P_2 A = dA = \frac{1}{p!} \partial_{[\nu} A_{\mu_1 \cdots \mu_p]} dx^{\nu} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

Here and throughout, we set $q = n - p$. The n-th order completely antisymmetric covariant Levi-Civita tensor is defined as $\varepsilon_{\mu_1 \cdots \mu_n} = \sqrt{-g}\, \tilde{\varepsilon}_{\mu_1 \cdots \mu_n}$, where g is the determinant of the spacetime metric. The partial derivative ∂_μ in the second equation can be replaced by the usual covariant derivative operator ∇_μ, satisfying $\nabla_\mu \nabla_\nu A_{\mu_1 \cdots \mu_p} = \nabla_{[\mu} \nabla_{\nu]} A_{\mu_1 \cdots \mu_p}$. Consequently, any differential form yields zero after two consecutive applications of the exterior derivative operator: $d^2 A = 0$.

We first consider how to construct new p-form fields with substantially changed components by applying the Hodge star operator and exterior derivative operator, either individually or in combination. Selecting any two of these operators yields four basic combination patterns:

$$O_1 = P_1 P_1, \quad O_2 = P_2 P_2, \quad O_3 = P_1 P_2, \quad O_4 = P_2 P_1$$

The operators $P_1 P_2$ and $P_2 P_1$ acting on a p-form field produce $(q-1)$- and $(q+1)$-forms respectively, differing in degree by 2. One can verify that $P_1 P_1$ and $P_2 P_2$ commute and that $P_2 P_2$ acting on any form yields zero. Thus, by applying $P_1 P_1$ to A and using the product relation for two Levi-Civita tensors and the contraction relations for generalized Kronecker delta symbols, we obtain

$$P_1 P_1 A = c_1 A, \quad \text{where} \quad c_1 = \operatorname{sgn}(g)(-1)^{p(n-p)}$$

The generalized Kronecker delta symbol is defined as $\delta^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_m} = m!\delta^{[\mu_1}{}_{\nu_1}\cdots\delta^{\mu_m]}{}_{\nu_m}$, and the contravariant Levi-Civita tensor satisfies $\varepsilon^{\mu_1\cdots\mu_n} = \operatorname{sgn}(g)/\sqrt{-g}\,\tilde{\varepsilon}^{\mu_1\cdots\mu_n}$. Equation (6) shows that $P_1 P_1$ preserves the degree of differential forms. Although it may introduce an overall sign when the coefficient $c_1 = -1$, it does not change component magnitudes. Since $P_2 P_2 A = 0$, these two operations are essentially trivial for constructing substantially different p-form fields and can be excluded.

Next, we analyze the commutation relations for $P_1 P_2$ and $P_2 P_1$:

$$P_1 P_2 A = (-1)^{p+1} P_2 P_1 A$$

If two operators generate differential forms of different degrees when their order is exchanged, we consider them non-comparable (e.g., P_1 and P_2 acting individually). Clearly, P_1 and P_2 anticommute when the spacetime dimension n is odd, but commute when n is even. That is, $P_1 P_2 = (-1)^{n+1} P_2 P_1$.

Given equations (4) and (7), one can prove that the minimal combinations from the five operators $\{1, P_1, P_2, P_1 P_2, P_2 P_1\}$ that generate substantially different p-form fields are $O_1 = P_1 P_2$ and $O_2 = P_2 P_1$. Their explicit actions on a p-form A are:

$$O_1 A = P_1 P_2 A = \frac{1}{(p-1)!} \nabla^{\mu} A_{\mu\mu_2\cdots\mu_p} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$$

$$O_2 A = P_2 P_1 A = \frac{1}{p!} \nabla_{[\mu_1} A_{\mu_2\cdots\mu_{p+1}]} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}$$

Applying P_1 to equation (9) and using (5) yields a new p-form field:

$$O_3A = P_1P_2P_1A = c_3\nabla^{\mu}\nabla^{\nu}A_{\nu\mu\mu_3\cdots\mu_p}dx^{\mu_3}\wedge\cdots\wedge dx^{\mu_p}$$

where $c_3 = \mathrm{sgn}(g)(-1)^{p(q-1)}$. This operator O_3 is related to the d' Alembertian operator. Similarly, applying P_2 to (10) gives another p-form:

$$O_4A = P_2P_1P_2A = c_4\nabla_{[\mu_1}\nabla^{\nu}A_{\nu\mu_2\cdots\mu_p]}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$$

Repeating this process generates a series of formally independent p-forms $\{A, O_3A, O_3^2A, \cdots\}$ and $\{O_1A, O_4A, O_4O_1A, \cdots\}$.

Since O_1 and O_2 preserve the degree of differential forms, we can consider their linear combination to define a new operator Ψ:

$$\Psi = \gamma_1O_1 + \gamma_2O_2$$

When $\gamma_1 = -\gamma_2 = 1$, Ψ becomes the Laplace-de Rham operator. To understand its significance, we examine its relation to the d' Alembertian. Using properties of the generalized Kronecker delta and the definition of the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$, we derive:

$$O_3A = -\Box A + \Omega$$

where $\Box = \nabla^{\mu}\nabla_{\mu}$ is the d' Alembertian, and Ω is a p-form quantity related to the Riemann tensor:

$$\Omega_{\mu_1\cdots\mu_p} = -\frac{p}{(p-1)!}R_{[\mu_1}{}^{\nu}A_{\nu\mu_2\cdots\mu_p]} + \frac{p(p-1)}{2}R_{[\mu_1\mu_2}{}^{\nu\rho}A_{\nu\rho\mu_3\cdots\mu_p]}$$

Thus, only in flat spacetime ($R^{\rho}{}_{\sigma\mu\nu} = 0$) does Ψ reduce to the d' Alembertian for arbitrary differential forms. For special choices, such as 1-forms with $R_{\mu\nu} = 0$ or p-forms satisfying $\Omega = 0$, the same reduction holds.

The interaction relationships among the five operators $\{1, P_1, P_2, O_1, O_2\}$ are summarized in Table 1 . In this table, all entries (except the first row and column) represent the sequential composition of the row' s first element with the column' s first element (the column operator acts first). Only diagonal elements preserve the degree of differential forms, confirming our earlier conclusion. The "1"represents the identity operator, and the sign coefficients c_1 and c_2 are defined as $c_1 = \mathrm{sgn}(g)(-1)^{p(n-p)}$ and $c_2 = \mathrm{sgn}(g)(-1)^{(p+1)(n-p-1)}$.

The operators O_1 and O_2 essentially perform divergence operations on differential forms, related to the codifferential $\delta = (-1)^{np+n+1} * d*$ by $O_1 = \delta$ and $O_2 = (-1)^{p+1}\delta$.

Applying the fundamental operators P_1 and P_2 to the three non-zero combinations O_1, O_2, and $P_1P_2P_1$ yields five independent composite operations with three fundamental operators:

$$P_1O_1, \quad P_2O_1, \quad P_1O_2, \quad P_2O_2, \quad O_1O_2$$

Their actions on p-form fields follow from Table 1. Only P_1O_1 and P_2O_2 produce results fundamentally distinct from O_1 and O_2.

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3. General Combination Rules for Arbitrary Hodge Star and Exterior Derivative Operators

Building on the previous section, we now comprehensively investigate the general combination rules when arbitrary numbers of Hodge star operators and exterior derivative operators act on differential form fields. We present unified expressions for all possible non-zero, formally independent composite operations and analyze their interactions and classification by the degree of generated forms.

3.1 Unified Expression for All Non-Zero Independent Combination Operators We extend the combination patterns of two and three fundamental operators to arbitrary N operators. Since P_1^2 and P_2^2 only change signs or vanish, two exterior derivatives cannot appear consecutively. Between adjacent d operators, there must be an odd number of Hodge stars (equivalently, $P_1P_2 = (-1)^{n+1}P_2P_1$). Thus, structures like $(P_1P_2)^k$ or $(P_2P_1)^k$ are permitted. Using the commutation relation (7), we can permute all P_1 operators to either end, while (4) excludes mixed P_1P_2 and P_2P_1 sequences. Therefore, any non-zero combination of N operators can be expressed in one of three forms:

$$U_1^{(m,j,k)} = (P_1P_2)^j P_1^m (P_2P_1)^k$$

$$U_2^{(m,j,k)} = (P_2P_1)^j P_2^m (P_1P_2)^k$$

$$U_3^{(m,j,k)} = (P_1P_2)^j (P_2P_1)^k$$

where $\tau = 0, 1$ indicates whether the number of P_1 operators is even or odd; m, j, k are non-negative integers satisfying $2(j+k)+m+\tau = N$. The integer j counts the total permutations needed to move all P_1 operators to the rightmost position. These three expressions are not completely independent: when $\tau = 1$, U_3 is covered by U_1 and U_2.

More specifically, when N is even, all $N/2$ non-zero independent operations can be expressed as U_1 and U_2 with $\tau = 0$:

$$\{(P_1P_2)^j P_1^m (P_2P_1)^k \mid j,k \geq 0,\, m \in \{0,1\},\, 2(j+k)+m = N\}$$

When N is odd, we require $\tau = 1$ in (22), and all formally independent non-zero combinations become:

$$\{(P_1P_2)^j (P_2P_1)^k \mid j,k \geq 0,\, 2(j+k)+1 = N\}$$

The total number of non-zero independent combinations is $(N+1)/2$. Although there are 2^N possible patterns with N operators, only $(N+1)/2$ are non-zero and independent. If sign differences are ignored, the count reduces further. For odd-dimensional spacetimes where P_1 commutes with P_1P_2 and P_2P_1, the number of independent non-zero combinations is only $(N+2)/2$.

The sign coefficient $\sigma = (-1)^{(n+1)j}$ accounts for permutations. When $j = 0$ (no P_1P_2 present), $\sigma = 1$; otherwise, $\sigma = (-1)^{n+1}$ when P_1, P_2, and P_1P_2 are mixed.

Equation (22) shows that any non-zero operation constructed from Hodge star and exterior derivative operators can be expressed using the five operators $\{1, P_1, P_2, P_1P_2, P_2P_1\}$. Using Table 1, we can compute their actions on differential forms.

Three examples illustrate (24) and (25): 1. For $N = 4$, applying P_1 and P_2 to the five $N = 3$ operations yields seven distinct non-zero operations:

$$\{P_1^2, P_2^2, P_1P_2P_1, P_2P_1P_2, (P_1P_2)^2, (P_2P_1)^2, P_1P_2P_1P_2\}$$

In odd dimensions, only five are truly independent.

2. For $N = 5$, we obtain nine independent operations:

   $$\{P_1(P_2P_1)^2, P_2(P_1P_2)^2, (P_1P_2)^2P_1, (P_2P_1)^2P_2, P_1P_2P_1P_2P_1, ...\}$$

   In odd dimensions, this reduces to six.

3. For $N = 6$, there are eleven independent combinations, with the identity operator used for simplification.

3.2 Interaction Relationships Between Operators Despite $(N+1)/2$ formally independent operations from N fundamental operators, only four types are substantially distinct when ignoring sign changes: $X_1^{(j)} = (P_1P_2)^j$, $X_2^{(j)} = (P_2P_1)^j$, $Y_1^{(j)} = P_1(P_2P_1)^j$, and $Y_2^{(j)} = P_2(P_1P_2)^j$. Here X_1, X_2 apply for even N with $j \in [0, N/2]$, while Y_1, Y_2 apply for odd N with $j \in [0, (N-1)/2]$.

The interaction relations are: - For X_i operators: $X_i^{(j)}X_i^{(k)} = X_i^{(j+k)}$ (closed algebra) - For Y_i operators: $Y_i^{(j)}Y_i^{(k)} = 0$ (nilpotent) - Mixed interactions: $Y_1^{(j)}Y_2^{(k)} + Y_2^{(k)}Y_1^{(j)} = 0$ (anticommutation)

The interactions with P_1 and P_2 are given by:

$$P_1Y_1^{(k)} = c_4X_1^{(k)}, \quad P_2Y_2^{(k)} = c_5X_2^{(k)}$$

where $c_4, c_5 = \pm 1$ depending on dimension and k. These relations, combined with Table 1, allow us to derive all interaction rules for the operators in (24) and (25).

3.3 Classification by Generated Differential Form Degree Regardless of how many times Hodge star and exterior derivative operators act on a p-form field A, the result is either zero or one of six possible degrees: $t \in \{p-1, p, p+1, q-1, q, q+1\}$ where $q = n-p$.

When the total number of fundamental operators N is even, only $(q-1)$-, p-, and $(q+1)$-forms can be generated, corresponding to the operator sets:

$$S_{q-1} = \{(P_2P_1)^j\}, \quad S_p = \{(P_1P_2)^jP_1^m\}, \quad S_{q+1} = \{(P_1P_2)^j\}$$

When N is odd, only $(p-1)$-, q-, and $(p+1)$-forms appear:

$$S_{p-1} = \{P_2(P_1P_2)^j\}, \quad S_q = \{(P_1P_2)^j\}, \quad S_{p+1} = \{P_1(P_2P_1)^j\}$$

Thus, operators preserving form degree must contain an even number of fundamental operators. The sign coefficients σ can be omitted when constructing essentially different forms.

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4. Application Example

As an application of the general theory, we consider all combinations with $N \leq 4$ acting on p-forms A and $(p-1)$-forms B to generate differential forms of the same degree.

For $N = 2, 4$, the operations generating $(q-1)$-, p-, and $(q+1)$-forms from A are:

$$W_{q-1} = \{P_2P_1\}, \quad W_p = \{1, P_1P_2, P_2P_1\}, \quad W_{q+1} = \{P_1P_2\}$$

For $N = 1, 3$, those generating $(p-1)$-, q-, and $(p+1)$-forms are:

$$W_{p-1} = \{P_2\}, \quad W_q = \{P_1\}, \quad W_{p+1} = \{P_1P_2P_1\}$$

Ignoring sign differences, there are only nine truly independent operations: $1, P_1, P_2, P_1P_2, P_2P_1, P_1P_2P_1, P_2P_1P_2, (P_1P_2)^2, (P_2P_1)^2$.

Four cases produce forms of equal degree: 1. **Both** $(p-1)$**-forms**: Operations on A and B are $\{P_2\}$ and $\{P_2P_1\}$ 2. **Both** p**-forms**: Operations are $\{1, P_1P_2\}$ and $\{P_1, P_2P_1\}$ 3. **Both** q**-forms**: Operations are $\{P_1, P_2P_1P_2\}$ and $\{P_1P_2, P_1\}$ 4. **Both** $(q+1)$**-forms**: Operations are $\{P_1P_2\}$ and $\{P_1P_2P_1\}$

For $p = 2$ (so $q = n-2$), consider a 2-form A and 1-forms B and J. The five independent $(n-1)$-forms are:

$$P_2A, \quad P_1B, \quad P_2P_1B, \quad P_1J, \quad P_2P_1J$$

We can construct equations from linear combinations of any two of these five forms. A physically meaningful selection yields Maxwell' s equations in differential form language. Taking B as the electromagnetic vector potential and J as the current, the three independent equations are:

$$d * A = \lambda_1 J, \quad d * B = \lambda_2 A, \quad d * J = \lambda_3 dB$$

These correspond to the Maxwell equations and the field equation of motion. The condition $\lambda_1 = \lambda_3$ ensures consistency, making the first and third equations identical field equations while the second defines the electromagnetic field tensor.

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5. Conclusion

This paper has comprehensively investigated the general combination rules for the Hodge star operator P_1 and exterior derivative operator P_2 (denoted as fundamental operators) acting on arbitrary differential form fields. We identified two minimal combination operators $O_1 = P_1P_2$ and $O_2 = P_2P_1$ that preserve form degree, with actions given by (11) and (12). Their linear combination Ψ (13) encompasses the Laplace-de Rham operator and the flat-spacetime d' Alembertian.

For arbitrary N fundamental operators, all $(N+1)/2$ non-zero independent operations can be expressed uniformly as U_1, U_2, U_3 in (22), or more specifically as (24) for even N and (25) for odd N. Ignoring sign differences, only four types X_1, X_2, Y_1, Y_2 in (29) are substantially distinct, with interactions given by (30)-(34).

We classified operators by their effect on form degree: even N generates $(q-1)$-, p-, and $(q+1)$-forms (37); odd N generates $(p-1)$-, q-, and $(p+1)$-forms (38).

Finally, we applied this theory to $N \leq 4$ operations on p- and $(p-1)$-forms, showing how the resulting independent $(n-1)$-forms can describe Maxwell' s equations for electromagnetic fields.

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Note: Figure translations are in progress. See original paper for figures.