# Regularity in the two-phase free boundary problems under non-standard growth conditions

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## Abstract

In this paper, we prove several regularity results for the heterogeneous, two-phase free boundary problems $\mathcal{J}_\gamma(u) = \int_\Omega \left(f(x, \nabla u) + (\lambda_+(u^+)^\gamma + \lambda_-(u^-)^\gamma) + gu\right)\mathrm{d}x \to \min$ under non-standard growth conditions. Included in such problems are heterogeneous jets and cavities of Prandtl-Batchelor type with $\gamma = 0$, chemical reaction problems with $0 < \gamma < 1$, and obstacle type problems with $\gamma = 1$. Our results hold not only in the degenerate case of $p > 2$ for $p-$Laplace equations, but also in the singular case of \$1

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## Full Text

## Preamble

### Regularity in the Two-Phase Free Boundary Problems Under Non-Standard Growth Conditions

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## Abstract

In this paper, we prove several regularity results for the heterogeneous, two-phase free boundary problems

$$J_\gamma(u) = \int (f(x, \nabla u) + (\lambda_+(u^+)^\gamma + \lambda_-(u^-)^\gamma) + gu)dx \to \min$$

under non-standard growth conditions. Included in such problems are heterogeneous jets and cavities of Prandtl-Batchelor type with $\gamma = 0$, chemical reaction problems with $0 < \gamma < 1$, and obstacle type problems with $\gamma = 1$. Our results hold not only in the degenerate case of $p > 2$ for $p$-Laplace equations, but also in the singular case of $1 < p < 2$, which are extensions of [?].

**Key words:** Free boundary problem; Two-phase; Non-standard growth; Minimizer; Regularity.

## Introduction

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ $(n \geq 2)$, and $g \in L^q(\Omega)$, $\psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\psi^+ = \max\{\pm\psi, 0\} \neq 0$ and $p \geq 2$, $q \geq n$. In [?], Leitão, de Queiroz and

Teixeira provided a complete description of the sharp regularity of minimizers to the heterogeneous, two-phase free boundary problems

$$J_\gamma(u) = \int (|\nabla u|^p + F_\gamma(u) + gu)dx \to \min,$$

over the set $\{u \in W^{1,p}(\Omega) : u - \psi \in W_0^{1,p}(\Omega)\}$, where $F_\gamma(u) = \lambda_+(u^+)^\gamma + \lambda_-(u^-)^\gamma$, $\gamma \in [0,1]$ is a parameter, $0 < \lambda_- < \lambda_+ < +\infty$, and by convention, $F_0(u) = \lambda_+\chi_{\{u>0\}} + \lambda_-\chi_{\{u\leq 0\}}$.

The lower limiting case, i.e., $\gamma = 0$, relates to jets and cavities problems. The upper case, i.e., $\gamma = 1$, relates to obstacle type problems. The intermediary problem, i.e., $0 < \gamma < 1$, can be used to model the density of certain chemical species in reaction with a porous catalyst pellet. The authors established local $C^{1,\alpha}$- and Log-Lipschitz regularities for minimizers of the functional $J_\gamma$ when $\gamma \in (0,1]$, $q > n$ and $\gamma = 0$, $q = n$ in (1) respectively, see [**?**].

Problem (1) was extended to a large class of the following heterogeneous, two-phase free boundary problems in [**?**, **?**]:

$$\int (A(|\nabla u|) + F_\gamma(u) + gu)dx \to \min,$$

over the set $\{u \in W^{1,A}(\Omega) : u - \psi \in W_0^{1,A}(\Omega)\}$, for given functions $g \in L^\infty(\Omega)$ and $\psi \in W^{1,A}(\Omega) \cap L^\infty(\Omega)$ with $\psi^+ \not\equiv 0$, where $W^{1,A}(\Omega)$ is the class of weakly differentiable functions with $\int_\Omega A(|\nabla u|)dx < \infty$. Under Lieberman' s condition on $A$, which allows for different behavior at 0 and at $\infty$, local Log-Lipschitz continuity and local $C^{1,\alpha}$-regularity of minimizers have been obtained for $\gamma = 0$ and $\gamma \in (0,1]$ respectively in the setting of Orlicz spaces, see [**?**, **?**].

The aim of this paper is to study the heterogeneous, two-phase free boundary problems

$$J_\gamma(u) = \int (f(x, \nabla u) + F_\gamma(u) + gu)dx \to \min,$$

over the set $\{u \in W^{1,p(\cdot)}(\Omega) : u - \psi \in W_0^{1,p(\cdot)}(\Omega)\}$ in the framework of Sobolev spaces with variable exponents, where $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function having the form:

$$L^{-1}|z|^{p(x)} \leq f(x,z) \leq L(1 + |z|^{p(x)}), \quad \forall x \in \Omega, z \in \mathbb{R}^n,$$

with $p : \Omega \to (1, +\infty)$ a continuous function and $L \geq 1$ a constant. We establish local Log-Lipschitz continuity and local $C^{1,\alpha}$-regularity for minimizers of $J_\gamma$ with $\gamma = 0$ and $\gamma \in (0,1]$ respectively.

To the knowledge of the author, the present paper seems to be a first regularity result for the heterogeneous, two-phase free boundary problems (2) with $p(x)$-growth. It should be mentioned that a large class of functionals and identical obstacle problems under non-standard growth conditions have been studied in [**?**, **?**], which provide the reference estimates and suitable localization and freezing techniques to treat the nonstandard growth exponents in the functional governed by (2).

The results obtained in this paper are not only extensions of one-phase obstacle problems under non-standard growth conditions (see, e.g., [**?**, **?**]), but also a supplement of the degenerate two-phase free boundary problems studied in [**?**], since our results contain the singular case of $1 < p < 2$.

The rest of this paper is organized as follows. In Section 2, we present some basic notations, definitions, assumptions, and the main results obtained in this paper, including existence and $L^\infty$-boundedness results (Theorem 2.1), and local Hölder, $C^{1,\alpha}$- and Log-Lipschitz regularities of minimizers (Theorem 2.2-2.4). In Section 3, we carry out the existence and $L^\infty$-boundedness for minimizers of the functional $J_\gamma$ ($\gamma \in [0,1]$). In Section 4, we establish the higher integrability for minimizers of the functional $J_\gamma$ ($\gamma \in [0,1]$). In Section 5, we address local $C^{0,\alpha}$-regularity for minimizers of the functional $\int (h(\nabla u) + F_\gamma(u) + gu)dx$ with $\gamma \in [0,1]$ (Theorem 2.2), where $h$ satisfies certain non-standard growth conditions. In Section 6, we prove local $C^{0,\alpha}$-regularity for minimizers of the functional $J_\gamma$ ($\gamma \in [0,1]$) (Theorem 2.3). In Section 7 and 8, we establish local $C^{1,\alpha}$-regularity for minimizers of the functional $J_\gamma$ ($\gamma \in (0,1]$) and local Log-Lipschitz continuity for minimizer of $J_0$ (Theorem 2.4) respectively.

## 2 Preliminaries and Statements

In this paper, $\Omega$ will denote an open bounded domain in $\mathbb{R}^n$ ($n \geq 2$) and $B_R(x)$ the open ball $\{y \in \mathbb{R}^n : |x-y| < R\}$ with centre $x \in \mathbb{R}^n$. If $u$ is an integrable function defined on $B_R(x)$, we will set $(u)_{x,R} = \frac{1}{|B_R(x)|}\int_{B_R(x)} u(x)dx$, where $|B_R(x)|$ is the Lebesgue measure of $B_R(x)$. Without confusion, we will write $B_R$ and $(u)_R$ instead of $B_R(x)$ and $(u)_{x,R}$ respectively. We may write $C$ or $c$ as a constant that may be different from each other, but independent of $\gamma$.

Let $p : \Omega \to (1, +\infty)$ be a continuous function. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u|^{p(x)} dx < +\infty \right\},$$

with the norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \lambda > 0; \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

Define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. We point out that, if $\Omega$ is bounded and $p(\cdot)$ satisfies (8), then the spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are all separable and reflexive Banach spaces. $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ is an equivalent norm on $W_0^{1,p(\cdot)}(\Omega)$. We refer to [**?**] for more details of the space $W^{1,p(\cdot)}(\Omega)$.

In this paper, we consider the following growth, ellipticity and continuity conditions:

$$f : \Omega\times\mathbb{R}^n \to \mathbb{R}, \quad f(x,z) \text{ is } C^2\text{-continuous in } x \text{ and } z, \text{ and convex in } z \text{ for every } x,$$

$$L^{-1}(\mu^2 + |z|^2)^{\frac{p(x)}{2}} \le f(x,z) \le L(\mu^2 + |z|^2)^{\frac{p(x)}{2}},$$

$$|f(x,z) - f(x_0,z)| \le L\omega(|x-x_0|)\left[(\mu^2+|z|^2)^{\frac{p(x)}{2}} + (\mu^2+|z|^2)^{\frac{p(x_0)}{2}}\right]\left[1+\log(\mu^2+|z|^2)\right],$$

for all $z \in \mathbb{R}^n$, $x$ and $x_0 \in \Omega$, where $L \ge 1$, $\mu \in [0,1]$, $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function, vanishing at zero, which represents the modulus of continuity of $p$, $|p(x) - p(y)| \le \omega(|x-y|)$ for all $x, y \in \Omega$, and satisfying $\limsup_{R\to 0} \omega(R)\log\left(\frac{1}{R}\right) < +\infty$, thus without loss of generality, assume that $\omega(R) \le L|\log R|^{-1}$, for all $R < 1$. Moreover, we assume that

$$1 < p^- = \inf_{x\in\Omega} p(x) \le p(x) \le \sup_{x\in\Omega} p(x) = p^+ < +\infty \quad \forall x \in \Omega.$$

Let $q : \Omega \to (1, +\infty)$ be a continuous function fulfilling the conditions of the type (6) and (7). We always make the following assumptions on $p(\cdot)$ and $q(\cdot)$:

$$q(x) \ge q^- \quad \forall x \in \Omega, \quad q^- > \frac{np^-}{n-p^-} \text{ if } p^- < 2, \quad q^- \ge \frac{np^-}{n-p^-} \text{ if } p^- \ge 2.$$

Given $\psi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, let

$$\mathcal{K} = \{u \in W^{1,p(\cdot)}(\Omega); u - \psi \in W_0^{1,p(\cdot)}(\Omega)\}.$$

We say that a function $u \in \mathcal{K}$ is a minimizer of the functional $J_\gamma(u)$ governed by (2) if $J_\gamma(u) \leq J_\gamma(v)$ for all $v \in \mathcal{K}$.

The first result obtained in this paper concerns the existence and $L^\infty$-boundedness of minimizers of $J_\gamma(u)$ governed by (2).

**Theorem 1.** Under assumptions (3)-(9), for each $0 \leq \gamma \leq 1$, there exists a minimizer $u_\gamma \in \mathcal{K}$ of the functional $J_\gamma(u)$ governed by (2). Furthermore, $u_\gamma$ is bounded. More precisely,

$$\|u_\gamma\|_{L^\infty(\Omega)} \leq C(n, L, q^-, p^\pm, \lambda^\pm, \Omega, \|\psi\|_{L^\infty(\partial\Omega)}, \|g\|_{L^{q(\cdot)}(\Omega)}).$$

Now let

$$H_\gamma(u) = \int (h(\nabla u) + F_\gamma(u) + gu)dx,$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a $C^2$-continuous and convex function satisfying for all $z \in \mathbb{R}^n$,

$$L^{-1}(\mu^2 + |z|^2)^{\frac{p(x)}{2}} \leq h(z) \leq L(\mu^2 + |z|^2)^{\frac{p(x)}{2}}.$$

We present then the regularity properties of minimizers of the functionals $H_\gamma$ and $J_\gamma$.

**Theorem 2.** Assume that (11) and (6)-(9) hold. If $u_\gamma \in \mathcal{K}$ is a minimizer of the functional $H_\gamma$ ($\gamma \in [0,1]$) governed by (10), then $u_\gamma \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0,1)$.

**Theorem 3.** Assume that (3)-(9) hold. If $u_\gamma \in \mathcal{K}$ is a minimizer of the functional $J_\gamma$ ($\gamma \in [0,1]$) governed by (2), then $u_\gamma \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0,1)$.

**Theorem 4.** Assume that (3)-(9) hold, and assume further that $\omega(R) \leq LR^\varsigma$ for some $\varsigma > \frac{n}{p^- - 1}$ and all $R \leq 1$. The following statements hold true:

(i) For each $\gamma \in (0,1]$, every minimizer $u_\gamma$ of the functional $J_\gamma$ governed by (2) is $C^{1,\alpha}_{\text{loc}}$-continuous for some $\alpha \in (0,1)$.

(ii) For each $\gamma = 0$, every minimizer $u_0$ of the functional $J_0$ governed by (2) is locally Log-Lipschitz continuous in $\Omega$, and therefore is $C^{0,\alpha}_{\text{loc}}$-continuous for any $\alpha \in (0,1)$.

## 3 Existence and $L^\infty$-boundedness of minimizers

In this section, we establish the existence and $L^\infty$-boundedness for minimizers of the functional $J_\gamma$ ($\gamma \in [0,1]$).

**Proof of Theorem 1.** Firstly we consider the existence of a minimizer of the functional $J_\gamma$. Let $I_0 = \min\{J_\gamma(u) : u \in \mathcal{K}\}$.

Initially we claim that $I_0 > -\infty$. Indeed, for any $u \in \mathcal{K}$, by Poincaré's inequality there exists a positive constant $C = C(n, p^\pm, \Omega)$ such that

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le \|u-\psi\|_{L^{p(\cdot)}(\Omega)}+\|\psi\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u-\nabla\psi\|_{L^{p(\cdot)}(\Omega)}+\|\psi\|_{L^{p(\cdot)}(\Omega)} \le C\big(\|\nabla u\|_{L^{p(\cdot)}(\Omega)}+\|\nabla\psi\|_{L^{p(\cdot)}(\Omega)}+\|\psi\|_{L^{p(\cdot)}(\Omega)}$$

which implies

$$\|\nabla u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \ge C_1\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} - \|\psi\|^{p^-}_{L^{p(\cdot)}(\Omega)} - \|\nabla\psi\|^{p^-}_{L^{p(\cdot)}(\Omega)},$$

$$\|\nabla u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \ge C_2\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} - \|\psi\|^{p^+}_{L^{p(\cdot)}(\Omega)} - \|\nabla\psi\|^{p^+}_{L^{p(\cdot)}(\Omega)},$$

where $C_1, C_2$ are positive constants depending only on $n, p^\pm, \Omega$.

Due to $q(x) \ge q^-$, we deduce by (9) and Hölder’ s inequality that

$$\left|\int_\Omega gu\,dx\right| \le C_3(p^+,p^-)\|g\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)}\|u\|_{L^{p(\cdot)}(\Omega)} \le C_4(p^+,p^-)\|g\|_{L^{q(\cdot)}(\Omega)}\|1\|_{L^{\frac{q(\cdot)}{q(\cdot)-1}}(\Omega)}\|u\|_{L^{p(\cdot)}(\Omega)}$$

$$\le C_4(p^+,p^-)(1+|\Omega|)\|g\|_{L^{q(\cdot)}(\Omega)}\|u\|_{L^{p(\cdot)}(\Omega)} \le \varepsilon\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} + C_5(\varepsilon,p^\pm,\Omega)\|g\|^{\frac{p^-}{p^--1}}_{L^{q(\cdot)}(\Omega)},$$

or

$$\le \varepsilon\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} + C_6(\varepsilon,p^\pm,\Omega)\|g\|^{\frac{p^+}{p^+-1}}_{L^{q(\cdot)}(\Omega)},$$

where in the last inequality we used Young’ s inequality and $\varepsilon \in (0,1)$ will be chosen later.

Now we consider two cases: (i) $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} > 1$, and (ii) $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \le 1$.

**(i)** If $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} > 1$, it follows from (4), (13) and (16) that

$$J_\gamma(u) \ge L^{-1}\int_\Omega |\nabla u|^{p(x)}dx - \left|\int_\Omega gu\,dx\right| \ge L^{-1}\|\nabla u\|^{p^-}_{L^{p(\cdot)}(\Omega)} - \varepsilon\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} - C_5(\varepsilon,p^\pm,\Omega)\|g\|^{\frac{p^-}{p^--1}}_{L^{q(\cdot)}(\Omega)}$$

$$\geq L^{-1}C_1\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}-L^{-1}(\|\psi\|^{p^-}_{L^{p(\cdot)}(\Omega)}+\|\nabla\psi\|^{p^-}_{L^{p(\cdot)}(\Omega)})-\varepsilon\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}-C_5(\varepsilon,p^{\pm},\Omega)\|g\|^{\frac{p^-}{p^--1}}_{L^{q(\cdot)}(\Omega)}.$$

Choose $\varepsilon\in(0,1)$ such that $L^{-1}C_1-\varepsilon>0$, then (18) yields

$$J_\gamma(u)>-L^{-1}(\|\psi\|^{p^-}_{L^{p(\cdot)}(\Omega)}+\|\nabla\psi\|^{p^-}_{L^{p(\cdot)}(\Omega)})-C_5(\varepsilon,p^{\pm},\Omega)\|g\|^{\frac{p^-}{p^--1}}_{L^{q(\cdot)}(\Omega)}>-\infty.$$

**(ii)** If $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}\leq 1$, we estimate by (4), (14) and (16)

$$J_\gamma(u)\geq L^{-1}\int_\Omega|\nabla u|^{p(x)}dx-\left|\int_\Omega gu\,dx\right|\geq L^{-1}\|\nabla u\|^{p^+}_{L^{p(\cdot)}(\Omega)}-\varepsilon\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}-C_6(\varepsilon,p^{\pm},\Omega)\|g\|^{\frac{p^+}{p^+-1}}_{L^{q(\cdot)}(\Omega)}$$

$$\geq L^{-1}C_2\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}-L^{-1}(\|\psi\|^{p^+}_{L^{p(\cdot)}(\Omega)}+\|\nabla\psi\|^{p^+}_{L^{p(\cdot)}(\Omega)})-\varepsilon\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}-C_6(\varepsilon,p^{\pm},\Omega)\|g\|^{\frac{p^+}{p^+-1}}_{L^{q(\cdot)}(\Omega)}.$$

Choose $\varepsilon\in(0,1)$ such that $L^{-1}C_2-\varepsilon>0$, then (19) gives

$$J_\gamma(u)>-L^{-1}(\|\psi\|^{p^+}_{L^{p(\cdot)}(\Omega)}+\|\nabla\psi\|^{p^+}_{L^{p(\cdot)}(\Omega)})-C_6(\varepsilon,p^{\pm},\Omega)\|g\|^{\frac{p^+}{p^+-1}}_{L^{q(\cdot)}(\Omega)}>-\infty.$$

Let us now prove existence of a minimizer of $J_\gamma(u)$. Let $u_j\in\mathcal{K}$ be a minimizing sequence. We shall show that $\{u_j-\psi\}$ (up to a subsequence) is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Without loss of generality, assume that $\|\nabla u_j\|_{L^{p(\cdot)}(\Omega)}>1$ (If not, then $\|\nabla u_j\|_{L^{p(\cdot)}(\Omega)}\leq 1$, which implies $\|u_j-\psi\|_{L^{p(\cdot)}(\Omega)}\leq C\|\nabla u_j-\nabla\psi\|_{L^{p(\cdot)}(\Omega)}\leq C+C\|\nabla\psi\|_{L^{p(\cdot)}(\Omega)}<\infty$). Now for $j\gg 1$, $J_\gamma(u_j)\leq I_0+1$. From (17), (15) and (12) and applying Young' s inequality with $\varepsilon$, we derive

$$\|\nabla u_j\|^{p^-}_{L^{p(\cdot)}(\Omega)}\leq\int_\Omega|\nabla u_j|^{p(x)}dx\leq LJ_\gamma(u_j)+L\left|\int_\Omega gu_j\,dx\right|\leq L(I_0+1)+LC_7(p^{\pm},\Omega,\|g\|_{L^{q(\cdot)}(\Omega)})\|u_j\|_{L^{p(\cdot)}(\Omega)}$$

$$\leq C_8(\|\nabla u\|_{L^{p(\cdot)}(\Omega)}+\|\nabla\psi\|_{L^{p(\cdot)}(\Omega)}+\|\psi\|_{L^{p(\cdot)}(\Omega)})+L(I_0+1)\leq\frac{1}{2}\|\nabla u_j\|^{p^-}_{L^{p(\cdot)}(\Omega)}+C_9(1+\|\nabla\psi\|_{L^{p(\cdot)}(\Omega)}+\|\psi\|_{L^{p(\cdot)}(\Omega)}),$$

where $C_8,C_9$ depend only on $L,I_0,p^{\pm},\Omega,\|g\|_{L^{q(\cdot)}(\Omega)}$. Therefore, we get

$$\|\nabla u_j\|^{p^-}_{L^{p(\cdot)}(\Omega)}\leq 2C_9(1+\|\nabla\psi\|_{L^{p(\cdot)}(\Omega)}+\|\psi\|_{L^{p(\cdot)}(\Omega)}).$$

Thus, using Poincaré inequality once more, we deduce that $\{u_j - \psi\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. By reflexivity, there is a function $u \in \mathcal{K}$ such that, up to a subsequence,

$$u_j \rightharpoonup u \quad \text{weakly in } W^{1,p(\cdot)}(\Omega), \quad u_j \to u \quad \text{in } L^{p(\cdot)}(\Omega), \quad u_j \to u \quad \text{a.e. in } \Omega.$$

With a slight modification of [**?**, Theorem 1.6], we deduce from (3) and (4) that

$$\int_\Omega f(x, |\nabla u|)dx \leq \liminf_{j\to\infty} \int_\Omega f(x, |\nabla u_j|)dx.$$

By pointwise convergence we have, in the case of $0 < \gamma \leq 1$,

$$\int_\Omega (F_\gamma(u) + gu)dx \leq \liminf_{j\to\infty} \int_\Omega (F_\gamma(u_j) + gu_j)dx.$$

For $\gamma = 0$, recalling that $\lambda_+ > \lambda_- > 0$, we have

$$\int_\Omega \lambda_- \chi_{\{u\leq 0\}}dx = \int_{\{u\leq 0\}} \lambda_- \chi_{\{u_j>0\}}dx + \int_{\{u\leq 0\}} \lambda_- \chi_{\{u_j\leq 0\}}dx \leq \int_{\{u\leq 0\}} \lambda_+ \chi_{\{u_j>0\}}dx + \int_{\{u\leq 0\}} \lambda_- \chi_{\{u_j\leq 0\}}dx,$$

which implies

$$\int_\Omega \lambda_- \chi_{\{u\leq 0\}}dx \leq \liminf_{j\to\infty} \left( \int_{\{u\leq 0\}} \lambda_+ \chi_{\{u_j>0\}}dx + \int_{\{u\leq 0\}} \lambda_- \chi_{\{u_j\leq 0\}}dx \right).$$

On the other hand, since $u_j \to u$ a.e. in $\Omega$, it follows from the Dominated Convergence Theorem that

$$\int_{\{u>0\}} \lambda_+ \chi_{\{u>0\}}dx = \int_{\{u>0\}} \lambda_+ (\lim_{j\to\infty} \chi_{\{u_j>0\}})dx = \lim_{j\to\infty} \int_{\{u>0\}} \lambda_+ \chi_{\{u_j>0\}}dx.$$

Hence

$$\int_\Omega (F_0(u) + gu)dx \leq \liminf_{j\to\infty} \int_\Omega (F_0(u_j) + gu_j)dx.$$

Now from (20), (21) and (22) we conclude that

$$J_\gamma(u) \le \liminf_{j\to\infty} J_\gamma(u_j) = I_0, \quad \forall 0 \le \gamma \le 1,$$

which proves the existence of a minimizer under the condition of $g \in L^{q(\cdot)}(\Omega)$.

Secondly, we establish the $L^\infty$-boundedness of $u_\gamma$, provided $g \in L^{q(\cdot)}(\Omega)$. Hereafter in this proof we will refer to $u_\gamma$ as $u$.

Let $j_0 := \lceil \sup_\Omega \psi \rceil$ be the smallest natural number above $\sup_\Omega \psi$. For each $j \ge j_0$, we define the truncated function $u_j : \Omega \to \mathbb{R}$ by

$$u_j = \begin{cases} j \cdot \operatorname{sgn}(u), & \text{if } |u| > j, \\ u, & \text{if } |u| \le j, \end{cases}$$

where $\operatorname{sgn}(u) = 1$ if $u \ge 0$ and $\operatorname{sgn}(u) = -1$ if $u < 0$. Define the set $A_j := \{|u| > j\}$. For $0 < \gamma \le 1$, in view of the minimality of $u$, we derive

$$\int_\Omega f(x, \nabla u)dx = \int_\Omega (f(x,\nabla u) - f(x, \nabla u_j))dx + \int_\Omega f(x, \nabla u_j)dx \le \int_\Omega g(u_j - u)dx + \int_\Omega \lambda_+((u_j^+)^\gamma - (u^+)^\gamma)dx + \int_\Omega \lambda_-((

For $\gamma = 0$ it suffices to notice that $u_j > 0$ and $u$ have the same sign. By the choice of the truncated function, we know that $(|u|-j)^+ \in W_0^{1,p(\cdot)}(A_j)$. Applying Hölder' s inequality and embedding theorem, we find

$$\int_{A_j} |g|(|u|-j)^+ dx \leq 2\|g\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(A_j)} \|(|u|-j)^+\|_{L^{p(\cdot)}(A_j)} \leq C\|g\|_{L^{q(\cdot)}(A_j)}\|1\|_{L^{t(\cdot)}(A_j)}\|(|u|-j)^+\|_{L^{p^*(\cdot)}(A_j)}\|1\|^{\frac{1}{n}}_{L^n(A_j)}$$

$$\leq C\|g\|_{L^{q(\cdot)}(\Omega)}|A_j|^{\frac{1}{t^-}}\|\nabla(|u|-j)^+\|_{L^{p(\cdot)}(A_j)} \leq C\|g\|_{L^{q(\cdot)}(\Omega)}|A_j|^{\frac{1}{n}}\|\nabla u\|_{L^{p(\cdot)}(A_j)},$$

where $t(x) = \frac{p(x)}{p(x)-1}$ and $p^*(\cdot) = \frac{np(\cdot)}{n-p(\cdot)}$, $t^- = \inf_{x\in A_j} t(x)$, $t^+ = \sup_{x\in A_j} t(x)$. The constant $C$ in the last inequality depends only on $p^{\pm}, q^-, n, \Omega, \|g\|_{L^{q(\cdot)}(\Omega)}$.

Collecting (23)-(26), we obtain

$$\int_{\Omega} f(x, \nabla u)dx \leq C|A_j|^{\frac{1}{n}}\|\nabla u\|_{L^{p(\cdot)}(A_j)} + L|A_j|,$$

where $C$ depends only on $p^{\pm}, q^-, n, \Omega, \|g\|_{L^{q(\cdot)}(\Omega)}$.

Now we consider two cases: (i) $\|\nabla u\|_{L^{p(\cdot)}(A_j)} > 1$, and (ii) $\|\nabla u\|_{L^{p(\cdot)}(A_j)} \leq 1$.

**(i)** If $\|\nabla u\|_{L^{p(\cdot)}(A_j)} > 1$, we estimate by (4), (27) and Young' s inequality

$$\|\nabla u\|^{p^-}_{L^{p(\cdot)}(A_j)} \leq \int_{A_j} |\nabla u|^{p(x)}dx \leq L\int_{A_j} f(x, \nabla u)dx \leq L\big(C|A_j|^{\frac{1}{n}}\|\nabla u\|_{L^{p(\cdot)}(A_j)} + L|A_j|\big) \leq \frac{1}{2}\|\nabla u\|^{p^-}_{L^{p(\cdot)}(A_j)} + C|A_j|^{\frac{p^-}{p^--1}}$$

which implies

$$\|\nabla u\|^{p^-}_{L^{p(\cdot)}(A_j)} \leq C|A_j|^{\frac{p^-}{p^--1}} + 2L^2|A_j|.$$

Therefore

$$\|\nabla u\|_{L^{p(\cdot)}(A_j)} \leq C|A_j|^{\frac{1}{p^--1}} + C|A_j|^{\frac{1}{p^-}} \leq C|A_j|^{\frac{1}{p^--1}},$$

where $C$ depends only on $L, p^{\pm}, q^-, n, \Omega, \|g\|_{L^{q(\cdot)}(\Omega)}$.

On the other hand, by an analogue argument as (26) and Young' s inequality, we obtain

$$\int_{A_j} (|u|-j)^+ dx \leq 2\|1\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(A_j)} \|(|u|-j)^+\|_{L^{p(\cdot)}(A_j)} \leq C|A_j|^{\frac{1}{n}}\|\nabla u\|_{L^{p(\cdot)}(A_j)} \leq C|A_j|^{\frac{1}{n}+\frac{1}{p^--1}}.$$

**(ii)** If $\|\nabla u\|_{L^{p(\cdot)}(A_j)} \leq 1$, analogously, we deduce that

$$\int_{A_j} (|u| - j)^+ dx \leq C|A_j|^{\frac{1}{n} + \frac{1}{p^+ - 1}}.$$

Now combining (29) and (30), we get

$$\int_{A_j} (|u| - j)^+ dx \leq C|A_j|^{1+\varepsilon_0},$$

where $\varepsilon_0 = \min\{\frac{1}{n}, \frac{1}{p^+-1}\} > 0$ and $C$ depends only on $L, p^\pm, q^-, n, \Omega, \|g\|_{L^{q(\cdot)}(\Omega)}$. Notice also that $\|u\|_{L^1(A_{j_0})} \leq \|u\|_{L^{p(x)}(A_{j_0})} \leq C$. Applying [**?**, Lemma 5.1], we obtain the desired result.

**Remark 1.** Note that in [**?**], the assumption that $\int_\Omega |\nabla u|^{p(x)} dx \leq M$ with some $M \geq 0$ is assumed in the establishment of local regularity for minimizers of a functional with the form $\int_\Omega f(x, u, \nabla u)dx$, while in this paper, we can show that any minimizer $u_\gamma$ of $J_\gamma(u)$ governed by (2) is uniformly bounded in $W^{1,p(\cdot)}(\Omega)$ by $L^\infty$-estimates of $u_\gamma$. Indeed, we have

$$\int_\Omega |\nabla u_\gamma|^{p(x)} dx \leq L \int_\Omega f(x, \nabla u_\gamma) dx \leq L(J_\gamma(\psi) - \int_\Omega F(u_\gamma)dx + \int_\Omega |gu_\gamma|dx) \leq LJ_\gamma(\psi) + C(L, n, p^\pm, \lambda^\pm, \Omega, \|\psi\|_{L^\infty(\partial\Omega}$$

where $M = M(L, n, q^-, p^\pm, \lambda^\pm, \Omega, \|\psi\|_{L^\infty(\partial\Omega)}, \|g\|_{L^{q(\cdot)}(\Omega)})$ is a positive constant. Therefore, we conclude by $u_\gamma - \psi \in W_0^{1,p(\cdot)}(\Omega)$ that $\|u_\gamma\|_{W^{1,p(\cdot)}(\Omega)} \leq C$, where $C$ is independent of $\gamma$.

## 4 High integrability

In this section we prove a higher integrability result for minimizers of functional in (2).

**Proposition 5.** Assume that (3)-(9) hold. Let $u \in \mathcal{K}$ be a minimizer of the functional $J_\gamma$ governed by (2). Then there exist two positive constants $C_0$ and $\delta_0 < q^-(1 - \frac{1}{p^-}) - 1$, both depending only on $n, p^\pm, \lambda^\pm, q^-, L, M, \Omega$, such that

$$\left(\frac{1}{|B_{R/2}|}\int_{B_{R/2}} |\nabla u|^{p(x)(1+\delta_0)} dx\right)^{\frac{1}{1+\delta_0}} \leq C_0 \left(\frac{1}{|B_R|}\int_{B_R} |\nabla u|^{p(x)} dx + C_0 \left(\frac{1}{|B_R|}\int_{B_R} (1 + |g|)^{\frac{p^-}{p^- - 1}(1+\delta_0)} dx\right)^{\frac{1}{1+\delta_0}}\right),$$

for all $B_R \Subset \Omega$.

In order to prove Proposition 4.1, we need the following iteration lemma.

**Lemma 6.** [?] Let $0 < \theta < 1$, $A > 0$, $B \geq 0$, $1 < p^- \leq p(x) \leq p^+ < +\infty$, and let $f \geq 0$ be a bounded function on $(r, R)$ satisfying

$$f(t) \leq \theta f(s) + A \int_{B_R} \frac{|h|^{p(x)}}{(s-t)^{p(x)}} dx + B, \quad \forall r \leq t < s \leq R,$$

where $h \in L^{p(\cdot)}(B_R)$. Then there exists a constant $C = C(\theta, p^+)$ such that

$$f(r) \leq C \int_{B_R} \frac{|h|^{p(x)}}{(R-r)^{p(x)}} dx + B.$$

**Proof of Proposition 5.** Let $0 < R < R_0 \leq 1$ and let $x_0 \in B_R$ with $B_{R_0}(x_0) \subset \Omega$. Let $t, s \in \mathbb{R}$ with $\frac{R}{2} < t < s < R$. Let $\eta \in C_0^\infty(B_s)$, $0 \leq \eta \leq 1$, be a cut-off function with $\eta \equiv 1$ on $B_t$, $\eta \equiv 0$ outside $B_s$ and $|\nabla \eta| \leq \frac{2}{s-t}$. We define the function $z = u - \eta(u - (u)_R)$. We deduce from (4) and minimality of $u$ that

$$\int_{B_s} |\nabla u|^{p(x)} dx \leq \int_{B_s} f(x, \nabla u) dx \leq \int_{B_s} f(x, \nabla z) dx + \int_{B_s} (F_\gamma(z) - F_\gamma(u)) dx + \int_{B_s} g(z-u) dx$$

$$\leq L \int_{B_s} (\mu^2 + |\nabla z|^2)^{\frac{p(x)}{2}} dx + \int_{B_s} (F_\gamma(z) - F_\gamma(u)) dx + \int_{B_s} g(z-u) dx,$$

where in the last but one inequality we used the fact that if $\phi \in W_0^{1,p(\cdot)}(\Omega)$ with $\operatorname{spt} \phi \Subset \Omega$, then there holds

$$\int_{\operatorname{spt} \phi} (f(x, \nabla u) + F_\gamma(u) + gu) dx \leq \int_{\operatorname{spt} \phi} (f(x, \nabla u + \nabla \phi) + F_\gamma(u+\phi) + g(u+\phi)) dx.$$

Indeed, it follows from the minimality of $u$ that

$$\int_{\operatorname{spt} \phi} (f(x, \nabla u) + F_\gamma(u) + gu) dx + \int_{\Omega \setminus (\operatorname{spt} \phi)} (f(x, \nabla u) + F_\gamma(u) + gu) dx$$

$$\leq \int_{\operatorname{spt} \phi} (f(x, \nabla u + \nabla \phi) + F_\gamma(u+\phi) + g(u+\phi)) dx + \int_{\Omega \setminus (\operatorname{spt} \phi)} (f(x, \nabla u + \nabla \phi) + F_\gamma(u+\phi) + g(u+\phi)) dx.$$

Since $\phi = 0$ on $\Omega \setminus (\operatorname{spt} \phi)$, we obtain the desired inequality.

We shall estimate each integration of (32). First,

$$\int_{B_s} |\nabla z|^{p(x)}dx \leq \int_{B_s} |(1-\eta)\nabla u-\nabla\eta(u-(u)_R)|^{p(x)}dx \leq C\int_{B_s\setminus B_t} |\nabla u|^{p(x)}dx+C\int_{B_s} \left|\frac{u-(u)_R}{s-t}\right|^{p(x)} dx,$$

where $C = C(p^+, p^-)$ is a positive constant.

A direct calculus shows that

$$\int_{B_s} (F_\gamma(z)-F_\gamma(u))dx = \lambda_+\int_{B_s} ((z^+)^\gamma-(u^+)^\gamma)dx+\lambda_-\int_{B_s} ((z^-)^\gamma-(u^-)^\gamma)dx \leq C\int_{B_s} |z-u|^\gamma dx,$$

where $C = C(\lambda_+, \lambda_-)$ is a positive constant.

Then we estimate from Young' s inequality that

$$\int_{B_s} (F_\gamma(z)-F_\gamma(u))dx \leq C\int_{B_s} \left|\frac{u-(u)_R}{s-t}\right|^\gamma dx \leq C\int_{B_s} \left|\frac{u-(u)_R}{s-t}\right|^{p(x)} dx+C|B_s|.$$

Similarly,

$$\int_{B_s} |g(z-u)|dx \leq \int_{B_s} |g||u-(u)_R|dx \leq C\int_{B_s} \left|\frac{u-(u)_R}{s-t}\right|^{p(x)} dx+C\int_{B_s} (|g||s-t|)^{\frac{p(x)}{p(x)-1}}dx.$$

Combining (32)-(35), we obtain

$$\int_{B_t} |\nabla u|^{p(x)}dx \leq C\int_{B_s\setminus B_t} |\nabla u|^{p(x)}dx+C\int_{B_s} \left|\frac{u-(u)_R}{s-t}\right|^{p(x)} dx+C\int_{B_s} (1+|g|)^{\frac{p(x)}{p(x)-1}}dx,$$

where the constant $C$ depends only on $L, p^\pm, \lambda^\pm$.

Now "filling the hole" , we get

$$\int_{B_t} |\nabla u|^{p(x)}dx \leq \frac{C}{1+C}\int_{B_s} |\nabla u|^{p(x)}dx+C\int_{B_s} \left|\frac{u-(u)_R}{s-t}\right|^{p(x)} dx+C\int_{B_s} (1+|g|)^{\frac{p(x)}{p(x)-1}}dx,$$

which and Lemma 4.2 imply

$$\frac{1}{|B_{R/2}|}\int_{B_{R/2}}|\nabla u|^{p(x)}dx \le C\frac{1}{|B_R|}\int_{B_R}\left|\frac{u-(u)_R}{R-R/2}\right|^{p(x)}dx+C\frac{1}{|B_R|}\int_{B_R}(1+|g|)^{\frac{p(x)}{p(x)-1}}dx.$$

Let $p_1=\min_{x\in B_R}p(x)$, $p_2=\max_{x\in B_R}p(x)$. By Sobolev-Poincaré' s inequality, there exists $\nu<1$ such that

$$\frac{1}{|B_R|}\int_{B_R}\left|\frac{u-(u)_R}{R}\right|^{p(x)}dx \le C\left(\frac{1}{|B_R|}\int_{B_R}(1+|\nabla u|^{p(x)})dx\right)^{\nu} \le C\left(\frac{1}{|B_R|}\int_{B_R}|\nabla u|^{p_1}dx\right)^{\frac{p_2}{p_1}\nu},$$

where in the last inequality we used Remark 1 and the fact that, by (7), $R^{\frac{(p_1-p_2)n}{p_1p_2}}$ is bounded.

Combining (36) and (37), we get

$$\frac{1}{|B_{R/2}|}\int_{B_{R/2}}|\nabla u|^{p(x)}dx \le C\left(\frac{1}{|B_R|}\int_{B_R}|\nabla u|^{p(x)\nu}dx\right)^{\frac{1}{\nu}}+C\frac{1}{|B_R|}\int_{B_R}(1+|g|)^{\frac{p^-}{p^--1}}dx,$$

where $C=C(n,p^\pm,\lambda^\pm,L,M,\Omega)$. We now apply Gehring' s lemma (see [?]) to deduce that there exists $0<\delta_0<q_1(1-\frac{1}{p^-})-1$ such that (31) holds.

## 5 Hölder estimates for minimizers of functional $H_\gamma$

In this section, we establish local $C^{0,\alpha}$-regularity for minimizers of the functional $H_\gamma$ ($\gamma\in[0,1]$) governed by (10). We always let $v\in W^{1,p(\cdot)}(B_R)$ with $v-u\in W_0^{1,p(\cdot)}(B_R)$ be a minimizer of the following local integral functional

$$H_\gamma(v)=\int_{B_R(x_0)}(h(\nabla v)+F_\gamma(v)+gv)dx,\quad B_R(x_0)\Subset\Omega,$$

and let $\tilde{v}(y)=\frac{1}{R}v(x_0+Ry)$. It is easy to check that $\tilde{v}$ is a minimizer of the functional

$$\tilde{H}_\gamma(\tilde{v})=\int_{B_1(0)}[h(\nabla\tilde{v})+R^\gamma F_\gamma(\tilde{v})+Rg\tilde{v}]dy,$$

in the class $\{\tilde{v}\in W^{1,\tilde{p}(\cdot)}(B_1):\tilde{v}-u_R\in W_0^{1,\tilde{p}(\cdot)}(B_1)\}$, where $\tilde{p}(y)=p(x_0+Ry)$.

Let $p_1=\min_{x\in B_R(x_0)}p(x)$, $p_2=\max_{x\in B_R(x_0)}p(x)$.

The following lemma is a slight version of [**?**, Lemma 7.1], and can be obtained by induction in the same way as in [**?**, Lemma 7.1]. We omit the proof here.

**Lemma 7.** Let $0 < a_1 \leq a_2$ and $\{\vartheta_i\}$ be a sequence of real positive numbers, such that

$$\vartheta_{i+1} \leq CB^i(\vartheta_i^{1+a_1} + \vartheta_i^{1+a_2}),$$

with $C > 1$ and $B > 1$. If $\vartheta_0 \leq (2C)^{-\frac{1}{a_1}}$, then we have $\vartheta_i \leq B^{-\frac{i}{a_1}}\vartheta_0$, and hence in particular $\lim_{i\to\infty} \vartheta_i = 0$.

**Lemma 8.** [**?**] Let $\phi(s)$ be a non-negative and non-decreasing function. Suppose that

$$\phi(r) \leq C_1\left(\frac{r}{R}\right)^{\alpha}\phi(R) + C_2R^{\beta}, \quad \forall r \leq R \leq R_0,$$

with $0 < \beta < \alpha$, $C_1$ positive constants and $C_2, \mu$ non-negative constants. Then, for any $\sigma \leq \beta$, there exists a constant $\mu_0 = \mu_0(C_1, \alpha, \beta, \sigma)$ such that if $\mu < \mu_0$, then for all $r \leq R \leq R_0$ it follows that

$$\phi(r) \leq C_3r^{\sigma},$$

where $C_3 = C_3(C_1, C_2, R_0, \phi, \sigma, \beta)$ is a positive constant.

**Lemma 9.** If $\tilde{v}$ is a minimizer of $\tilde{H}_{\gamma}$ governed by (39), then $\tilde{v}$ is locally bounded and satisfies the estimates

$$\sup_{B_{1/2}} |\tilde{v}| \leq C\left(\left(\int_{B_1} |\tilde{v}|^{p_2}dy\right)^{\frac{1}{p_2}} + 1\right),$$

$$\sup_{B_{1/2}} \tilde{v} \leq C\left(\frac{1}{|A_{0,1}|^{\frac{1}{p_2}}}\left(\int_{B_1} (\tilde{v}^+)^{p_2}dy\right)^{\frac{1}{p_2}} + 1\right),$$

for some $\alpha > 0$, where $A_{0,1} = \{y \in B_1(0) : \tilde{v}(y) > 0\}$, $C = C(n, L, p^{\pm}, \lambda^{\pm}, q^-, M, \Omega, \|g\|_{L^{q(\cdot)}(\Omega)})$ is a positive constant.

**Proof of Lemma 9.** Without loss of generality, we may assume that $R \leq 1$. The proof proceeds in three steps.

**First step: De Giorgi type estimates.** For any $k \in \mathbb{R}$, we define the sets $A_{k,\sigma} = \{y \in B_{\sigma}(0) : \tilde{v}(y) > k\}$, $B_{k,\sigma} = \{y \in B_{\sigma}(0) : \tilde{v}(y) < k\}$.

We claim that for any $k \in \mathbb{R}$, $\tilde{v}$ satisfies the inequalities

$$\int_{A_{k,\tau}} |\nabla\tilde{v}(y)|^{\tilde{p}(y)}dy \le C_1\int_{A_{k,\sigma}}\left|\frac{\tilde{v}(y)-k}{\tau-\sigma}\right|^{\tilde{p}(y)}dy+C_2\int_{A_{k,\sigma}}(1+(|g||\tau-\sigma|)^{\frac{\tilde{p}(y)}{\tilde{p}(y)-\gamma}})dy,$$

$$\int_{B_{k,\tau}} |\nabla\tilde{v}(y)|^{\tilde{p}(y)}dy \le C_1\int_{B_{k,\sigma}}\left|\frac{\tilde{v}(y)-k}{\tau-\sigma}\right|^{\tilde{p}(y)}dy+C_2\int_{B_{k,\sigma}}(1+(|g||\tau-\sigma|)^{\frac{\tilde{p}(y)}{\tilde{p}(y)-\gamma}})dy,$$

for any $\frac{1}{2}\le\sigma<\tau\le 1$, where $C_i=C_i(L,\lambda^{\pm},p^{\pm})$. Indeed, for $\eta\in C_0^{\infty}(B_\tau)$ with $\operatorname{spt}\eta\subset B_\sigma$, $0\le\eta\le 1$, $\eta\equiv 1$ on $B_s(0)$, $|\nabla\eta|\le\frac{2}{\tau-\sigma}$ be a standard cut-off function. Set $\tilde{z}(y)=\tilde{v}(y)-\eta\tilde{w}(y)$, where $\tilde{w}(y)=\max\{\tilde{v}(y)-k,0\}$. In view of minimality of $\tilde{v}$, we obtain

$$\int_{A_{k,\tau}} |\nabla\tilde{v}(y)|^{\tilde{p}(y)}dy \le \int_{A_{k,\tau}} |\nabla\tilde{z}(y)|^{\tilde{p}(y)}dy+\int_{B_1} R^{\gamma}(F_\gamma(\tilde{z})-F_\gamma(\tilde{v}))dy+\int_{B_1} Rg(\tilde{z}-\tilde{v})dy$$

$$\le\int_{A_{k,\tau}} |(1-\eta)\nabla\tilde{v}-\nabla\eta\cdot(\tilde{v}-k)|^{\tilde{p}(y)}dy+C\int_{A_{k,\tau}}\left|\frac{\tilde{v}-k}{\tau-\sigma}\right|^{\tilde{p}(y)}dy+\int_{B_1}(F_\gamma(\tilde{z})-F_\gamma(\tilde{v}))dy+\int_{B_1} g(\tilde{z}-\tilde{v})dy,$$

where $C=C(\tilde{p}_1,\tilde{p}_2)$ is a positive constant. We remark that $\tilde{p}_1=\min_{y\in B_1(0)}\tilde{p}(y)=p_1$, $\tilde{p}_2=\max_{y\in B_1(0)}\tilde{p}(y)=p_2$. Therefore $C=C(\tilde{p}_1,\tilde{p}_2)=C(p_1,p_2)$. Moreover, we can let $C$ depend only on $p^{\pm}$.

In view of (34) and (35), we derive

$$\int_{B_1}(F_\gamma(\tilde{z})-F_\gamma(\tilde{v}))dy\le C\int_{A_{k,\tau}}\left|\frac{\tilde{v}-k}{\tau-\sigma}\right|^{\tilde{p}(y)}dy,$$

$$\int_{B_1}|g(\tilde{z}-\tilde{v})|dy\le C\int_{A_{k,\tau}}\left|\frac{\tilde{v}-k}{\tau-\sigma}\right|^{\tilde{p}(y)}dy+C\int_{A_{k,\tau}}(|g||\tau-\sigma|)^{\frac{\tilde{p}(y)}{\tilde{p}(y)-1}}dy.$$

Therefore (44) becomes

$$\int_{A_{k,\tau}} |\nabla\tilde{v}(y)|^{\tilde{p}(y)}dy \le C\int_{A_{k,\sigma}} |\nabla\tilde{v}(y)|^{\tilde{p}(y)}dy+C\int_{A_{k,\sigma}}\left|\frac{\tilde{v}-k}{\tau-\sigma}\right|^{\tilde{p}(y)}dy+C\int_{A_{k,\sigma}}(|g||\tau-\sigma|)^{\frac{\tilde{p}(y)}{\tilde{p}(y)-1}}dy.$$

“Filling the hole” and using Lemma 4.2, we obtain the desired result (42). (43) follows by an analogue argument.

**Second step: Boundedness of $\tilde v$: estimate (40).** We start by showing that

$$\sup_{B_{1/2}} \tilde v \le C \left( \left( \int_{B_1} (\tilde v^+)^{p_2} dy \right)^{\frac{1}{p_2}} + 1 \right).$$

Without loss of generality we assume that $p_1 < n$, otherwise the assertion directly follows by the Sobolev Embedding Theorem.

For $\frac{1}{2} \le \rho < r \le 1$, let $\eta$ be a function of class $C_0^\infty(B_r)$, with $\eta \equiv 1$ on $B_\rho$ and $|\nabla \eta| \le \frac{4}{r-\rho}$. Denoting by $p_1^* = \frac{np_1}{n-p_1}$ the Sobolev conjugate of $p_1$, we introduce the quantities

$$\varepsilon = 1 - \frac{p_2}{p_1^*}, \quad \beta = \varepsilon + \frac{p_2}{p_1} = 1 + \frac{p_2}{p_1} - \frac{p_2}{p_1^*}, \quad \theta = \varepsilon + \frac{p_2}{p_1 - 1} = 1 + \frac{p_2}{p_1 - 1} - \frac{p_2}{p_1^*}.$$

Thanks to assumption (9), we have $p_2 \le p_1^*$, $\theta > 1$.

Now we define $\Phi_{k,\rho} = \int_{A_{k,\rho}} (\tilde v - k)^{p_2} dy$. We claim that for arbitrary $h < k$ there holds

$$\Phi_{k,\rho} \le C \Phi_{h,r}^{\beta} \left( \frac{|k-h|^{p_2}}{|r-\rho|^{p_2}} + \frac{|k-h|^{p_2 \theta}}{|r-\rho|^{p_2}} \right) + C \Phi_{h,r}^{\theta} \left( \frac{|k-h|^{p_2}}{|r-\rho|^{p_2}} + \frac{|k-h|^{p_2}}{|r-\rho|^{p_1}} \right).$$

Indeed, as in [**?**, pp. 1413], we obtain

$$\int_{A_{k,\rho}} (\tilde v - k)^{p_2} dy \le C \left( \int_{A_{k,r}} |\nabla \tilde v|^{\tilde p(y)} dy + \int_{A_{k,r}} \left| \frac{\tilde v - k}{r - \rho} \right|^{\tilde p(y)} dy \right)^{\frac{p_2}{\tilde p_1}} |A_{k,r}|^{\varepsilon},$$

where $C = C(p^+, p^-)$ is a positive constant.

Combining (42) and (47), we derive for any $k \in \mathbb{R}$

$$\int_{A_{k,\rho}} (\tilde v - k)^{p_2} dy \le C |A_{k,r}|^{\varepsilon} \left( \int_{A_{k,r}} \left| \frac{\tilde v - k}{r - \rho} \right|^{\tilde p(y)} dy + \int_{A_{k,r}} (1 + (|g||r-\rho|)^{\frac{\tilde p(y)}{\tilde p(y) - \gamma}}) dy \right)^{\frac{p_2}{\tilde p_1}} + C |A_{k,r}|^{\beta}.$$

Next, for $h < k$ we deduce from $u - h > k - h$ on $A_{k,r}$ that

$$|A_{k,r}| \le \int_{A_{k,r}} \left| \frac{\tilde v - h}{k - h} \right|^{p_2} dy \le \int_{A_{h,r}} \left| \frac{\tilde v - h}{k - h} \right|^{p_2} dy,$$

and, moreover, we have

$$\int_{A_{k,\rho}} (\tilde{v}-k)^{p_2} dy \leq \int_{A_{h,r}} (\tilde{v}-h)^{p_2} dy \leq \Phi_{h,r}.$$

By (48)-(50), we obtain

$$\Phi_{k,\rho} \leq C\Phi_{h,r}^{\beta}\left(\frac{|k-h|^{p_2}}{|r-\rho|^{p_2}}+\frac{|k-h|^{p_2\theta}}{|r-\rho|^{p_2}}\right)+C\Phi_{h,r}^{\theta}\left(\frac{|k-h|^{p_2}}{|r-\rho|^{p_2}}+\frac{|k-h|^{p_2}}{|r-\rho|^{p_1}}\right).$$

Our aim is now to deduce a decay estimate for the quantity $\Phi_{k,\rho}$ to decreasing levels $k$ on balls of increasing radii $\rho$. For this purpose we will make use of Lemma 5.1. Let us define the sequence of levels and radii

$$k_i = 2d(1-2^{-i-1}), \quad \rho_i = \frac{1}{2}(1+2^{-i}),$$

and the quantity

$$\vartheta_i = d^{-p_2}\Phi_{k_i,\rho_i} = d^{-p_2}\int_{A_{k_i,\rho_i}} (\tilde{v}-k_i)^{p_2} dy,$$

where $d \geq 1$ is a constant that will be chosen later. First, we note that $k_{i+1}-k_i = d2^{-i-1}$, $\rho_i-\rho_{i+1} = \frac{1}{4}2^{-i}$. Exploiting (46) with the choice $k = k_{i+1}$, $h = k_i$, $\rho = \rho_{i+1}$, $r = \rho_i$ and the fact that $d \geq 1$, we derive

$$\vartheta_{i+1} = d^{-p_2}\Phi_{k_{i+1},\rho_{i+1}} \leq Cd^{-p_2}\Phi_{k_i,\rho_i}^{\beta}(d2^{-i-1})^{p_2\varepsilon}\left(\frac{(4\cdot 2^i)^{p_2}}{(d2^{-i-1})^{p_2}}+\frac{(d2^{-i-1})^{p_2\theta}}{(d2^{-i-1})^{p_2}}\right)+Cd^{-p_2}\Phi_{k_i,\rho_i}^{\theta}(d2^{-i-1})^{p_2\theta}\Bigg(\frac{(4\cdot 2^{i}}{(d2^{-i-}}$$

$$\leq C(d^{-p_2}\Phi_{k_i,\rho_i})^{\beta}d^{-p_2(1-\beta)}2^{ip_2\beta}\vartheta_i^{\beta}+C(d^{-p_2}\Phi_{k_i,\rho_i})^{\theta}d^{-p_2(1-\theta)}2^{ip_2\theta}\vartheta_i^{\theta}.$$

Due to Theorem 2.1, there exists a constant $C = C(M, p_1, p_2)$ such that

$$d^{-p_2(1-\beta)} \leq C\left(1+\frac{1}{d^{p_2(p_2-p_1)}}\right).$$

Consequently, (51) becomes

$$\vartheta_{i+1} \leq c\left(1+\frac{1}{d^{p_2(p_2-p_1)}}\right)2^{ip_2(\beta+\theta)}(\vartheta_i^{\beta}+\vartheta_i^{\theta}),$$

where $c = c(\lambda^{\pm}, p^{\pm}, q^{-}, n, M, p_1, p_2, \Omega, \|g\|_{L^{q(\cdot)}(\Omega)})$ is a positive constant.

On the other hand, the choice of $d$ and the fact that $d \geq 1$ immediately yield

$$\vartheta_0 = d^{-p_2} \int_{A_{k_0,\rho_0}} (\tilde{v} - k_0)^{p_2} dy \leq d^{-p_2} \int_{B_1} \tilde{v}^{p_2} dy \leq d^{-p_2} M.$$

We apply Lemma 5.1 with $B = 2^{p_2(\beta+\theta)} > 1$, $C = c(1 + d^{-p_2(p_2-p_1)}) > 1$, $0 < a_1 = \theta - 1 < \beta - 1 = a_2$. To guarantee that the condition $\vartheta_0 \leq (2C)^{-\frac{1}{a_1}}$ is satisfied, we have to choose the quantity $d$ in such a way that

$$d^{-p_2} M = (2C)^{-\frac{1}{\theta-1}},$$

i.e.,

$$d^{p_2(\beta-1)} = 2cB\left(1 + d^{-p_2(p_2-p_1)}\right).$$

Note that, since $\beta = \varepsilon + \frac{p_2}{p_1}$, we always have that $p_2(\beta - 1) > p_2(p_2 - p_1)$, which guarantees that equation (52) has a unique solution $0 < d \equiv d(\lambda^{\pm}, p^{\pm}, q^{-}, n, M, p_1, p_2, \Omega, \|g\|_{L^{q(\cdot)}(\Omega)}) < \infty$. In addition, we remark that global boundedness $p^{\pm}$ for $p(\cdot)$ imply that $p_2(\beta - 1) \in [0, p^{+}(p^{+} - p^{-})]$ and $p_2(p_2 - p_1) \in [0, p^{+}(p^{+} - p^{-})]$. Furthermore, the solution $d$ of equation (52) depends continuously on the parameters $p^{-}$ and $p^{+}$.

Now Lemma 5.1 gives $\lim_{i\to\infty} \vartheta_i = 0$, which, noting that $\lim_{i\to\infty} \rho_i = 1$ and $\lim_{i\to\infty} k_i = 2d$, directly translates into $|A_{2d,1}| = 0$ and therefore $\sup_{B_{1/2}} \tilde{v} \leq 2d$. Taking into account the choice of $d$, we end up with

$$\sup_{B_{1/2}} \tilde{v} \leq C \left( \left( \int_{B_1} (\tilde{v}^{+})^{p_2} dy \right)^{\frac{1}{p_2}} + 1 \right),$$

where $C = C(\lambda^{\pm}, p^{\pm}, q^{-}, n, M, \Omega, \|g\|_{L^{q(\cdot)}(\Omega)})$.

An argument similar to the preceding one with the function $-\tilde{v}$, using (43) instead of (42) yields

$$\sup_{B_{1/2}} (-\tilde{v}) \leq C \left( \frac{1}{|B_{1/2}|^{\frac{1}{p_2}}} \left( \int_{B_1} ((-\tilde{v})^{+})^{p_2} dy \right)^{\frac{1}{p_2}} + 1 \right).$$

Therefore (45) and (53) yield the desired estimate (40).

**Third step: Boundedness of $\tilde{v}$: estimate (41).** Firstly we choose some constants we will use for our proof. By (9), we know that $\theta = \varepsilon + \frac{p_2}{p_1 - 1}$, thus we can find a positive constant $\tilde{\alpha}$ small enough such that

$$\frac{n+1-p_2}{p_1-1} > p_2\left(1-\frac{1}{n}\right) = p_2\theta + \tilde{\alpha},$$

and $\varepsilon + \tilde{\alpha} > \tilde{\alpha}$. Then we can find positive constants $\tilde{\beta}, \tilde{\theta}$ small enough such that

$$\theta + \tilde{\beta} \leq \tilde{\theta} \leq \theta + \tilde{\alpha},$$

and

$$\beta - \tilde{\beta} - \frac{p_2}{p_1} + \tilde{\alpha} = \varepsilon - \tilde{\beta} + \tilde{\alpha} \geq \tilde{\alpha}\left(\frac{p_2}{p_1} + \tilde{\beta}\right),$$

where the third inequality implies $\theta - \tilde{\theta} + \tilde{\alpha} \geq \tilde{\alpha}\tilde{\theta}$.

For the above constants, it follows from (48) that

$$\Phi_{k,\rho}|A_{k,\rho}|^{\tilde{\alpha}} \leq C|A_{k,r}|^{\varepsilon}|A_{k,\rho}|^{\tilde{\alpha}}\left(\int_{A_{k,r}}\left|\frac{\tilde{v}-k}{r-\rho}\right|^{\tilde{p}(y)}dy\right)^{\frac{p_2}{\tilde{p}_1}} + C|A_{k,r}|^{\beta}|A_{k,\rho}|^{\tilde{\alpha}}|r-\rho|^{\frac{p_2}{p_1}} + C|r-\rho|^{\frac{p_2}{p_1}}|A_{k,r}|^{\theta}|A_{k,\rho}|^{\tilde{\alpha}} + C|A$$

In the following estimates we use (49), (50), and the fact that $|A_{k,\rho}| \leq |A_{k,r}| \leq |A_{h,r}|$.

Let $\tilde{\Phi}_{k,t} = \Phi_{k,t}|A_{k,t}|^{\tilde{\alpha}}$. Collecting the estimates, we obtain

$$\tilde{\Phi}_{k,\rho} \leq C\tilde{\Phi}_{h,r}^{\tilde{\beta}+\frac{p_2}{p_1}}\left(\frac{|k-h|^{p_2}}{|r-\rho|^{p_2}}\right)^{\tilde{\theta}} + C\tilde{\Phi}_{h,r}^{\tilde{\theta}}\left(\frac{|k-h|^{p_2}}{|r-\rho|^{p_2}} + \frac{|k-h|^{p_2}}{|r-\rho|^{p_1}}\right),$$

where $C$ depends only on $n, q^-, \lambda^{\pm}, p^{\pm}, \|g\|_{L^{q(\cdot)}(\Omega)}$.

To apply Lemma 5.1, taking $d \geq 1$ to be chosen later and setting $k_i = d(1-2^{-i})$, $r_i = \frac{1}{2} + 2^{-i-1}$, we have $k_{i+1} - k_i = d2^{-i-1}$, $r_i - r_{i+1} = \frac{1}{4}2^{-i}$. Rewriting (60) with $\rho = r_{i+1}$, $r = r_i$, $k = k_{i+1}$, $h = k_i$ and $\vartheta_i = d^{-p_2}\tilde{\Phi}_{k_i,r_i}$ and exploiting again the fact that $d \geq 1$, we deduce that

$$\vartheta_{i+1} \leq C\left(1 + \frac{1}{d^{p_2(p_2-p_1)}}\right)2^{ip_2(\tilde{\beta}+\frac{p_2}{p_1}+\tilde{\theta})}(\vartheta_i^{\tilde{\beta}+\frac{p_2}{p_1}} + \vartheta_i^{\tilde{\theta}}).$$

We now choose $d = 1 + \tilde{A}\left(\int_{B_1}\tilde{v}^{p_2}dy\right)^{\frac{1}{p_2}}$, where $\tilde{A}$ will be fixed a bit later. Analogously to the preceding argument we observe that

$$\vartheta_0 = d^{-p_2}\tilde{\Phi}_{k_0,r_0} \leq \tilde{A}^{-p_2}.$$

We apply Lemma 5.1 with $B = 2^{p_2\tilde{\theta}} > 1$, $C = c(1 + \tilde{A}^{p_2-p_1}) > 1$, $0 < a_1 = \tilde{\beta} + \frac{p_2}{p_1} - 1 \leq \tilde{\theta} - 1 = a_2$. To guarantee that the condition $\vartheta_0 \leq (2C)^{-\frac{1}{a_1}}$ is satisfied, we have to choose the quantity $\tilde{A}$ in such a way that

$$\tilde{A}^{-p_2} = (2C)^{-\frac{1}{\tilde{\beta}+\frac{p_2}{p_1}-1}},$$

i.e.,

$$\tilde{A}^{p_2(\tilde{\beta}+\frac{p_2}{p_1}-1)} = 2cB^{-1}(1 + \tilde{A}^{p_2-p_1}).$$

We note that $\tilde{\beta} > 0$ which guarantees equation (61) has a unique solution $0 < \tilde{A} < \infty$. Here $\tilde{A} \equiv \tilde{A}(n, q^-, M, p^\pm, \lambda^\pm, \|g\|_{L^{q(\cdot)}(\Omega)})$.

By Lemma 5.1, we conclude that $\lim_{i\to\infty} \vartheta_i = 0$, which, noting that $\lim_{i\to\infty} r_i = \frac{1}{2}$ and $\lim_{i\to\infty} k_i = d$, directly translates into $|A_{d,1/2}| = 0$ and therefore we deduce that

$$\sup_{B_{1/2}} \tilde{v} \leq d = C\left(\left(\int_{B_1} (\tilde{v}^+)^{p_2} dy\right)^{\frac{1}{p_2}} + 1\right),$$

with $C = C(\tilde{A}, n, q^-, M, p^\pm, \lambda^\pm, \|g\|_{L^{q(\cdot)}(\Omega)})$. We should note that the constant $C$ may be replaced by a constant $C = C(n, q^-, M, p^\pm, \lambda^\pm, \|g\|_{L^{q(\cdot)}(\Omega)})$.

Now we turn to prove local boundedness for minimizers of the functional $H_\gamma$.

**Lemma 10.** Let $v$ be a minimizer of $H_\gamma$ governed by (38). Then $v$ is locally bounded and satisfies the estimates

$$\sup_{B_{R/2}(x_0)} \pm v \leq C\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} ((\pm v)^+)^{p_2} dy\right)^{\frac{1}{p_2}} + CR,$$

$$\sup_{B_{R/2}(x_0)} v \leq C\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} ((v-\kappa_0)^+)^{p_2} dy\right)^{\frac{1}{p_2}} + R + \kappa_0,$$

for some $\alpha > 0$, for all $\kappa_0 \leq \sup_{B_R(x_0)} v$, where $C = C(n, L, q^-, M, p^\pm, \lambda^\pm, \|g\|_{L^{q(\cdot)}(\Omega)})$.

**Proof of Lemma 10.** Indeed, by the definition of $\tilde{v}$ and Lemma 9, it follows that

$$\sup_{x\in B_{R/2}} v(x) = R \sup_{y\in B_{1/2}} \tilde{v}(y) \leq CR\left(\left(\int_{B_1} (\tilde{v}^+)^{p_2} dy\right)^{\frac{1}{p_2}} + 1\right) \leq C\left(\left(\frac{1}{|B_R|}\int_{B_R} (v^+)^{p_2} dx\right)^{\frac{1}{p_2}} + R\right).$$

Estimate (62) can be obtained via (41) by a similar argument, taking into account that $|A_{0,R}| = R^n |A_{0,1}|$ and then writing $v - \kappa_0$ instead of $v$.

**Lemma 11.** Let $v$ be a minimizer of $H_\gamma$ governed by (38). Then for every couple of balls $B_\rho \subset B_r \subset B_R$ having the same center $x_0$ and for every $k \in \mathbb{R}$ the following two estimates hold

$$\int_{B_\rho} |\nabla v|^{p(x)} dx \leq C \int_{B_r} \left| \frac{v-k}{r-\rho} \right|^{p(x)} dx + Cr^{\lambda_0 + n} + Cr^n,$$

$$\int_{B_\rho} |\nabla v|^{p(x)} dx \leq C \int_{B_r} \left| \frac{v-k}{r-\rho} \right|^{p(x)} dx + Cr^{\lambda_0 + n} + Cr^n,$$

with $\lambda_0 = \min\{\frac{\gamma p_2}{p_2 - \gamma}, \frac{p_1}{p_1 - 1}(1 - \frac{n}{q^-})\} \geq 0$, $C = C(n, L, p^\pm, \lambda^\pm, \|g\|_{L^{q(\cdot)}(\Omega)})$.

**Proof.** We employ an argument similar to the one used to obtain (42), getting

$$\int_{B_\rho} |\nabla v|^{p(x)} dx \leq C \int_{B_r \setminus B_\rho} |\nabla v|^{p(x)} dx + C \int_{B_r} \left| \frac{v-k}{r-\rho} \right|^{p(x)} dx + C \int_{B_r} \frac{|r-\rho|^{\frac{\gamma p(x)}{p(x)-\gamma}}}{|r-\rho|^{p(x)}} dx + C \int_{B_r} (|g||r-\rho|)^{\frac{p(x)}{p(x)-1}} dx.$$

"Filling the hole" and using Lemma 4.2, we obtain the desired estimate.

Now we shall prove Hölder regularity for the minimizers of the functional $H_\gamma$.

**Proof of Theorem 2.** Let $v$ be a minimizer of the functional $H_\gamma$ governed by (38). Let $\operatorname{osc}(v, \rho) = \sup_{B_\rho} v - \inf_{B_\rho} v$. Due to Lemma 11, one may proceed exactly as in [**?**, Lemma 4.10], to see that the minimizer $v$ has also an estimate as (4.40) in [**?**, Lemma 4.10]. Again, due to Lemma 10 and proceeding as in [**?**, Proposition 4.11], we have

$$\operatorname{osc}(v, \rho) \leq c \left( \frac{\rho}{r} \right)^{\alpha_1} \operatorname{osc}(v, r) + \rho^{\alpha_1}, \quad \forall \rho < r < R,$$

for some $0 < \alpha_1 < 1$. By a slight modification of proof of [**?**, Proposition 4.12], (63) gives

$$\int_{B_\rho} |v - (v)_\rho|^{p_2} dx \leq C \left( \frac{\rho}{R} \right)^{n + p_2 \alpha_1} \int_{B_R} |v - (v)_R|^{p_2} dx + C\rho^{n + p_2 \alpha_1},$$

$$\int_{B_\rho} |\nabla v|^{p(x)} dx \leq C \left( \frac{\rho}{R} \right)^{n - p_2 + p_2 \alpha_1} \int_{B_R} |\nabla v|^{p(x)} dx + C\rho^{n - p_2 + p_2 \alpha_1}.$$

It follows from Lemma 5.2 that

$$\int_{B_\rho} |v-(v)_\rho|^{p_2} dx \le C\rho^{n+p_2\alpha_1}, \quad \int_{B_\rho} |\nabla v|^{p(x)} dx \le C\rho^{n-p_2+p_2\alpha_1}.$$

Notice that each of the above inequalities combining with covering theorem implies $v \in C^{0,\alpha_1}_{\mathrm{loc}}(\Omega)$. This concludes the proof.

## 6 Hölder estimates for minimizers of functional $J_\gamma$

**Proof of Theorem 3.** We proceed in five steps.

**First step: Localization.** Let $\delta_1 < \min\{p^- - 1, \delta_0\}$ that will be chosen much smaller a bit later. Fix a ball $B_{R_0} \Subset \Omega$ with the property $\omega(8R_0) < \delta_1$. Let $B_{4R} \Subset B_{R_0}$. Define $p_2 = \max_{B_{4R}} p(x)$, $p_1 = \min_{B_{4R}} p(x)$. We remark that by continuity of $p(x)$, there exists $x_0 \in B_{4R}$, not necessarily the center, such that $p_2 = p(x_0)$. Consequently we obtain

$$p_2 - p_1 \le \omega(8R) \le \delta_1.$$

Furthermore we note the localization together with the bound (7) for the modulus of continuity yields for any $8R \le R_0 \le 1$:

$$R^{-n}\omega(R) \le \exp(nL) = c(n,L), \quad R^{-\frac{n}{1+\omega(R)}}\omega(R) \le c(n,L).$$

In the following proofs we consider all the balls with the same center $x_0$.

**Second step: Higher integrability.** By our higher integrability result (Proposition 4.1) and localization, it holds that

$$\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla u|^{p_2(1+\frac{\delta_1}{4})} dx \le C_0 \left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla u|^{p(x)} dx\right)^{1+\frac{\delta_1}{4}} + C_0 \left(\frac{1}{|B_{2R}|}\int_{B_{2R}} (1+|g|)^{\frac{p^-}{p^- -1}(1+\frac{\delta_1}{4})} dx\right)^{\frac{1}{1+\frac{\delta_1}{4}}}.$$

**Third step: Freezing.** Let $v \in W^{1,p_2}(B_R)$ with $v-u \in W^{1,p_2}_0(B_R)$ be a minimizer of the functional

$$G(v) = \int_{B_R} f(x_0, \nabla v) dx = \int_{B_R} \tilde{h}(\nabla v) dx.$$

Note that by Remark 1 and the growth condition (4), we obtain the following estimate for the $p_2$ energy of $v$

$$\int_{B_R} |\nabla v|^{p_2} dx \le L^2 \int_{B_R} (1+|\nabla u|^{p_2}) dx < \infty.$$

Moreover, in view of [**?**, Lemma 3.1], there exist $C = C(p^{\pm}, L)$, $\delta_2 = \delta_2(p^{\pm}, L)$ with $0 < \delta_2 < \frac{q^-}{p_2}$ such that

$$\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla v|^{p_2(1+\delta_2)}dx\right)^{\frac{1}{1+\delta_2}} \leq C\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla v|^{p_2}dx\right)^{\frac{1}{p_2}} + C\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla u|^{q}dx\right)^{\frac{1}{q}},$$

for $q = p_2(1 + \frac{\delta_1}{4}) > p_2$. By the proof of Theorem 2, and the boundedness of $v$, which is guaranteed by the boundedness of $u$, there exists some $\alpha_2 \in (0, 1)$ such that

$$\int_{B_\rho} |\nabla v|^{p_2}dx \leq C\left(\frac{\rho}{R}\right)^{n-p_2+p_2\alpha_2}\int_{B_R} |\nabla v|^{p_2}dx + C\rho^{n-p_2+p_2\alpha_2}, \quad \forall \rho \text{ with } 2\rho < R.$$

**Fourth step: Comparison estimate.** We prove the following comparison estimate

$$\int_{B_R} (\mu^2+|\nabla u|^2+|\nabla v|^2)^{\frac{p_2-2}{2}}|\nabla u-\nabla v|^2dx \leq C\left(\omega(R)\log\left(\frac{1}{R}\right) + R^{\theta_1} + R^{\theta_2}\right)\int_{B_R} (1+|\nabla u|^{p_2})dx + C\omega(R)\log\left(\frac{1}{R}\right)$$

for some $0 < \lambda_1 < n$, $\lambda_2 > n$, $\lambda_3 > n$.

A similar argument to the one in [**?**, (4.10)] yields

$$\int_{B_R} (\tilde{h}(\nabla u) - \tilde{h}(\nabla v))dx \geq C\int_{B_R} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p_2-2}{2}}|\nabla u - \nabla v|^2dx.$$

On the other hand, we derive

$$\int_{B_R} (\tilde{h}(\nabla u)-\tilde{h}(\nabla v))dx = \int_{B_R} (f(x_0, \nabla u)-f(x, \nabla u))dx + \int_{B_R} (f(x, \nabla u)-f(x, \nabla v))dx + \int_{B_R} (f(x, \nabla v)-f(x_0, \nabla v))$$

We estimate $I^{(1)}$, using the continuity of the integrand with respect to the variable $x$ (see (2.3)),

$$I^{(1)} \leq C\int_{B_R} \omega(|x-x_0|)\left[(\mu^2 + |\nabla u|^2)^{\frac{p(x)}{2}} + (\mu^2 + |\nabla u|^2)^{\frac{p(x_0)}{2}}\right][1 + |\log(\mu^2 + |\nabla u|^2)|]\,dx.$$

Arguing exactly as [**?**, Section 4], we obtain

$$I^{(1)} \leq C\omega(R)\int_{B_R} |\nabla u|^{p_2} \log\left(e+\frac{|\nabla u|^2}{\||\nabla u|^2\|_{L^1(B_R)}}\right) dx + C\omega(R)\int_{B_R} |\nabla u|^{p_2} dx + C\omega(R)R^n.$$

Now we estimate the first term, using first [**?**, (3.3)], which is a basic estimate for the $L\log L$ norm, then exploiting higher integrability,

$$I^{(1)} \leq C\omega(R)R^n\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla u|^{p_2(1+\frac{\delta_1}{4})} dx\right)^{\frac{1}{1+\frac{\delta_1}{4}}} + C\omega(R)R^n$$

$$\leq C\omega(R)R^n\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla u|^{p(x)} dx\right)^{1+\omega(R)} + C\omega(R)R^n\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} (1+|g|)^{\frac{p^-}{p^--1}(1+\frac{\delta_1}{4})} dx\right)^{\frac{1}{1+\frac{\delta_1}{4}}} + C\omega(R)R^n$$

$$\leq C\omega(R)\int_{B_R} (1+|\nabla u|^{p_2}) dx + C\omega(R)R^{\lambda_1},$$

where $\lambda_1 = n - \frac{n}{1+\delta_1} + n\left[1-\frac{1}{p_1-1}\left(1+\frac{\delta_1}{4}\right)\right]\frac{1}{1+\frac{\delta_1}{4}}$. Notice that $\delta_1 < \delta_0 < q^-(1-\frac{1}{p^-})-1$, therefore $0<\lambda_1<n$.

Thus, all together we obtain

$$I^{(1)} \leq C\omega(R)\log\left(\frac{1}{R}\right)\int_{B_R} (1+|\nabla u|^{p_2}) dx + C\omega(R)R^{\lambda_1}.$$

We shall estimate $I^{(2)}$. By the minimizing property of $u$ and arguing as in Section 4, we have

$$I^{(2)} \leq \int_{B_R} (F_\gamma(v)-F_\gamma(u)+g(v-u)) dx \leq C\int_{B_R} |v-u|^\gamma dx + C\int_{B_R} |g||v-u| dx.$$

Using Young' s inequality and the boundedness of $v$ and $u$, we get

$$I^{(2)} \leq C(R^{\theta_1}+R^{\theta_2})\int_{B_R} (|v|^{p_2}+|u|^{p_2}) dx + CR^{\lambda_2} + CR^{\lambda_3},$$

where $\lambda_2 = n + \frac{p_2\gamma}{p_2-\gamma} - \frac{\gamma\theta_1}{p_2-\gamma} \geq n$, $\lambda_3 = n\left(1+\frac{p_2}{p_2-1}\left(\frac{1}{n}-\frac{1}{q^-}\right)\right) - \frac{\theta_2}{p_2-1} > n$ and in the last inequality we used (64).

We deal with $I^{(3)}$ in a similar way to $I^{(1)}$. Estimating in exactly the same way as in (70) with $v$ instead of $u$ and doing the same splitting into $I^{(1)}$ to $I^{(3)}$, we use higher integrability of $v$ and $u$ ((65) and Proposition 4.1) to obtain

$$I^{(3)} \leq C\omega(R)\log\left(\frac{1}{R}\right)\int_{B_R}(1+|\nabla u|^{p_2})dx + C\omega(R)R^{\lambda_1}.$$

From (68) to (71), one may obtain (67).

**Fifth step: Conclusion.** Now we turn to prove a decay estimate for the $p_2$ energy of $u$. We split as follows:

$$\int_{B_\rho}|\nabla u|^{p_2}dx \leq \int_{B_\rho}(\mu^2+|\nabla u|^2)^{\frac{p_2-2}{2}}|\nabla u|^2dx+C\int_{B_\rho}(\mu^2+|\nabla v|^2)^{\frac{p_2-2}{2}}|\nabla v|^2dx+C\int_{B_\rho}(\mu^2+|\nabla u|^2+|\nabla v|^2)^{\frac{p_2-2}{2}}|\nabla u-$$

For $A$, we deduce from (64) and (66) that

$$A \leq C\rho^n+C\left(\frac{\rho}{R}\right)^{n-p_2+p_2\alpha_2}\int_{B_R}|\nabla v|^{p_2}dx \leq C\left(\frac{\rho}{R}\right)^{n-p_2+p_2\alpha_2}\int_{B_R}(1+|\nabla u|^{p_2})dx+C\rho^{n-p_2+p_2\alpha_2}.$$

For $B$, by the comparison estimate (67), it follows that

$$B \leq C\left(\omega(R)\log\left(\frac{1}{R}\right)+R^{\theta_1}+R^{\theta_2}\right)\int_{B_R}(1+|\nabla u|^{p_2})dx+C\omega(R)\log\left(\frac{1}{R}\right)R^{\lambda_1}+CR^{\lambda_2}+CR^{\lambda_3}.$$

Note that $\lambda_1 < n < \lambda_2, \lambda_3$, then we have

$$\int_{B_\rho}|\nabla u|^{p_2}dx \leq C\left(\left(\frac{\rho}{R}\right)^{n-p_2+p_2\alpha_2}+\omega(R)\log\left(\frac{1}{R}\right)+R^{\theta_1}+R^{\theta_2}\right)\int_{B_R}(1+|\nabla u|^{p_2})dx+CR^{\lambda_1}+CR^{n-p_2+p_2\alpha_2}.$$

On the other hand, by (9) we have

$$\lambda_1 = n-\frac{n}{1+\delta_1}+n\left[1-\frac{1}{p_1-1}\left(1+\frac{\delta_1}{4}\right)\right]\frac{1}{1+\frac{\delta_1}{4}} \geq n-p_2+p_2\alpha_2.$$

Thus $p_1-1 > n-p_2$, therefore we may choose $\delta_1$ and $\alpha_2$ small enough such that

$$\int_{B_\rho}|\nabla u|^{p_2}dx \leq C\left(\left(\frac{\rho}{R}\right)^{n-p_2+p_2\alpha_2}+\omega(R)\log\left(\frac{1}{R}\right)+R^{\theta_1}+R^{\theta_2}\right)\int_{B_R}(1+|\nabla u|^{p_2})dx+CR^{n-p_2+p_2\alpha_2}.$$

In order to apply Lemma 5.2, we may take $R_1 > 0$ small enough such that $\omega(R)\log(\frac{1}{R}) + R^{\theta_1} + R^{\theta_2}$ is smaller than $\mu$ in Lemma 5.2 for any $0 < R \leq R_1$. Thus there holds

$$\int_{B_\rho} |\nabla u|^{p_2} dx \leq C\rho^{n-p_2+p_2\alpha_3} \leq C\rho^{n-p_1+p_1\alpha_3}, \quad \forall 0 < \alpha_3 < \alpha_2.$$

By a standard covering argument we deduce that $u \in L^{p^-,\lambda}(\Omega)$ with $\lambda = n + p^-\alpha_3$, where $L^{p,\lambda}(\Omega)$ denotes Campanato' s spaces, the definition of which can be found in [**?**], for instance. Poincaré inequality and a well-known property of functions in Campanato' s spaces (see [**?**] for instance) imply that $u \in C^{0,\alpha_3}_{\text{loc}}(\Omega)$.

## 7 $C^{1,\alpha}$ estimates for minimizers of $J_\gamma$ $(\gamma \in (0,1])$

**Proof of Theorem 4 ($0 < \gamma \leq 1$).** The proof consists of three steps.

**First step: localization and freezing.** Firstly, by (2.10), we can choose $\delta_3 > 0$ small enough such that $\varsigma > \frac{n}{p^- - 1}$. Now let $\delta = \min\{\delta_0, \delta_1, \delta_2, \delta_3\}$. We adopt the same localization argument as the proof of Theorem 2.3. In this case all the balls $B_{CR}$ and the exponents $p_1, p_2$ that we consider here are the same as in the proof of Theorem 2.3 (replace $\delta_1$ with $\delta$ in Section 6). Let $v \in W^{1,p_2}(B_R)$ with $v - u \in W^{1,p_2}_0(B_R)$ be a minimizer of the functional

$$G_0(v) = \int_{B_R} f(x_0, \nabla v) dx = \int_{B_R} \tilde{f}(\nabla v) dx.$$

We note that since $v$ is a minimizer of the functional $G_0$ with boundary data $u$ on $\partial B_R$, where $u|_{\partial B_R}$ is the trace of a Hölder continuous function. By Theorem 7.8 in [**?**], we conclude that $v \in C^{0,\alpha_4}$ for some $\alpha_4 \in (0,1)$. Therefore, for the rest of the proof we assume that

$$|v(x) - v(y)| \leq [v]_{\alpha_4} |x-y|^{\alpha_4} \leq C|x-y|^{\alpha_4},$$

holds for all $x, y \in \overline{B}_R$. We remark that for simplicity we will use the same Hölder exponent for the functions $v$ and $u$, which is not restrictive. Let us remark that, since $v$ minimizes the functional (72), by the growth condition (4), higher integrability and Remark 3.2, we obtain the following estimate for the $p_2$ energy of $v$

$$\int_{B_R} |\nabla v|^{p_2} dx \leq L^2 \int_{B_R} (1 + |\nabla u|^{p_2}) dx < \infty.$$

**Second step: Comparison estimate.** We will show that

$$\int_{B_R} |\nabla u - \nabla v|^{p_2} dx \leq CR^{\theta_5} \int_{B_R} (1 + |\nabla u|^{p_2}) dx,$$

for some $\theta_5 > 0$.

Firstly we prove

$$G_0(u) - G_0(v) \leq C \left( \omega(R) \log\left(\frac{1}{R}\right) + R^{\theta_1} + R^{\theta_2} \right) \int_{B_R} (1 + |\nabla u|^{p_2}) dx + C\omega(R) \log\left(\frac{1}{R}\right) R^{\lambda_1} + CR^{\lambda_2} + CR^{\lambda_3},$$

for some $0 < \lambda_1 < n$, $\lambda_2 > n$, $\lambda_3 > n$.

Indeed, since $u$ is a minimizer of the functional (2), we obtain

$$\int_{B_R} f(x, \nabla u) dx \leq \int_{B_R} f(x, \nabla v) dx + \int_{B_R} (F_\gamma(v) - F_\gamma(u)) dx + \int_{B_R} g(v - u) dx,$$

which implies

$$\int_{B_R} f(x_0, \nabla u) dx \leq \int_{B_R} f(x_0, \nabla v) dx + \int_{B_R} (f(x_0, \nabla u) - f(x, \nabla u)) dx + \int_{B_R} (f(x, \nabla u) - f(x, \nabla v)) dx + \int_{B_R} (f(x, \nabla v) -$$

Arguing as $I^{(1)}$, $I^{(3)}$, $I^{(2)}$ in Section 6, we obtain

$$I^{(4)} + I^{(5)} \leq C\omega(R) \log\left(\frac{1}{R}\right) \int_{B_R} (1 + |\nabla u|^{p_2}) dx + C\omega(R) R^{\lambda_1},$$

where $0 < \lambda_1 = n - \frac{n}{1+\delta_1} + n \left[1 - \frac{1}{p_1 - 1} \left(1 + \frac{\delta_1}{4}\right)\right] \frac{1}{1 + \frac{\delta_1}{4}} < n$.

$$I^{(6)} + I^{(7)} \leq C(R^{\theta_1} + R^{\theta_2}) \int_{B_R} (1 + |\nabla u|^{p_2}) dx + CR^{\lambda_2} + CR^{\lambda_3},$$

where $\lambda_2 = n + \frac{p_2 \gamma}{p_2 - \gamma} - \frac{\gamma \theta_1}{p_2 - \gamma} > n$, $\lambda_3 = n \left(1 + \frac{p_2}{p_2 - 1} \left(\frac{1}{n} - \frac{1}{q^-}\right)\right) - \frac{\theta_2}{p_2 - 1} > n$.

Therefore we may conclude (75) from (76) to (78). Since $\varsigma > \frac{n}{p^- - 1}$, we may choose $\theta_3 > 0$ small enough such that $\varsigma \geq \frac{n}{p^- - 1} + \theta_3$. Again we may choose $0 < \theta_4 < \theta_3$ such that

$$\varsigma + \lambda_1 - \theta_4 \geq \frac{n}{p^- - 1} + \theta_3 - \theta_4 > \frac{n}{p^- - 1}.$$

By the assumption that $\omega(R) \leq LR^{\varsigma}$, we get

$$\omega(R) \log \left(\frac{1}{R}\right) R^{\lambda_1} \leq LR^{\varsigma} R^{\lambda_1} R^{-\theta_4} R^{\theta_4} \log \left(\frac{1}{R}\right) \leq CLR^{\varsigma+\lambda_1-\theta_4},$$

for $R$ small enough.

We deduce from (75), (79) and (80) that

$$G_0(u) - G_0(v) \leq CR^{\theta_5} \int_{B_R} (1 + |\nabla u|^{p_2}) dx,$$

where $0 < \theta_5 = \theta_5(\theta_1, \theta_2, \theta_3, \theta_4, \lambda_1, \lambda_2, \lambda_3, n, q^-, p^{\pm}, \varsigma, \delta)$, $C$ is independent of $\theta_5$ and $\gamma$.

Since the integrand is of class $C^2$, we conclude from [**?**, pp. 131, 137-138] that

$$\int_{B_R} |\nabla u - \nabla v|^{p_2} dx \leq CR^{\theta_5} \int_{B_R} (1 + |\nabla u|^{p_2}) dx,$$

which completes the proof of (74).

**Third step: Conclusion.** Firstly applying Jensen' s inequality we get

$$\int_{B_\rho} |(\nabla u)_\rho - (\nabla v)_\rho|^{p_2} dx \leq \int_{B_\rho} \left| \frac{1}{|B_\rho|} \int_{B_\rho} (\nabla u - \nabla v) dx \right|^{p_2} dx \leq \int_{B_\rho} \frac{1}{|B_\rho|} \int_{B_\rho} |\nabla u - \nabla v|^{p_2} dx\, dx = \int_{B_\rho} |\nabla u - \nabla v|^{p_2} d$$

Secondly, by [**?**, (3.20)], we have

$$\int_{B_\rho} |\nabla v - (\nabla v)_\rho|^{p_2} dx \leq C \left(\frac{\rho}{R}\right)^{n+\beta p_2} \int_{B_R} (1 + |\nabla v|^{p_2}) dx,$$

where $C > 0$, $0 < \beta < 1$ and both $C$ and $\beta$ depend only on $p^{\pm}, L$.

Now combining comparison estimate with (74) and (82), we deduce for any $0 < \rho < \frac{R}{2} < R_1$

$$\int_{B_\rho} |\nabla u - (\nabla u)_\rho|^{p_2} dx \leq C \int_{B_\rho} |\nabla u - \nabla v|^{p_2} dx + C \int_{B_\rho} |\nabla v - (\nabla v)_\rho|^{p_2} dx + C \int_{B_\rho} |(\nabla v)_\rho - (\nabla u)_\rho|^{p_2} dx$$

$$\leq C \left(\frac{\rho}{R}\right)^{n+\beta p_2} \int_{B_R} (1 + |\nabla u|^{p_2}) dx + CR^{\theta_5} \int_{B_R} (1 + |\nabla u|^{p_2}) dx.$$

On the other hand, we obtain (see [**?**, pp. 133] for more details),

$$\int_{B_\rho} |\nabla v|^{p_2} dx \le C\left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla v|^{p_2} dx + CR^n.$$

Therefore it follows from (73) that

$$\int_{B_\rho} |\nabla u|^{p_2} dx \le C\left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla u|^{p_2} dx + C\omega(R)\log\left(\frac{1}{R}\right)\int_{B_R} |\nabla u|^{p_2} dx + CR^{\theta_5}\int_{B_R}(1+|\nabla u|^{p_2})dx + CR^n.$$

Thus for small $R$, applying Lemma 5.2, we obtain

$$\int_{B_\rho} |\nabla u|^{p_2} dx \le C\rho^{n-\tau}, \quad \forall \tau \in (0,1).$$

Now let $\rho = \frac{1}{2}R^{1+\theta_6}$ with $\theta_6 = \frac{\theta_5}{2(n+\beta p_2)}$, and let $\tau = \frac{\theta_5 \beta p_2}{n+\beta p_2}$. Then we deduce from (84) that

$$\int_{B_\rho} |\nabla u - (\nabla u)_\rho|^{p_2} dx \le C\rho^{\theta_7},$$

with $\theta_7 = \frac{n+\theta_5\beta p_2}{n+\beta p_2+\theta_5}$. Since we can choose $\theta_5$ sufficiently small, thus we conclude that $Du \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1 - \frac{n-\theta_7}{p_2}$, which completes the proof of Theorem 2.4 with $0 < \gamma \le 1$.

## 8 Log-Lipschitz estimates for minimizers of $J_0$

**Proof of Theorem 4 ($\gamma = 0$).** We proceed along the lines of proof in Section 7. Notice that $\lambda_2 = n + \frac{p_2\gamma}{p_2-\gamma} - \frac{\gamma\theta_1}{p_2-\gamma} = n$ with $\gamma = 0$. Therefore (81) becomes

$$\int_{B_R} |\nabla u - \nabla v|^{p_2} dx \le CR^{\theta_5'}\int_{B_R}(1+|\nabla u|^{p_2})dx + CR^n,$$

where $0 < \theta_5' = \theta_5'(\theta_1, \theta_2, \theta_3, \theta_4, \lambda_1, \lambda_3, n, q^-, p^\pm, \varsigma, \delta)$, $C$ is independent of $\theta_5'$.

Thus, (83) becomes

$$\int_{B_\rho} |\nabla u - (\nabla u)_\rho|^{p_2} dx \le C\left(\frac{\rho}{R}\right)^{n+\beta p_2} \int_{B_R} |\nabla u - (\nabla u)_{4R}|^{p_2} dx + CR^{\theta_5'}\int_{B_R}(1+|\nabla u|^{p_2})dx + CR^n.$$

Now Lemma 5.2 implies that for any fixed subdomain $\Omega' \Subset \Omega$, there holds

$$\int_{B_\rho} |\nabla u - (\nabla u)_\rho|^{p_2} dx \le C\rho^n,$$

which shows that the gradient of $u$ lies in BMO space and

$$\|\nabla u\|_{BMO(\Omega)} \le C(\Omega', n, p^\pm, \lambda^\pm, \|g\|_{L^{q(\cdot)}(\Omega)}, M).$$

Then arguing exactly as in [?], one has

$$|u(x) - u(x_0)| \le C|x - x_0| \cdot |\log|x - x_0||.$$

The proof of Theorem 2.4 is concluded.

**Remark 2.** It should be mentioned that the regularity results in [?], where Ekeland' s variational principle was applied to the establishment of regularity in the obstacle problem associated with the functional $\int_\Omega f(x, u, \nabla u)dx$, are stronger than the corresponding one in [?]. We believe that Ekeland's variational principle can be also applied to the following heterogeneous, two-phase free boundary problem

$$\int (f(x, u, \nabla u) + F_\gamma(u) + gu)dx \to \min,$$

under non-standard growth conditions, and obtain stronger regularities than the results in this paper.

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